Unit 3 *Functions* Introduction

Learning objectives

This unit is focused on functions, looking at a more mathematical way of analysing their properties. After completing this unit you should

- understand the concepts of the domain and range of a function and 1 : 1 mappings
- understand and be able to use composite functions
- have confidence in finding the inverse of a function
- be able to transform graphs of functions.

Introduction

The Persian mathematician, Sharaf al-Din al Tusi

The history of the function concept in mathematics dates back to the Persian mathematician, *Sharaf al-Din al Tusi*, in the 12th century. In his analysis of the equation $x^3 + d = bx^2$ for example, he begins by changing the equation's form to $x^2(b - x) = d$. He then states that the question of whether the equation has a solution depends on whether or not the 'function' on the left side reaches the value d. To determine this, he finds a maximum value for the function. Sharaf al-Din then states that if this value is less than d, there are no positive solutions; if it is equal to d, then there is one solution; and if it is greater than d, then there are two solutions.

The German mathematician and philosopher, Gottfried Leibniz

As a mathematical term, 'function' was coined by *Gottfried Leibniz* in a 1673 letter, to describe a quantity related to a curve, such as a curve's slope at a specific point. The functions Leibniz considered are today called differentiable functions. For this type of function, one can talk about limits and derivatives; both are measurements of the output or the change in the output as it depends on the input or the change in the input. Such functions are the basis of calculus.

Johann Bernoulli, by 1718, had come to regard a function as any expression made up of a variable and some constants. *Leonhard Euler*, during the mid-18th century, used the word to describe an expression or formula involving variables and constants, for example, $x^2 + 3x + 2$, whilst *Alexis Claude Clairaut* (in approximately 1734) and Euler introduced the familiar notation '$f(x)$.'

The German mathematician Dirichlet and the Russian, Lobachevsky, are traditionally credited with independently giving the modern 'formal' definition of a function as a relation in which every first element has a unique second element; that is, it maps values from the domain onto the co-domain.

It is this modern version of a function that is used in this unit of work.

Key points

- A function is defined for a specified range of values; this is called the *domain* of the function
- The *range* of a function is the set of values that the function maps onto.
- A function has an inverse only if it is a 1 : 1 mapping.

Unit 3 *Functions* Introduction

Facts to remember

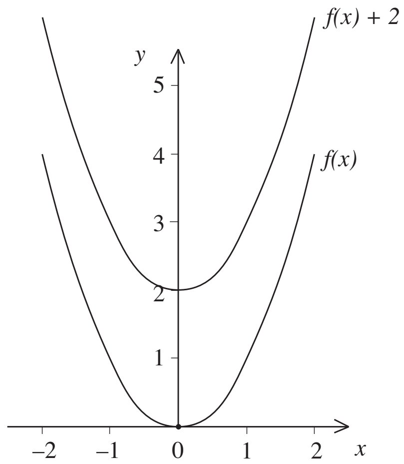
- An alternative function notation is $f : x \rightarrow f(x)$

  For example, you could write $y = x^2$ as $f : x \rightarrow x^2$

- The composite function fg (or $fg(x)$) means $f(g(x))$

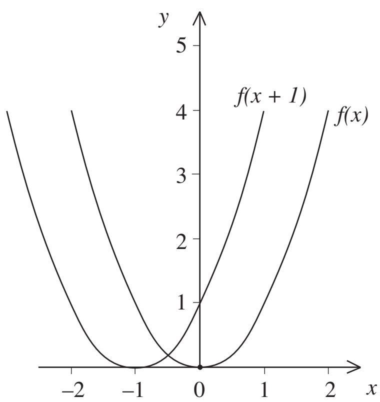
- The inverse function of *f* is denoted by f^{-1} and

  $$f f^{-1}(x) = f^{-1} f(x) = x$$



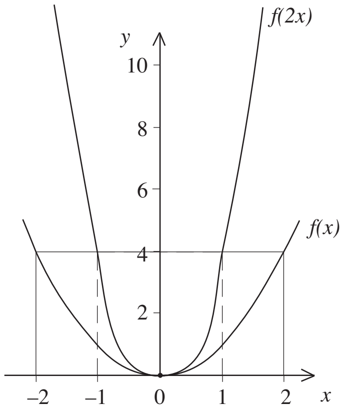
The graph of $y = f(x)$ is mapped onto the graph of $y = f(x) + 2$ by translating it up 2 units.

> In general $f(x) + a$ moves a curve up *a* units and $f(x) - a$ moves it down *a* units, where *a* is a positive number.



The graph of $y = f(x)$ is mapped onto $f(x + 1)$ by a translation of 1 unit to the left.

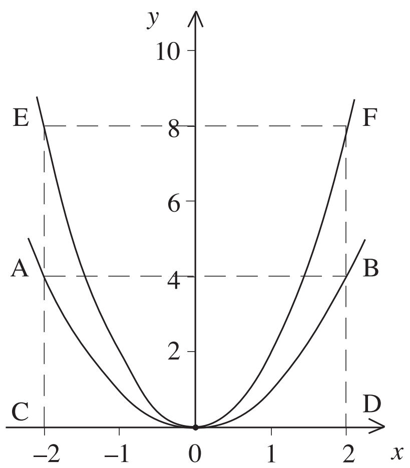
> In general $f(x + a)$ translates a curve *a* units to the left and $f(x - a)$ translates a curve *a* units to the right, where *a* is a positive number.



The curve for $f(2x)$ is much steeper than for $f(x)$. This is because the curve has been compressed by a factor of 2 in the *x*-direction. Compare the rectangles ABCD and EFGH.

> In general the curve of $y = f(kx)$ will be compressed by a factor of *k* in the *x*-direction, where $k > 1$.

Unit 3 *Functions* Introduction



Here the curve $y = f(x)$ has been stretched by a factor of 2 in the vertical or y-direction to obtain the curve $y = 2f(x)$. Compare the rectangles ABCD and CDFE.

> In general the curve of $y = kf(x)$ stretches the graph of $y = f(x)$ by a factor of k in the y-direction where $k > 1$.

Note that if k is negative and $k < -1$, the curve will be stretched and reflected in the x-axis, while if $-1 < k < 1$, it is compressed.

Glossary of terms

The *domain* of a function is the value of x for which the function is defined.

For example, $f(x) = x^2 + 1$ for $x \geq 0$ (domain is $x \geq 0$)

$$g(x) = \frac{1}{x-1} + \frac{1}{x-2} \quad \text{for } 1 < x < 2 \quad (\text{domain is } 1 < x < 2)$$

The *range* of a function is the set of values that the function maps onto.

For example, if $f(x) = x^2,\ 0 \leq x \leq 5$,

the range of f is $0 \leq f(x) \leq 25$.

1 : 1 mapping is a function for which every value of $f(x)$ is unique; that is, if $f(a) \neq f(b)$ unless $a = b$.

For example, $f(x) = x + 1$ is a 1 : 1 mapping, but

$f(x) = x^2$ is *not* a 1 : 1 mapping as, for example,

$f(2) = 4 = f(-2)$

Unit 3 *Functions* Introduction

The *composite function* fg or $fg(x)$ is defined as $f(g(x))$

For example, if $f(x) = x + 1$ and $g(x) = x^2$, then

$$fg(x) = f\left(x^2\right) = x^2 + 1$$

and

$$gf(x) = g(x + 1) = (x + 1)^2$$

The *inverse function* of f is denoted by f^{-1} and is such that

$$ff^{-1}(x) = x = f^{-1}f(x)$$

The inverse only exists if the function is a 1 : 1 mapping.