Remembering Zoltan Dienes, a Maverick of Mathematics Teaching and Learning: Applying the Variability Principles to Teach Algebra

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Abstract

This paper is written in honor of Zoltan Paul Dienes, an internationally renowned mathematician and educator, who passed away in January 2014. It is an attempt to describe, analyze and apply Dienes' theory on how mathematical structures can be taught by applying his four principles of learning upon which he believed a teacher can base concept development around the use of multiple embodiments through manipulatives, how such usage leads to abstraction, and the implications for teaching mathematics in today's mathematics classroom. It illustrates how abstraction can result from the passage of concrete manipulations of objects to representational mapping of such manipulations and then to formalizing such representations into rule structures through the application of Dienes' four principles to teach the concepts and processes of simplification of algebraic expressions.

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On January 11, 2014, we received the sad news that Zoltan Paul Dienes, internationally renowned mathematician and educator, who was both a public figure and a much-loved family man, has passed away peacefully at the age of 97 years old (http://www.zoltandienes.com/obituary, January 19, 2014). While most in the mathematics education field at the college level remembered him, putting into context the meaning of the passing of someone who, along with Jean Piaget and Jerome Bruner, was one of the greatest contributors of the field of mathematics education (Post, 1981; Sriraman, 2008; English, 2008), many in today's world of mathematics teaching and learning at the elementary and secondary levels, may not have even known about his contributions. Yet, they might have taught place value with the widely used Multi-base Arithmetic Blocks (MAB) he created, or algebra and logic with materials that sowed the seeds of contemporary uses of manipulative materials in mathematics instruction (Sriraman, 2008). They might have also used instructional ideas applying his theories. I was once, one of such teachers who taught algebra in middle school using Base Ten Blocks and applied Dienes' theories without knowing of their origin until my thesis advisor suggested that I read "Building Up Mathematics" after the presentation of my first Ph.D. thesis proposal in 1996. I then realized how much of what I was doing and believed in was tied to and guided by Dienes' work and belief in the effectiveness of constructive and discovery learning. I went on to read almost all of Dienes' books and related papers and write the proposal that led to the Ph.D. thesis titled: "*The use of manipulatives in middle school algebra: Application of Dienes' variability principles*." (Gningue, 2000) Such in-depth knowledge about Dienes' theories of teaching and learning along with those of Piaget and Bruner eased my transition from teaching students in middle school to teaching in-service and pre-service teachers in a college of education.

This paper is an attempt to describe, analyze and apply Dienes' theory on how mathematical structures can be taught by applying his four principles of learning upon which he believed a teacher can base concept development around the use of multiple embodiments through manipulatives, how such usage leads to abstraction, and the implications for teaching mathematics in today's mathematics classroom. We will illustrate how abstraction can result from the passage of concrete manipulations of objects to representational mapping of such manipulations and then to formalizing such representations into rule structures through the application of Dienes' four principles to teach the concepts and processes of simplification of algebraic expressions. The activities describing and applying the Dynamic, Constructivity, Perceptual Variability, and Mathematical Variability Principles were originally examined and developed as part of the author's Ph.D. thesis (Gningue, 2000), before being refined and reapplied with two groups of 6^{th} and 7^{th} grade students taught by two different teachers and redescribed through a 2006 Lehman College faculty grant.

Theoretical Framework

In the early part of the 20th century up until the mid 50's and even early 90's, the predominant view was that mathematics teaching is a show-and-tell process as well as a supervision of drill and practice (Davis, 1988). School mathematics focused primarily on arithmetic and computational fluency in the elementary grades, followed by a largely procedural approach to algebra from middle grades onward (Blanton et al., 2008). It was the result of a learning theory focused on the outcomes of learning (behaviors) rather than on how learning occurs (Thorndike, 1920). In this view, it was assumed that learning occurs by passively, but rationally, reflecting on stimuli from the environment. This approach has been unsuccessful however, in terms of student achievement **(**U.S. Department of Education & National Center for Education Statistics, 1998) and has compromised the ability of US schools to compete internationally in mathematics (Hiebert, et al, 2005; Heitin, 2013).

In the past thirty-five years however, a new view emerged that regards learning of mathematics as a process of building up mental representations and acquiring skills in using and modifying these representations, and synthesizing new ones. In this view, where understanding is given the highest priority, the child constructs his/her knowledge, learns from his/her own experiences, and builds his/her own interpretive framework for making sense of the world (Grouws & Shultz, 1996; National Council of Teachers of Mathematics [NCTM], 2000). Such ideas about the constructive nature of the learning process find their psychological foundations from many learning theories, including Piaget's theories of cognitive development, Dienes' variability principles, and Bruner's modes of representations (Post, 1981; English & Halford, 1995). These theories provided the framework for much of the research on conceptual understanding beginning in the 1950's. Findings from such research have now led to the realization, after centuries, that: (a) abstraction and generalization take far longer than previously considered and require more work seemingly unrelated to the concept before the direction thought can be determined (Dienes, 1971), and (b) the transition from using numbers to using symbols is much more difficult for many students than has been assumed (Mathematical Association of America [MAA], 2008).

Preparing students for the increasingly complex mathematics of this century requires indeed an approach different from the traditional methods of teaching arithmetic as show-and-tell (MAA, 2008). Following the call by both the National Council of Teachers of Mathematics ([NCTM], 2000) and the National Research Council ([NRC], 1996) for students to be able to use various forms of representations to investigate and communicate about real world applications, the MAA (2008) emphasized the need to develop ways teachers could effectively use to facilitate students' transition from work with numbers to work with symbols. For the MAA (2008), "preparing and sustaining teachers in implementing good instructional practices and curricular materials will enable them to transfer their own learning into new teaching practices thus ensuring a smoother transition to symbolization." (p.11)

The new considerations about teaching and learning make Dienes' contribution to the field even more meaningful and relevant today. Implementing good instructional practices that provides for individualized learning rates and learning styles was at the core of Dienes' work (Behr, Harel, Post, & Lesh, 1992; Post, 1981). The creation of mathematics laboratories was for a large part inspired by Dienes' work (Post, 1981; Reys & Post, 1973; English, 1995). They have been a part of many mathematics classrooms since the mid-70s, with teachers using hundreds of activities he designed (Sriraman, 2008; http://www.zoltandienes.com/obituary). Dienes' theory on how mathematical structures can be taught from the early grades onwards using multiple embodiments through manipulatives, games, stories and dance, made him an early pioneer in what was later to be called sociocultural perspectives and democratization of learning" (English & Graeme, 1995; Sriraman, 2008).

Describing Dienes' Theory

Dienes believed that the understanding or non-understanding of a mathematical idea depends on the method of communication used by the teacher to transmit the mathematical idea to the student. Communicating a mathematical idea from concrete manipulations of objects to representational mapping of such manipulations and then to formalizing such representations into rule structures, will lead to abstraction, generalization, and transfer (Dienes, 1971) and thus understanding of a mathematical idea. This passage constitutes the essential features on which Dienes based his four general principles for teaching concepts to children.

In the first, the Dynamic Principle, Dienes believed that preliminary, structured, practice and/or reflective types of games/activities must be provided as necessary experiences from which mathematical concepts can eventually be built, as long as each game/activity is played/introduced with concrete material at the appropriate time. The games/activities should be "devised to form a connection with the child's developmental process which, he believed, devolves in successive cycles." (Dienes, 1971, p. 201)

In the second, the Constructivity Principle, Dienes asserts that the games/activities should be always structured to allow construction to precede analysis, which is usually absent from children's learning until the age of 12. Piaget's and Bruner's findings that children develop constructive thinking long before analytical thinking, reinforced Dienes' belief in the effectiveness of constructive and discovery learning. Dienes identified two kinds of thinkers: the constructive thinker and the analytical thinker. The constructive thinker can be described as the child who is at Piaget's concrete operational stage; the analytical thinker is the child who is at the formal operational level. For Dienes, it is better to teach a new concept by formulating a situation that will lead to constructive rather than analytical thinking and understanding (Dienes, 1971; *as cited in Gningue, 2000*).

In the Perceptual Variability Principle or Multiple Embodiment Principle, Dienes prescribes the utilization of a variety of physical contexts or embodiments to maximize conceptual learning (Dienes, 1971). The provision of multiple experiences (not the same experience many times), using a variety of materials, is designed to promote abstraction of the mathematical concept. Dienes believed that each child may see the world differently, approach it differently, and understand it differently. Therefore, to ensure that all children learn a concept with understanding, along with children's active participation in its development, Dienes prescribes the use of various representations of the concept rather than a single representation (Dienes, 1971). For example, in teaching the simplification of algebraic expressions using *algebraic tiles*, *base ten blocks, clips, counters,* and *pencils,* the concept of like and unlike terms could derive from the different perceptions. The materials are different, but the same concepts are inherent to all of them (Gningue, 2000).

Finally, the Mathematical Variability Principle implies that children need to experience many variations of "*irrelevant attributes*" (I call them *complicating factors*) linked to the concept structure, in order to single out the general mathematical concept which is constant to all manipulations (Dienes 1971; Gningue, 2000 & 2006). In applying the mathematical variability principle to teaching the simplification of algebraic expressions, for example, there are *irrelevant attributes* (*complicating factors*) inherent to the concept of like and unlike terms. Concepts of like terms do not depend, for instance, on the nature of the coefficients or signs. By varying the signs and the coefficients using *whole numbers, decimals or fractions*, and keeping constant the *relevant attributes* (e.g., same variable to the same exponents) students will become conscious of what happens to different numbers in the similar situations while ensuring an understanding of like terms and unlike terms. For instance, in learning the principle that ax + bx = (a + b)x, students should understand that $a + b$ could represent fractions, decimals, negative integers as well as whole numbers. They should also understand that any combination of variables to a particular power, such as x^2, xy^3, or y, could replace the variable denoted by "x."

Dienes considers the learning of a mathematics concept to be difficult because it is a process involving abstraction and generalization. He suggests that the two variability principles be used together since they are designed to promote the complementary processes of abstraction and generalization, both of which are crucial aspects of conceptual development.

Whereas Piaget's theories were mostly descriptive, Dienes theories were more prescriptive and provided opportunities to develop students' understanding of mathematics concepts through the use of very specific manipulatives (Dienes, 1971; English & Halford, 1995; English, 2008). After Piaget elaborated and refined his theory of stages of intellectual development, Dienes (1973) refined his four principles by identifying *six stages* of teaching and learning mathematical concepts. He asserted that mathematical concepts are learned in progressive stages, analogous to Piaget's stages of intellectual development: *(1) free play, (2) games, (3) searching for communalities;* then *(4) representation, (5) symbolization,* and *(6) formalization* (pp. 6-9). Learning a new concept is described as an evolutionary process involving the learner in two three-sequentially ordered stages, or cycles. The completion of the first learning cycle (Cycle I), which leads to abstraction, is necessary before the mathematical concept becomes operational for the learner during the second cycle (Cycle II) when generalization is expected to occur (Dienes, 1971; Dienes & Golding, 1971).

In Cycle I, the learning concept will start with unstructured and undirected activities to permit students to experiment with and manipulate physical abstract representations of some elements of the concepts to be learned (*Stage 1*). They form mental ingredients they will use later to put together the final concept. They are then provided with a great number of experiences of varying structures, but all leading to the concept, allowing them to observe patterns, regularities, and restrictions that are embodied in the concept. For Dienes, students should notice that certain *rules* govern events, that some things are possible, some others are not, and once they find the rules and patterns which determine the events, they are ready to play games. Games allow students to experiment with the parameters and variations within the concept, to begin analyzing the mathematical structures of the concept (*Stage 2*).

The *third stage* comes after the child has played many games, using different physical representations of the concept, with some having the same structure. Even after that, the child may not discover the mathematical structure which is common to all representations of that concept. For Dienes (1971), until a child becomes aware of the common properties in the representations, s/he will not be able to classify *examples* and *non-examples* of the concept. The child will make an abstraction after realizing the *communality* of the concept in the different representations. Dienes suggests that teachers can help students see the communality of structure in the examples of the concept by showing them how each example can be translated into every other example without altering the abstract properties common to all examples. This amounts to pointing out the common properties found in each example by considering several examples at the same time.

Cycle II of Dienes' learning cycle is the transition from manipulative materials to more abstract representations, such as pictorial models and graphs, and finally to mathematical symbols. This second cycle starts with (4) the *representation stage*, which occurs after the child has observed the common elements in each example of the concept. This phase makes the child fully aware of the abstraction and allows her/him to talk about it. The child needs to develop, or to receive from the teacher, a single representation of the concept that embodies all the common elements found in each example. This could be a *diagrammatic* representation of the concept, a *verbal* representation, or an *inclusive* example. Students need a representation in order to sort out the common elements present in all examples of the concept. A representation of the concept will be usually more abstract than the examples and will bring students closer to understanding the abstract mathematical structure underlying the concept (Gningue, 2000).

In the *fifth or symbolization stage*, the child describes his representation of the concept using an appropriate *verbal* and mathematical system of *symbols*. It is important for each child to invent an individual symbolic representation of each concept; however, teachers have to intervene in the students' selection of symbol systems. One possibility is to permit students to first make up their own symbolic representations, and then have them compare their symbolization with those in the textbook. Students should be shown the value of a good symbol system in solving problems, proving theorems, and explaining concepts.

The *sixth and final formalization stage* allows students to set up a formal system. After students have learned a concept and related mathematics structures, they must choose some of its fundamental properties, order them, and consider their consequences. Dienes (1971) considers the fundamental properties of a mathematical structure to be the ***axioms*** of the system. Other properties derived from the fundamental properties are the ***theorems***, and the procedures and paths used to go from axioms to theorems are the mathematical ***proofs***. In this stage, students examine the consequences of the concept to solve pure and applied mathematics problems.

In recapitulating Dienes' theory (Hirstein, 2008), one can identify the six stages necessary for concept formation as: (1) the Free Play (trial and error activity), followed by (2) rule-bound play (games) which leads to (3) the identification of many different games possessing the same structure (search for isomorphisms) and the discarding of the irrelevant features found in the many games. In the representation stage (4), the child is now fully aware of the abstraction allowing her/him to "*talk*" about it, through *writing* or *drawing*, with the teacher helping draft a single representation of the concept that embodies all the common elements found in each example. Finally, the child describes her/his (5) *representation* of the concept using an appropriate verbal and mathematical system of symbols, before *formally* choosing fundamental properties of the symbol system, order them, and consider their consequences.

Applying Dienes' Theory

Dienes' theory of learning has had a great impact on the teaching of mathematics, and many of Dienes' instructional ideas are still applied today (English & Graeme, 1995; English, 2008). Some implications of Dienes' work have been the creation of mathematics laboratories (Post, 1980; Reys & Post, 1973). The application of the variability principles provides for individualized learning rates and learning styles (Behr et al., 1992; Post, 1980). Behr et al. (1992) developed a model that applies Dienes' principles to the instruction of concepts such as rational numbers, ratio, and proportion. They constructed a two-dimensional matrix representing Dienes' mathematical and variability principles, with one of the variability principles defining each dimension. Fraction circles, Cuisenaire Rods, number lines, paper folding, and chips were used to represent different embodiments or ***perceptual variates***. The ***mathematical variates*** were identified in the different activities as part-whole, measure, ratio, decimal, and operator. For Behr et al., Dienes' model of variability requires the learner's very active physical and mental involvement (p. 327) and a use of intellectual processes not commonly found in mathematics instruction (p. 328). Even though Behr (1976) found a significant gap between manipulative aids and symbols, Behr and al. (1992) found that the levels of student achievement and understanding were gratifying. Behr et al. went further in extending the application of Dienes' variability principles to the way teachers should be exposed to various aspects of teaching, that is, in a wide variety of conditions or contexts. By considering an extension of Dienes' principles to an instructional setting for teacher education, Behr et al. suggest the broadening of the teacher's role to those of an instructor of large and small groups, of a tutor, of a student, of an interviewer, of a diagnostician, of a confidant and so forth (p.329).

Dienes' four principles also found applications in STEM learning environments (Glancy & Moore, 2013). The constructivity principle and the dynamic principles are modelled in activities with windmills and gears, in the context of the "Wind Turbine" problem, where students investigating the effect of blade size and configuration on the power of the windmill, are given table-top models of the windmills, in order to manipulate the number and configuration of blades, as well as the shape, size, and material of the blades. Citing the many variables that can be adjusted, "for example the number of teeth on the gears or the number of gears in a chain," (*perceptual variability*; p.12), Glancy and Moore (2013) compare the *mathematical variability* to what they call *structural variability*. Since the principle of mathematical variability requires that systematic variation of the important variables be used to expose those structures, in this case, varying the number of teeth on different gears (structural variability) leads to the discovery of the underlying structure, the relationships between the number of teeth and the gears.

In applying Dienes' six stages for concept learning to the planning of a mathematical lesson however, Bell (1978) cautions the teacher to "use the model as a guide, not as a set of regulations to be followed slavishly" (Bell, 1978, p. 127). For Bell, one stage, possibly the free play stage, may not be appropriate for the students or that the activities of first two or three stages could be combined into a single activity. It may be necessary to plan unique learning activities for each stage when teaching younger elementary school students; however, other secondary students may be able to omit certain stages in learning such concepts. Structured games should be followed by practical and meaningful exercises (Bell, 1978).

In this paper, using a model that is in line with Bell's suggestion, and considering that Dienes did not believe in a curriculum for young children, since for him, "what matters is that children learn how to think" (Sriraman & Lesh, 2007), we present a concrete example of how Dienes' theories could be applied to the concepts of "***Simplification of Algebraic Expressions***." The lesson could be taught to 6^{th} and 7^{th} grade students (Gningue, 2000) and to any developmentally appropriate groups of students, with the first three stages combined into one (Conceptualization) and the last three stages into two as the fourth (Pictorial-representation) and fifth (Symbolization) stages are implemented together.

We will present two types of materials (***Perceptual Variates***) to introduce the concept of simplifying algebraic expressions: the Lab Gear Blocks (Picciotto, 1990) [**fig. 1**], and the Base Ten Blocks (**fig. 2**). Behr et al. (1992), in an experiment with the variability principles on fraction and ratio concepts, used the terms "***perceptual variates***" to describe the different materials used, and the term "***mathematical variates***" to describe the many *complicating factors* (fractions, decimals, integers as coefficients) whose variations do not change the general mathematical concept of *like* and *unlike terms*, and of the processes and structures underlying the concepts.

The first ***perceptual variate*** (Lab Gear Blocks) is utilized to introduce the algebraic concepts *without reference to negative integers*, whereas the second perceptual variate (Base Ten Blocks) is designed to *incorporate the use of both positive and negative integers* as coefficients. The learning of integers presents difficulties that, when added to those intrinsic to the learning of new algebraic processes, could make the learning of those algebraic processes more difficult for younger students. Consequently, the use of the two materials to introduce the concept will allow the teacher to disassociate the difficulties that come with the use of negative integers with those that prevent an understanding of the different processes involved in learning the new algebraic processes (Gningue, 2000). **Figure 1** and **Figure 2** describe the Perceptual and Mathematical Variates used in these activities.

Figure 1:
*Perceptual and Mathematical Variates of the **First Activity** on Algebraic Expressions*

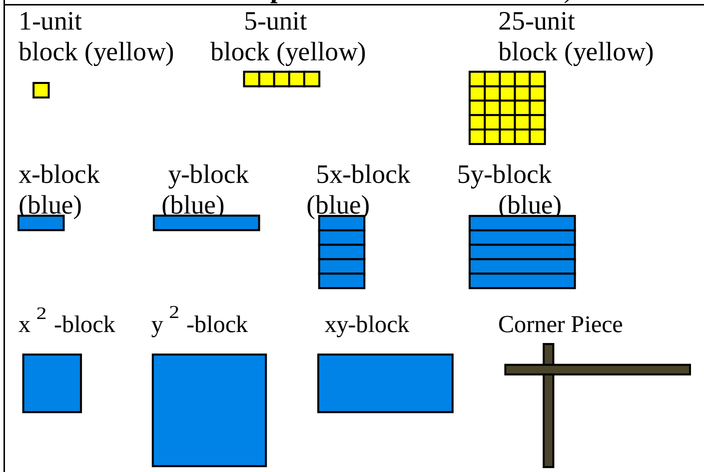
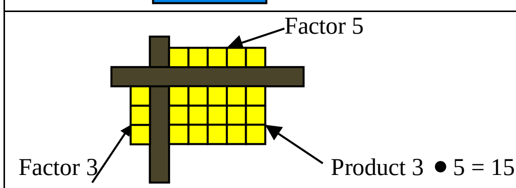
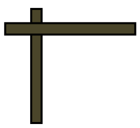
First Perceptual Variates: Lab Gear Blocks (The Units (1-25) are Yellow, while **Blocks are blue**: **All** used as **positive** values or variables)	**Mathematical Variates**
	**MV1** - Like terms and unlike terms (**A1**) **MV2** - Use of fractions as coefficients (**N2**) **MV3** - Use of decimals as coefficients (**N2**) **MV4** - Use of variables with exponents equal to 2 (**N2**) **MV5** - Use of variables with exponents higher than 2(**N2**) **(Note that all the variables are positive)** **MV1, 2,3…** = Mathematical Variate 1,2,3… **A** = Algebraic Variate **N** = Numerical Variate
	The Corner Piece is used for multiplication and division. Example 3 ● 5 = 15. Place 3 units on one side. Place 5 units on the other side. The resulting product should be a 3 by5 rectangle whose area equals 15 squares.

Figure 2:

Perceptual and Mathematical Variates of the ***Second Activity*** *on Algebraic Expressions*

Second Perceptual Variates: Base Ten Blocks **(**For **Positive and Negative** values and Variables**)**	**Mathematical Variates**
Blue 1-unit Block (+1) (shaded) **Yellow 1-unit Block** (-1) (non-shaded) **Blue x- Block** (+x) **Yellow x- Block** (-x) **Blue x^2 -Block** ($+x^2$) **Yellow x^2 -Block** ($-x^2$) Corner Piece  **When <u>drawing</u>, *Blue blocks* (*positive*)** are **<u>represented</u>** as ***shaded*** and *Yellow (Negative)* blocks as ***Non-Shaded.***	**MV1** - The expression contains Like and unlike terms (**A**) **MV2** - The expression contains parentheses (**A)** **MV3** - Use of fractions as coefficients (**N**) **MV4** - Use of decimals as coefficients (**N**) **MV5** - Use of variables with exponents equal to 2 (**N**) **MV6** - Use of variables with exponents higher than 2 (**N2**) **MV7** - Use of negative coefficients (N2) **(Negative Variables are used with this second material)** **<u>MV1, 2,3…</u>** = Mathematical Variate 1,2,3… **<u>A</u>** = Algebraic Variate **<u>N</u>** = Numerical Variate

While perceptual variates represent the different materials utilized as manipulatives, two types of mathematical variates are identified: a *Numerical Variate* and an *Algebraic Variate* (Gningue, 2000). A numerical variate is a decimal, a fraction, a negative integer, or any number other than a positive integer used as a coefficient. A numerical variate could also be a variable of exponent higher than 2. An algebraic variate is a *process* conceptually linked to the algebraic process to be performed, but not necessary for understanding the concept. For instance, the expression may contain parentheses requiring therefore an understanding of the distributive property to simplify; or the expression may require more than two steps to combine the like terms and simplify. In each case, the presence of a numerical or algebraic variate has the potential to add a degree of difficulty to a problem. Therefore, the application of Dienes' Principles enables the teacher to present work that progresses from easier to more difficult ones while working from totally concrete (In Cycle I) to symbolic (In Cycle II). Varying the signs and the coefficients of the variables using whole numbers first, then decimals or fractions, and keeping constant the relevant attributes (same variable to the same exponents) will make students become conscious of what happens to different numbers in the similar situations while ensuring an understanding of like and unlike terms. As we use the two principles together (Dienes, 1971), the two perceptual variates should lead to abstraction of the concept whereas the generalization of the mathematical concept will derive from the use of the mathematical variates (Gningue, 2000).

Using the First Perceptual Variate: *Lab Gear Blocks* (Activity 1)

The activity begins with students familiarizing themselves with, identifying, and naming (**fig. 1 & 2**) the materials ("*Play*" stage for younger children; "*familiarization*" stage for older students). **Questions 1, 2,** and **3** below (**fig. 3**) are designed to lead them to discover the meaning of each block and consequently the meaning of algebraic terms such as 5x, 5y, x^2, y^2, and xy through meaningful manipulations. For instance, through trial and error, x^2 can be shown to be the resulting *product of* ***x by itself*** (**not x + x**) using the Corner Piece; and 5x can be shown to be the *product of* ***5 and x*** (**not 5 + x**). The term 5x can also be shown to be the addition of five x's or x added five times. The meaning of the other variables can be similarly discovered. Understanding these terms and expressions is crucial for students. The juxtaposition of symbols in arithmetic that leads to the problem of *concatenation* in algebra has been identified in prior research as being problematic for students (Matz, 1980; Booth, 1984). While the expression ***4a*** means "4 times a" in algebra, ***42*** does not mean 4 times 2 in arithmetic. Students dispel misconceptions such as “$2 \bullet x = x^2$”or “$x \bullet x = 2x$,” a necessary step in the abstraction process (Gningue, 2000).

Questions 4 and **5** (**fig. 4**) represent an application, variation and extension that derives from what students just learned about the new algebraic terms embodied in the materials. For instance, 5^2 (the product of 5 by itself or $5 \bullet 5$) has the same meaning as x^2 thus derives from it. The same could be said about 6x and 5x. Students will see different situations in which the concepts of coefficients and variables can be applied. The teacher may include questions such as $(a + b)^2$ [written as (a + b)(a + b)] that will be studied later in multiplication of polynomials. Coefficients and exponents are varied to include decimals, fractions, and integers so that students become conscious that the structures of the manipulations are similar despite the different forms and numbers. This is the stage of the search for isomorphisms, enabling the child to find “links” to the materials and discard many irrelevant features on their way to be able to use symbolic language.

Figure 3.
Making Sense of the Materials

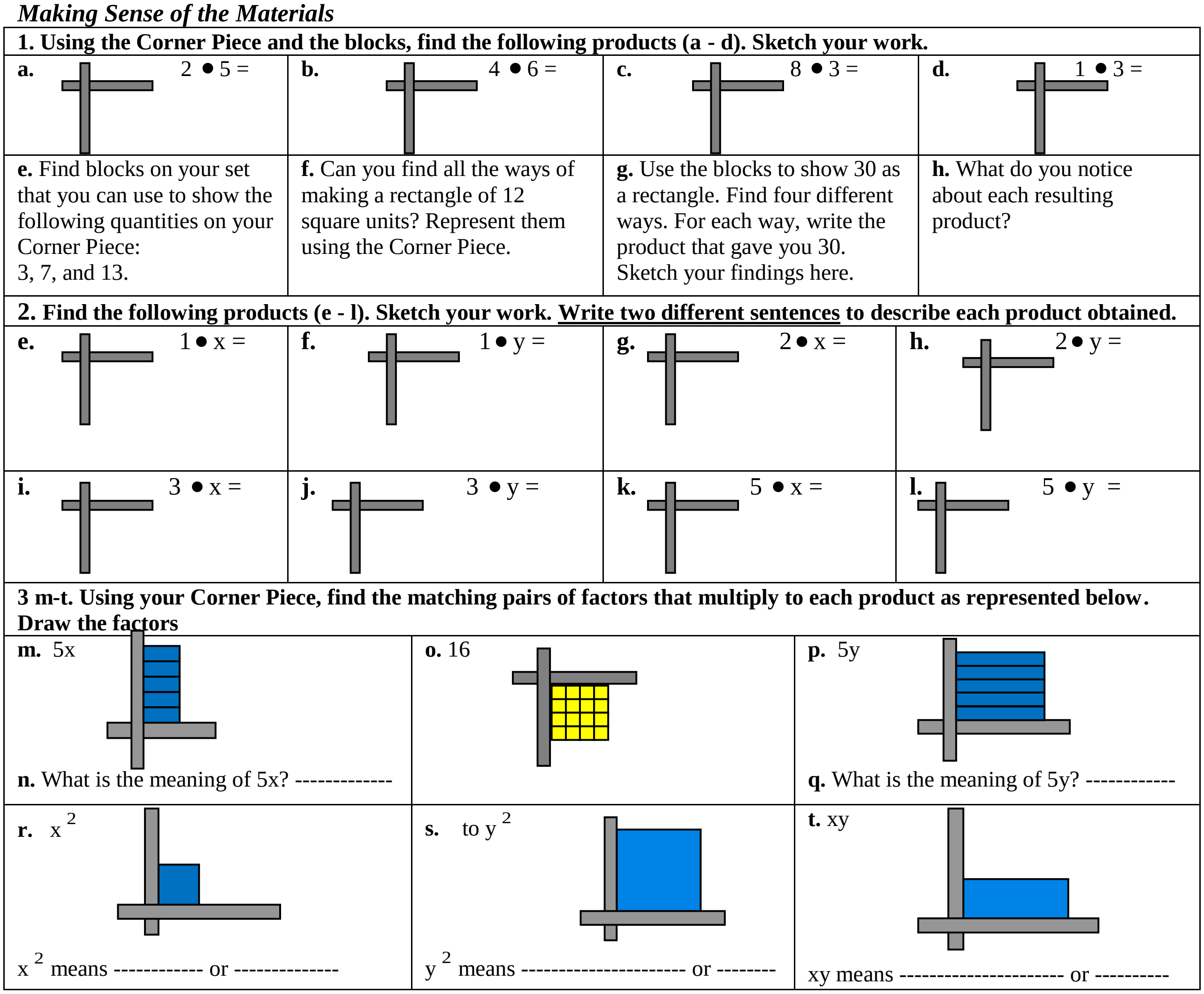


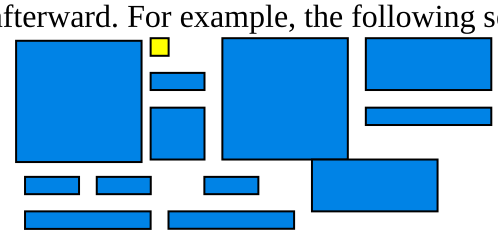
Figure 4
Searching for Isomorphisms

4. Write the meaning of each expression. Do not compute**.**						
a. 5^{2} =	**b.** 6x =	**c.** 7^{2} =	**d.** xy =	**e.** ab =	**f.** -8t	**g.** 9xy =
h. $(x + 2)^{2}$ =	**i.** $(a + b)^{2}$ =	**j.** 5^{3} =	**k.** a^{3} =	**l.** $(xy)^{3}$ =	**m.** 7^{4} =	**n.** $(6x)^{3}$ =

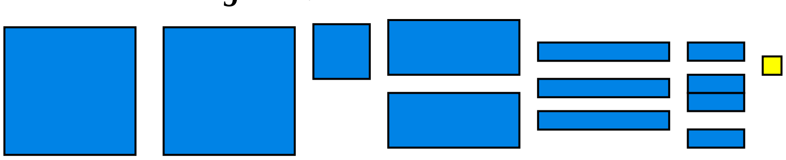
5. Determine the coefficient and variable in each expression										
	3x	5y	$\frac{2}{3}x$	-x	5 + 7x	(2 + 7)x	-9z	$0.7y^{2}$	8a	-4x
Meaning										
Coefficient										
Variable										

The concepts of like and unlike terms are introduced through a modelling problem in which students do an **inventory** of chairs and desks in two different rooms [**Question 6** (**fig. 5**)]. The end-result of the inventory should be a different classification of chairs and desks in the two rooms. For instance, students should realize that Hector's answer of 90 chairs and desks can be written as 1 desk and 89 chairs, 2 desks and 88 chairs etc…Thus Hector's report should distinguish the chairs and desks because they are not alike (like terms). It should give a more accurate answer of 50 chairs and 40 desks.

In **Question 7** (**fig. 5**), students are asked to do an **inventory** of a given set of blocks, rearrange them according to their shapes and sizes or resemblance, and describe in writing what they see, in details first and in simplest form afterward. For example, the following set of blocks



could be ***rearranged*** as:



described as: $y^{2} + y^{2} + x^{2} + xy + xy + y + y + y + x + x + x + x + 1$; and

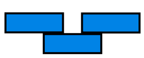
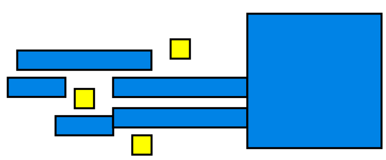
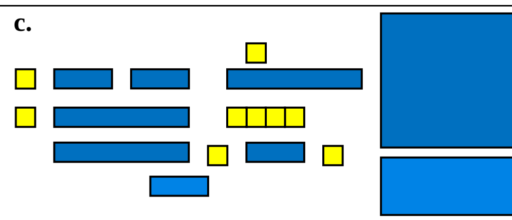
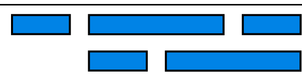
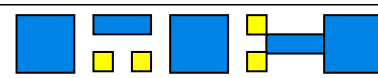
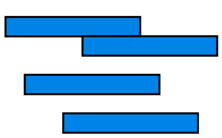
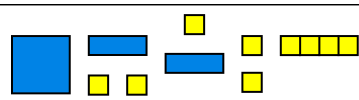
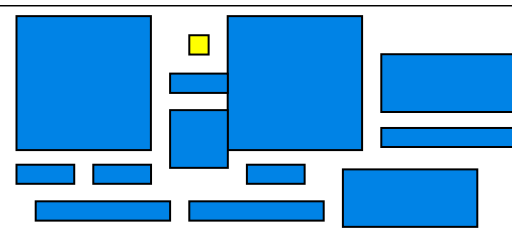
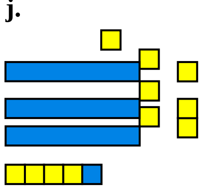
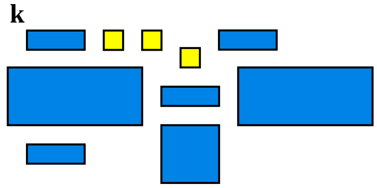
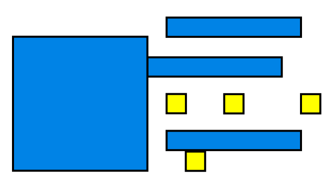
simplified as: $2y^{2} + 1x^{2} + 2xy + 3y + 4x + 1$. Students are expected to realize, in describing what they *see*, that: (1) this last form represents the farthest they can go; (2) all the blocks that were combined were of the same shape (3) the different shapes represent all different terms; none of the terms can be combined with the other remaining ones; and (4) like terms are represented by blocks of the same shape and size and defined as terms with the *same variables* to the *same exponent*.
Question 7 (a-l) and **Question 8 (a-i)** illustrate how, in Cycle II of Dienes' theory, the pictorial (stage 4) and symbolic representations and manipulations (stage 5) are combined. Students move from the concrete manipulations to pictorial and symbolic representations of the concepts of like and unlike terms (**fig. 5**) and vice versa **(fig. 6)**. Conceptualizing such ideas is also a crucial

aspect in the teaching and learning of simplification of algebraic expressions. Students' perception of algebraic expressions as incomplete statements has been identified as a cognitive obstacle for children's understanding of algebraic concepts (Collis, 1975; Kuchemann, 1981). Students are not usually receptive to the idea of an algebraic expression as a possible answer to a problem. Collis (1974) explained this in terms of the student's inability to hold unevaluated operations in suspension. If in arithmetic"3 + 4" can be replaced by "7," an expression such as "x + 4" cannot be replaced by another number, it is the answer. Operations performed on variables alone without reference to any number will make little sense to the students. Collis referred to this difficulty in holding unevaluated operations in suspension as students' inability to accept the *Lack of Closure*. Students also perceive that algebra and arithmetic are different (Chalouh & Herscovics, 1988; Davis, 1975). Davis (1975) referred to it as the *Name-Process Dilemma*; that is, in algebra, an expression such as 4+x describes both the process and the answer. In arithmetic, however, an expression like 5+9 represents the problem and 14 is the answer.
Manipulating algebraic expressions concretely with the blocks first before doing it symbolically will help students conceptualize the ideas of like and unlike terms and consequently minimize the *Lack of Closure* and *Name Process Dilemma* difficulties.

Figure 5:
Pictorial-Symbolic: Conceptualizing the Concept of Like and Unlike Terms Using the Lab Gear Blocks (Cycle I)

6. Introductory Problem
Hector the custodian has been asked by the principal to do the inventory of desks and chairs in room 206 and In room 211. He found that there were 30 chairs and 25 desks in room 206, and 20 chairs 15 desks in room. He reported to the principal that there were 90 chairs and desks in the two rooms. Was his report accurate? Explain your answer.

7. Like Hector the custodian, **you are asked to do the inventory** of the sets of blocks below. Represent each quantity in the simplest form possible, using the name of the blocks.

a. 	**b.** 	**c.** 
d. 	**e.** 	**f.** 
g. 	**h.** 	**i.** 
j. 	**k** 	**l.** 

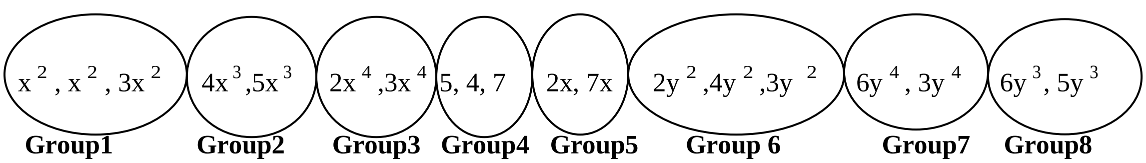
m. What do you notice when rearranging the blocks?

Figure 6

Symbolic-Pictorial: Conceptualization of the Concept of Like and Unlike Terms Using the Lab Gear Blocks

8. Use the blocks to represent each expression below. Sketch your work below each expression. Represent each expression in the simplest (shortest) form possible.			
a. $2x + 6 + 3$	**b.** $x + y + x$	**c.** $x + 5$	**d.** $y + 12$
e. $2x + 3y + 6 + 4x + y + 2$	**f.** $3x^2 + 2x + 7 + x^2 + 3x + 5 + x$	**g.** $2y^2 + xy + 3x + 3 + 2xy + x^2 + 6$	**h.** $x^2 + 5 + x^2 + x + 4$ $x^2 + 2x + y^2 + y + 3x^2$
i. Based on your experience with the blocks, rearrange the expression below into **groups of terms that seem to be alike**. Form as many groups as you need. $x^2 + 5 + x^2 + 4x^3 + 2x^4 + 2x + 2y^2 + 4 + 5y^3 + 3x^2 + 4y^2 + 7 + 6y^3 + 5x^3 + 3y^2 + 7x + 3y^4 + 3x^4 + 6y^4$ … … (x^2) **group1** ( ) **group2** ( ) **group3** ( ) **group4** …			
j. What did you base the choice of your groups on?			

Question 8i (**fig. 6**) introduces expressions containing variables with exponents of 2, 3, and 4 (mathematical variates). Students are given the expression $x^2 + 5 + x^2 + 4x^3 + 2x^4 + 2x + 2y^2 + 4 + 5y^3 + 3x^2 + 4y^2 + 7 + 6y^3 + 5x^3 + 3y^2 + 7x + 3y^4 + 3x^4 + 6y^4$ and asked to use their experience with the blocks to rearrange it into groups of terms that seem alike, despite the higher exponents (2 and 3) with variables that did not have a physical representation. It is expected that students will: (1) create eight different groups; (2) recognize that each group contains terms that are alike within the groups, but unlike to terms in other groups; and (3) explain their reasoning, that is, they based their choice of like terms as variables to the same exponents.



Finally, the series of problems in **Questions 9, 10** and **11** (**fig. 7**) introduce new mathematical variates in the form of decimals and fractional coefficients, and in the use of different letters as variables or of variables with higher exponents. As students realize that the variations of such attributes are irrelevant to and do not affect the constancy of the main concept- *same variables to the same exponent*, generalization of the concepts of like and unlike terms takes place, fulfilling therefore the purpose of Dienes mathematical variability principle.

Figure 7

Formalization of the Concepts of Like Terms and Unlike Terms Using the Lab Gear Blocks – Introducing New Mathematical Variates

9. For each problem below, decide which are like terms, combine them, and then simplify the expression. You **may or may not use** the Blocks.

a. $2x + 5 + 4x + 3$	**b.** $3y + 2 + y + 4$	**c.** $3x + 5 + 2y + x + 3y + 1$	**d.** $x^2 + 3 + 4x + 1 + 3x + 2x^2$
e. $x^3 + 4x + 2x^2 + x + x^2$	**f.** $2n + 3m + 4 + 4m + 5n + 8$	**g.** $3b + 2m + 3 + 4m + 9b + 1$	**h.** $x^3 + 2x + 7 + x^3 + x^2 + 3x + 5$
i. $3xy + 2x + 4y + 2xy + 6 + x$	**j.** $6ab + 2ac + 3a + 5 + 4ac + 3ab + ac + 3$	**k.** $\frac{2}{5}x + 8 + \frac{1}{5}x + 4$	**l.** $\frac{3}{4}y^2 + 1 + \frac{1}{2}x^2 + \frac{2}{3}y^2 + 6 + \frac{3}{5}x^2 + \frac{2}{3}x + \frac{4}{5}y + 2x + 5y$
m. Explain in writing how to simplify algebraic expressions.			

10. Use your algebra blocks to represent in simplest form the sum of:

a. $(x^2 + 3x + 2)$ and $(2x^2 + 4x + 5)$	**b.** $(5y^2 + 2y + 1)$ and $(3y + y^2 + 7)$
c. $(6 + 2y + y^2)$ and $(9 + 8y^2 + y)$	**d.** $(2 + x^2 + 3x)$ and $(x^2 + x + 8)$

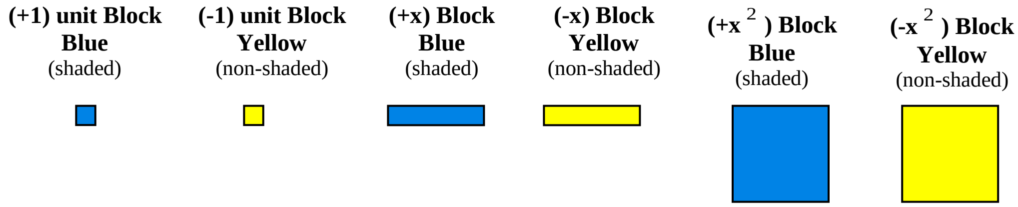
11. Find the sums. Use the blocks as you wish and when possible.

d. $(2y^4 + 2y + 1)$ and $(y^4 + 3y + 7)$	**f.** $(5 + y + y^2)$ and $(1 + y^2 + 3y)$	**f.** $(2x^4 + x^2 + 3x + 1)$ and $(3x^4 + 5x^2 + 6x + 8)$
h. $(\frac{1}{3}x^2 + \frac{1}{5}x + 2)$ and $(\frac{1}{3}x^2 + \frac{2}{5}x + 5)$	**k.** $(.25y^2 + 2y + 1.6)$ and $(.7y + .75y^2 + 3.8)$	**l.** $(\frac{5}{8} + \frac{3}{4}y + \frac{5}{6}y^2)$ and $(\frac{1}{4} + \frac{2}{3}y^2 + \frac{1}{2}y)$
m. $(2 + x^2 + 3x)$ and $(x^2 + x + 8)$	**n.** $(\frac{5}{8}y^4 + \frac{1}{2}y + 1)$ and $(\frac{2}{3}y^4 + 3y + 7)$	**o.** $(.12 + 1.5y + .45y^2)$ and $(.6 + .8y^2 + 2.6y)$
p. $(2x^4 + .4x^2 + 3x + 1)$ and $(3x^4 + 5x^2 + 6x + 8)$		

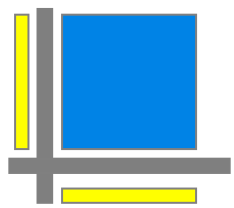
Using the Second Perceptual Variate: *Base Ten Blocks* (Activity 2)

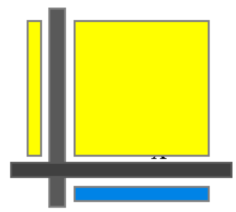
This second activity on algebraic expressions begins with the introduction of the Base Ten Blocks as the **second set of *Perceptual Variates*** with students again familiarizing themselves with, identifying, and naming (**fig. 2**) the materials ("*Play*" stage for younger children; "*familiarization*" stage for older students). Dienes (1971) developed the Multi-Base Arithmetic Blocks (MAB or Base Ten Blocks) to introduce concepts of arithmetic bases. A set of MAB is composed of (a) units-blocks, represented by small cubes, (b) tens-blocks, represented by bars of 10 units long, (c) hundreds-blocks, represented by squares of 10 bars or 100 units, and (d) thousands-blocks, represented by bigger cubes of 10 one-hundred blocks or 100 tens-blocks or 1000 units-blocks.

The MAB can be utilized however, to teach algebraic concepts. For that purpose, two different colors are used: a ***blue color*** for positive integers or variables, and a ***yellow color*** for negative integers or variables. When representing the integers and variables on paper, **without color pencils**, ***blue blocks are shaded***, while ***yellow blocks are non-shaded***.



In addition to the MAB, the **Corner Piece**, used with the Lab Gear Blocks can be utilized again to show the meaning of x^2 and of $-x^2$. The teacher can lead students to discover (based on their prior knowledge about integer multiplication) that two same colors (*x multiplied by x* or as *-x multiplied by -x*) yield a blue block (x^2), whereas two different colors (*x multiplied by - x* or *-x multiplied by x*) yield a yellow block ($-x^2$).

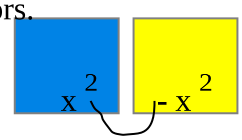




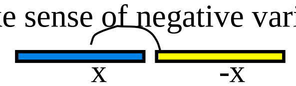
x times x $= x^2$ (-x) times (-x) $= x^2$ x times (-x) $= -x^2$

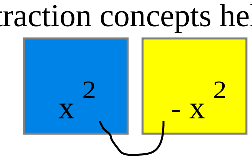
By using the Base ten Blocks as a second Perceptual Variate to satisfy Dienes' variability principle, students are introduced to **four new elements** all related to understanding the concept of simplification of algebraic expressions.

1. The first new element is the concept **of negative variable**. Variables –x and $–x^2$ are introduced as the opposites of x and x^2 by using different colors.



2. The second new element is the concept of **zero pairs using variables**. The variable x can be cancelled by the variable –x and x^2 cancelled by $–x^2$. A preliminary work with integers using blue and yellow cubes might be needed to reinforce integer subtraction concepts helping students make sense of negative variables.

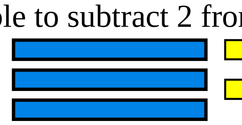




Zero Pair Value = 0 Zero Pair Value = 0

$(x + (-x)) = 0$ $(x^2 + (-x^2)) = 0$

3. The third element is the **possible representation of subtraction problems with variables using a "renaming" process**. The renaming enables students to represent expressions like 3x – 2 as 3x + (-2). It is not concretely possible to subtract 2 from 3x unless 3x – 2 is "renamed" to 3x + (-2) by the addition of zero pairs.

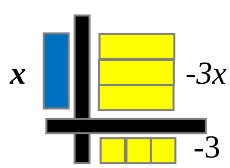


4. Finally, the fourth element is the fact **that students work with materials representing variables x and x^2 only**. No other letter is used to represent variables.

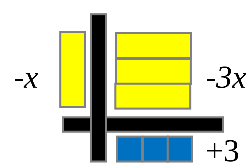
The series of problems in **Figures 8**, **9**, and **10** should lead students to discover these new elements by using expressions and models containing positive and negative variables. Because in Activity 1 with the Lab Gear Blocks, students learned to identify what could and could not be simplified, the problem of recognizing like and unlike terms should not be an issue. The issue could be, however, in the symbolic representations of models containing negative variables. Another type of concatenation (juxtaposition of symbols) (Matz, 1979; Booth, 1981) issue may arise. For example, Gningue (2000) found that when asked to describe a set of three yellow bars,

many students said, in words, "*three negative x*" and wrote either "*3-x*" or "*3(-x)*". One student explained his reasoning by stating that because "*we have –x three times, we should describe it as three negative x.*" He however wrote "*3-x*" first, before changing it to 3(-x) after another student remarked that "*3-x*" *looked like more a subtraction of x from 3 rather than a multiplication*. After some discussion about it, all students agreed with one girl who suggested that the use of "–3x" was the best way to represent "three negative x's", leading students to recognize the similarity between 3*x* and –3*x* (Gningue, 2000).

The corner piece method could be also used to clarify why when describing a set of three yellow bars, the answer -3x makes sense, since multiplication is defined as a repeated addition. Using yellow and blue pieces could be used to consider the equivalence of the two products $3 \times (-x)$ and $(-3) \times x$ possibly leading to better acceptance of the notation –3*x*, and helping students distinguish the result (as yellow pieces) from the blue of 3x.



$(-3) \times x$ = -3*x*



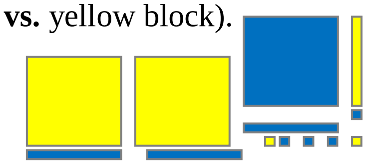
$3 \times (-x)$= -3*x*

The two products are both negative
Since the two factors have opposite signs

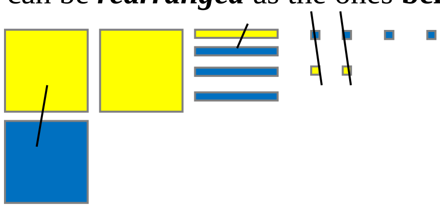
Students will not also automatically differentiate a subtraction symbol from a negative sign. They should become more aware of the renaming process by changing any subtraction problem to addition of the opposite, and by using parentheses around negative terms except for the case when a negative term is written first in the expression. For example, the expression x^2 - 4x - 2 contains a subtraction (minus 4x) which should be renamed to x^2 + (-4x) + (-2). The expression -2x^2 - 5x - 3 contains -2x and two subtractions (minus 5*x* and minus 3). The renamed expression would be -2x^2 + (-5x) + (-3).

The series of problems in **Figures 8**, **9**, **10,** and **11** are designed to address these issues. Students rearrange, combine, simplify, and write the algebraic expressions representing different sets of blocks. They familiarize themselves with the different types of manipulations involving algebraic expressions, **manually, pictorially,** and **symbolically (fig. 8, 9, 10, 11, & 12)**.

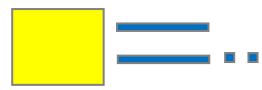
In the "Play" (manual) phase (**Fig. 8)**, students manipulate the blocks by combining the ones that are alike to simplify the expression. Because the idea of combining like terms was already studied in Activity 1 with the Lab Gear Blocks, the difficulty may lie in the simplification of "opposite blocks" (x vs. –x) differentiated by the color (blue block **vs.** yellow block). For example, the set of blocks on the ***right***



can be ***rearranged*** as the ones **below** (***left***),



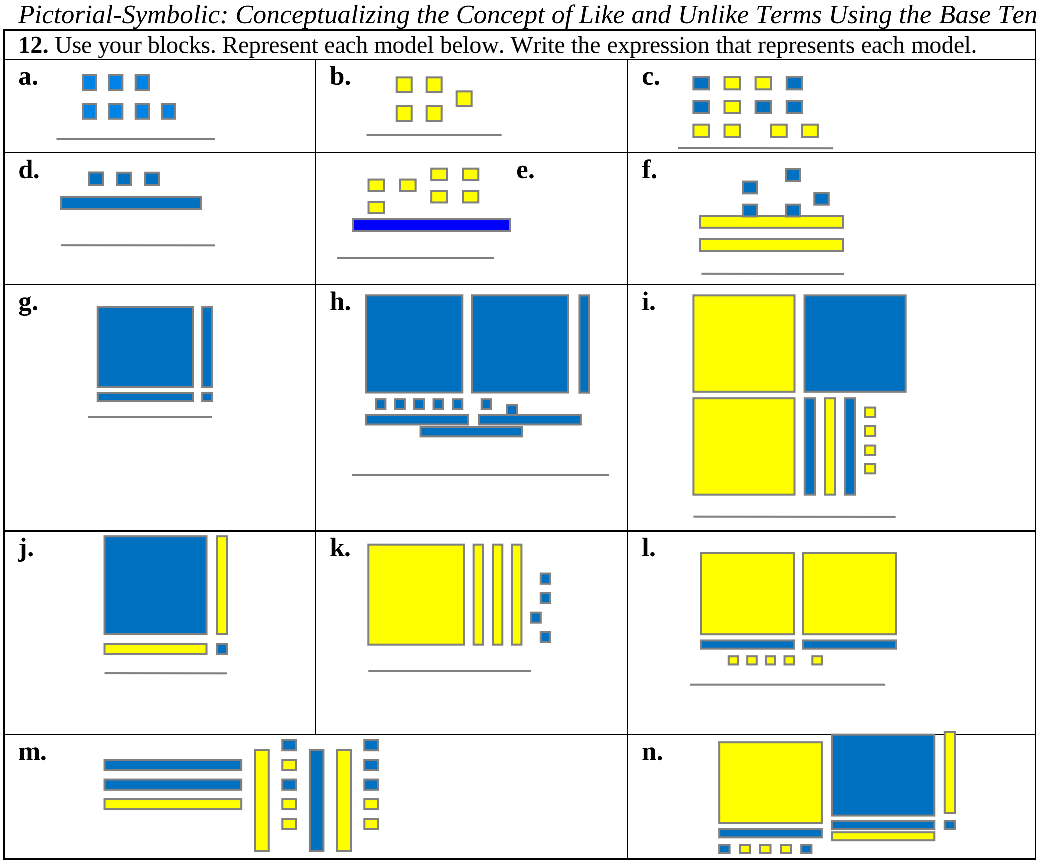
simplified as



and ***described*** as: $-1x^2 + 2x + 2$ as zero pairs are eliminated.

Figure 8

Pictorial-Symbolic: Conceptualizing the Concept of Like and Unlike Terms Using the Base Ten Blocks (Cycle I)



In the combined pictorial and symbolic stages (**Fig. 9, 10, and 11**), students **sketch the** blocks and write the expressions representing each given set and vice versa (**Fig. 9, Question 13 a-h**). Then in **Question 13 i-n**, they add a group of blocks vertically and horizontally, sketch the answers, and write the corresponding algebraic expressions. As suggested by Dienes' variability principles, to ensure that generalization of the concept of like terms takes place, students are exposed to different types of manipulations involving the largest possible number of variables by going back and forth from pictorial to symbolic representations.

The introduction of elements such as fraction, integers, and decimals as coefficients in the formalization stage (**fig. 12)** will complete the application of the mathematical variability principle enabling students to generalize the processes involved in the addition of algebraic expressions. This **second activity** will allow students to focus on other structural processes related to the concept of simplification of polynomials rather than on the recognition of like and unlike terms. That goal of recognizing like and unlike terms was achieved through the first activity. The problems in Figures 11 and 12 also represent a mean of reinforcing the concepts of manipulations of integers and their properties**.**

Figure 9

Symbolic-Pictorial: Conceptualization of the Concept of Like and Unlike Terms Using the Base Ten Blocks (Cycle II)

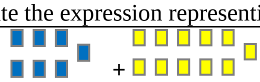
13. Use your **Base Ten Blocks** to represent each expression below. **Sketch** your work here.

	Sketch		**Sketch**
a. $x^2 + 2x + 4$		**b.** $3x - 2$	
c. $2x^2 - x + (-4)$		**d.** $-3x + 2$	
e. $x^2 - 3x + 2$		**f.** $2x^2 + x - 3$	
g. $-2x^2 + 2x - 5$		**h.** $-x^2 - x - 1$	

In this section:

- -Write the expression representing each group of blocks.
- -Perform the addition.
- -Sketch the answer.
- -Write the expression representing the answer.

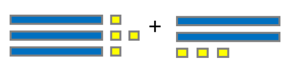
i.



=

Expression .... + ... = ...

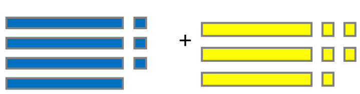
j.



=

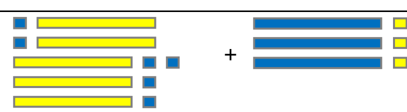
.... + ... = ...

k.



-------------------- + --------------------- =

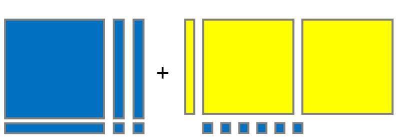
l.



=

-------------------- + --------------------- = -------

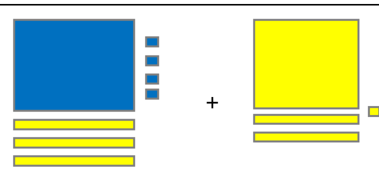
m.



=

.... + ... = -------

n.



=

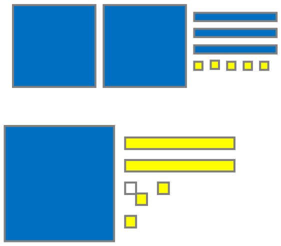
.... + ... = ...

Figure 10

Symbolic-Pictorial: Conceptualization of the Concept of Like and Unlike Terms Using the Base Ten Blocks

14. - Write the expression representing each group of blocks.
- Perform the addition.
- Sketch the answer.
- Write the expression representing the answer.

a.



+

= ……………………………

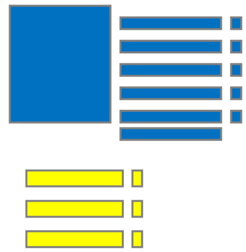
Expression

……………………

+ ……………………

= ……………………

b.



+

= ……………………………

Expression

……………………

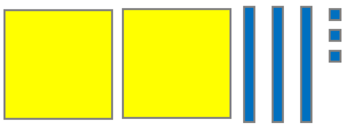
+ ……………………

……………………

=

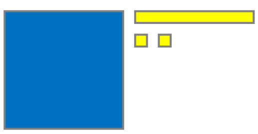
15. Draw a model for the ***missing addend***. Write the expressions for each model.

c.



+

=



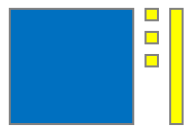
Expression

……………………

+ ……………………

= ……………………

d.



+

=

Expression

……………………

+ ……………………

= ……………………

Figure 11

Symbolic-Pictorial: Conceptualization of the Concept of Like and Unlike Terms Using the Base Ten Blocks

16. Use your Base Ten Blocks to find the sums. -Sketch your work. -Write the answer using algebraic symbols.	**Sketch**
a. $(4x + 1) + (-3x + 4) =$	
b. $(x^2 + 3x + 2) + (2x^2 + x - 3) =$	
c. $(-2x^2 - 4x + 5) + (-x^2 + 5x - 3) =$	
d. $(x^2 + 3x - 4) + (2x - 3) =$	
e. $2x^2 + 4x - 1 +$ $x^2 - x + 4$	
f. $-4x^2 + 3x + 2$ $+ \quad 2x + x - 6$	
g. $x^2 + 3$ $+ \quad 2x^2 + 5x - 7$	

Figure 12:
Formalization of the Concepts of Like Terms and Unlike Terms Using the Base Ten Blocks – Introducing New Mathematical Variates

17. Now simplify the expressions by combining the like terms. You may or may not use the base ten blocks.	
a. $2x^2 + 5x - 7 + 6x + 3 + x^2$	
b. $x^3 + 4x + 2x^2 - 5x + 6 + 2x + 2x^3 + 1$	
c. $3x^4 + 2x^3 + x + 9 + 2x^4 + x^3 - x - 2$	
d. $x^4 + 2x^2 - 2x^3 + 4 - 2x + 5x^2 - 7$	
e. $(3x^4 + 2x^3 + 5x^2) + (2x^4 - 2x^2 + 6x^3 + 4x - 1)$	
f. $x^4 + 3x^3 + 5x^2 - 7$ + $2x^4 + 2x^2 + 6$ =	**g.** $2x^3 - 4x^2 - 8$ + $2x^4 + 5x^2 + 6x - 3$ =
h. $-3x^4 + x^3 - 5x^2 + 3$ + $-2x^4 - x^3 + 2x^2 - 6$ =	**i.** $-2x^3 - 4x^2 + 6x - 1$ + $4x^3 + 5x^2 + x - 3$ =
j. $(\frac{1}{3}x^2 - \frac{1}{5}x + 2) + (\frac{1}{3}x^2 + \frac{2}{5}x - 5) =$	**k.** $(.25y^2 + 2y + 1.6) - (.7y - .75y^2 + 3.8)$
l. $(\frac{5}{8}y^3 - \frac{3}{4}y^4 + \frac{5}{6}y^2) + (\frac{1}{4}y^3 - \frac{2}{3}y^2 - \frac{1}{2}y^4) =$	**m.** $(2 + x^2 + 3x^3) - (x^2 + x^4 + 8) =$
n. $(\frac{5}{8}y^4 + \frac{1}{2}y^2 + 1) + (\frac{2}{3}y^4 + 3y^2 + 7) =$	**o.** $(.12 + 1.5y^3 + .45y^2) + (.6 + .8y^2 + 2.6y^3)$

Implications

One of the concerns that have been raised in applying Dienes' principles in the classroom is the "noise" created by the use of too many materials to introduce one concept. Wheeler (1996), for instance, describes Dienes' principles of learning as "noisy" in the sense that they may be too full of distractions to achieve their objective. For Wheeler, the use of too many different embodiments makes Dienes' approach extremely "busy," and the presence of too much "stuff" that is not meant to be assimilated can lead to chaos in the classroom. In introducing and teaching algebra, for example, Wheeler suggests not only taking a careful look at the relationships between context and noise, but also identifies the amount of noise that can be tolerated.

In this age of technological advances, however, the development of a plethora of virtual manipulatives on the Internet (National Library or Virtual Manipulatives ([NLVM], 2008) makes the application of Dienes' principles in most areas of mathematics even more appealing. Virtual manipulatives are interactive, that is, the learner can manipulate the same objects and create the same mental representations of the objects using the computer mouse, in a short amount of time, minimizing thus the "noise" issue raised by Wheeler. In today's technology-enriched environment, this could be appealing not only to teachers at the elementary and secondary levels, but also at the college level in remedial community college classrooms. Such applications would require however, a considerable investment in technology, especially in urban areas with a large concentration of students from a low socio-economic background, and the training of teachers and faculty as well as their willingness to embrace technology in teaching. (Gningue, Menil, & Fuchs 2014)

Finally, in light of the Common Core's Learning Standards (CCLS) in New York State, Dienes' principles of teaching and learning process of moving from concrete manipulations of objects to representational mapping of such manipulations and then to formalizing such representations into rule structures, emphasizes conceptual understanding (*comprehension of mathematical concepts, operations and relations*) and procedural fluency (*skill in carrying out procedures flexibly, accurately, efficiently and appropriately*) and connects the Standards for Mathematical Practice (MP) to the Standards for Mathematical Content in the domain of algebra. Students construct their own understanding of the concepts of variable (*Figures 1 & 2; Activity 1, Questions 2-7; Activity 2, Questions 12*), like and unlike terms and algebraic expressions (*Activity 1, Questions 2-8; Activity 2, Questions 13-16*), make verbal, pictorial and symbolic descriptions of important features and relationships (*MP1, MP3*), make and verify conjectures about like and unlike terms (*MP1, MP3*) before generalizing (Questions 9m,17), and communicate their findings through interaction with their peers.

The concepts taught can be seen in the CCLS content areas of: "*Apply and extend previous understandings of arithmetic to algebraic expressions*" (6.EE.A.1); "*Write expressions that record operations with numbers and with letters standing for numbers*" (6.EE.A.2.B); "*Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity*" (6.EE.A.2.C); "*Apply the properties of operations to generate equivalent expressions*" (6.EE.A.4); "*Use properties of operations to generate equivalent expressions*" (7.EE.A.1); and "*Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients*" (7.EE.A.2). Understanding these processes is of utmost importance as they are the needed prerequisites for high school algebra. They give students the ability to ***interpret*** "the structure of expressions (HSA.SSE.A.1), parts of an expression, such as terms, factors, and coefficients (HSA.SSE.A.1.A), and complicated expressions by viewing one or more of their parts as a single entity (HSA.SSE.A.1.B) to just name a few.

The CCLS emphasizes a learning process that is concerned about "how" children learn as with "what" it is they learn, and a classroom where individuals are allowed and encouraged to interact personally with various aspects of their environment. Such principle finds its foundations in the work of Piaget, Bruner and Dienes, validating their theories more than 50 years after they established them. Such a legacy represents to me, one of the greatest contributions to the field of mathematics teaching and learning.

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