Mathematical Practices in Social Learning Environment Guided by the Hypothetical Learning Trajectory for Quadrilaterals

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The purpose of the current article is to analyze the effects of the hypothetical learning trajectory designed for teaching quadrilaterals. This article reports the classroom mathematical practices that emerged in a social learning environment that was developing preservice middle school mathematics teachers' understanding of quadrilaterals. Ten preservice teachers participated in a five-week instructional sequence guided by the hypothetical learning trajectory designed by problem-based learning strategy about quadrilaterals. This teaching experiment study focused on their learning and understanding of quadrilaterals through argumentations. Using case study design, qualitative data was collected through whole class and peer group discussions, field notes, written worksheets, group meetings, and interviews. Data were analyzed by three-phase methodology of Rasmussen and Stephan (2008) based on Toulmin's model of argumentation (1969). Three mathematical practices emerged through the five-week instructional sequence: reasoning on definitions of quadrilaterals, reasoning on relationship among quadrilaterals and reasoning on properties of quadrilaterals. Moreover, it was observed that preservice teachers developed their learning and understanding of quadrilaterals by the hypothetical learning trajectory.

Key words: Argumentation, mathematical practices, quadrilaterals, teaching experiment

Geometrical concepts include figural and conceptual components so they can be explained as the figures possessing special properties and usage in the space. Based on this view, we can say that if students draw figures by knowing and analyzing their properties, they engage in geometry actively. This is because geometry activities can be performed by following three connected phases; visualization, construction by tools and reasoning (Laborde, Kynigos & Strasser, 2006). In this respect, it is important to teach geometry by providing opportunities that enhance these connected processes to students. By this way, they can acquire experiences about logical reasoning, understanding the role and place of mathematical tools in the real world, and the work of mathematicians (Gonzalez & Herbst, 2006). To line with this view, enacting geometry teaching effectively necessitates well-educated teachers; for that reason, teachers should attain necessary knowledge and skills in their preservice years in order to help students obtain these skills.

In accordance with these explanations, it is important to design and organize instructional sequences and learning environments for preservice teachers to acquire necessary knowledge and skills about geometry and geometry teaching. If necessary conditions and adequate situations are provided, it is possible to transform mathematics classrooms into learning places supported by beneficial tasks and arguments. Therefore, the current study focused on designing and analyzing the effects of learning trajectory on the preservice middle school mathematics teachers (PMSMT) understanding about quadrilaterals, as a framework guiding teaching and learning process, enhancing the preservice teachers' development of knowledge and understanding about quadrilaterals as a geometrical concept. Moreover, in the current study, preservice teachers' developmental process on understanding about a geometrical concept in a social context was analyzed. Hence, the effects of these activities and discussions could be identified. Afterwards, they can be integrated into instructions in geometry lessons because of the importance of language in geometric thinking (Lai & White, 2012).

This special language can be acquired in social learning environments effectively by developing learning and geometric thinking. In line with this view, the current study was focused on mathematical practice, taken-as-shared ideas formed in discussion process, about the geometrical concept of quadrilaterals emerged through the enacted instructional sequence guided by the designed hypothetical learning trajectory. Hence, mathematical practices can enhance formation and analysis of this kind of learning process and providing feedback into instructional sequence and activities.

Mathematical practices have been examined in different grade levels for different subjects (Stephan & Akyuz, 2012; Stephan & Cobb, 2003; Stephan & Rasmussen, 2002; Uygun & Akyuz, 2019). The researchers of these previous research explored the mathematical practices about integers, measurement, circles and triangles. Based on the mathematical practices' important effects and role of illustrating social learning environment clearly on instructional sequence as established in previous research, this study focused on the mathematical practices about quadrilaterals.

In accordance with these explanations, the research problem of the study is "What are the mathematical practices emerging in social learning environment guided by the hypothetical learning trajectory designed for teaching quadrilaterals to preservice middle school mathematics teachers (PMSMT)?" By answering this question, it was aimed to enhance the PMSMT understanding of a particular concept of quadrilaterals, provide them beneficial instructional strategies about their understanding and determine the impacts of using guide such as hypothetical learning trajectory (HLT) to improve their understanding and geometry content knowledge in the current study. In other words, by the HLT designed and examined in the current study, is it possible to provide insights into how the PMSMT can be supported in the process of learning and understanding about quadrilaterals in a social learning environment?

Theoretical Framework

Teaching and Learning about Quadrilaterals

Quadrilaterals are not only shapes but also figures with their conceptual parts needing to be examined carefully. Quadrilaterals taken an important place in the middle school mathematics curriculum (Ministry of National Education, 2013). Therefore, it is essential that teachers and PMSMT have both a deep understanding and sufficient knowledge about quadrilaterals since they are expected to teach mathematics to secondary school students. More specifically, they are expected to enhance the students' understanding of geometric ideas about quadrilaterals by exploring their characteristics to form their definitions, and identify relationship between the types of them and their properties (NCTM, 2000). By this view, the teachers and preservice teachers need to have deep knowledge about quadrilaterals to help their students execute this expectation. Despite this consideration, some research have shown that different grade level students, including preservice mathematics teachers, cannot successfully identify the connection among different types of quadrilaterals, their definitions and properties (Erez & Yerushalmy, 2006; Fujita & Jones, 2008).

A number of have focused on identifying the difficulties in learning quadrilaterals, their reasons and suggestions to overcome these difficulties. Some of these studies have explained that different grade level learners have difficulties in defining and classifying quadrilaterals (Monaghan, 2000; Pickreign, 2007; Erez & Yerushalmy, 2006). For example, Pickreign

(2007), and Jones, Mooney and Harries (2002) have stated that most trainee teachers in their study groups could not produce formal definitions of quadrilaterals accurately. Also, Fujita (2012) found that although preservice teachers could define some of the quadrilaterals, they were unable to conceptualize the inclusion relationships (such as, trapezoids include parallelograms; parallelograms include rhombi, squares and rectangles; rhombi include squares; rectangles include squares) among them because they focused on their prototypical examples and definitions. In this respect, it is important to analyze the similarities and differences among different types of quadrilaterals, their inclusion relationships based on their properties and definitions. Erez and Yerushalmy (2006) have also stated that identification of critical and non-critical attributes of different types of quadrilaterals has effect on students' difficulties in learning and understanding of quadrilaterals.

Most of the studies examining the teaching and learning of quadrilaterals pay attention to their properties (Jones, 2000/2001; Wares, 2004). Some of these studies emphasize classification of types of quadrilaterals (Jones, 2001) and learning the definition of them (De Villiers, 2004) in the process of learning their properties. Zazkis and Leiken (2008) state the positive effects of producing and using definitions of geometrical shapes in teacher education programs. They also add that preservice teachers have various views and skills about the definitions of the concepts, and this variety can result from their deficiencies about the concepts and their definitions.

These problems could have been addressed through the process of teacher education programs (Chesler, 2012; Zazkis & Leiken, 2008). The aforementioned studies provide suggestions for students and (pre- and in-service) teachers about how to examine, learn and develop the understanding of the properties of quadrilaterals by connecting them necessarily. These explanations show that quadrilaterals may be difficult to learn because of its complex and hierarchical nature. This difficulty also obstructs mathematics educators such as teacher trainers and researchers to provide beneficial suggestions about learning and teaching quadrilaterals. Also, it is explained that the complex nature of quadrilaterals, and their hierarchical relationship and disagreements about it cause learner difficulties.

This situation leads a problem for mathematics education needing further research (Jones, 2000). More specifically, in spite of these studies in the literature, there is still need to investigate different grade level of students' understanding of quadrilaterals, their properties and the relationships among them (Athanasopoulou, 2008).

In line with this view, the current study explored the design and examination of the learning environment about the concept of quadrilaterals. Since PMSMT are expected to have deep knowledge of quadrilaterals and need to attain this knowledge through their preservice years, the learning of PMSMT was the focus of the current study.

To conclude, given the requirements and explanations above, this study was focused on the design of a hypothetical learning trajectory about quadrilaterals and analyzing its effects on the PMSMT's understanding about quadrilaterals through an instructional sequence. Moreover, it was observed that most of the studies in this field have predominantly focused on the identification and development of learning and understanding about quadrilaterals as in this study. Differently, because formation and development of ideas through classroom discussions could provide contribution and insight into the studies about quadrilaterals, the process of development of learning was focused on the ideas of a classroom community rather than individual processes including cognitive changes.

Hypothetical Learning Trajectory

With the support of effective learning environments and instructions, teachers can help their students explore geometry concepts by providing evidence for existence of particular relationships among the quadrilaterals and forming geometric shapes such as quadrilaterals considering their properties (Jones, 2000; Santos-Trigo, 2004). Therefore, in the current study, it was aimed to design a learning environment in order to teach quadrilaterals to preservice teachers instructed by a mathematics educator and to analyze its effects in detail. In this way, it was attempted to address the questions arising from the theoretical considerations and to remove the obstacles explained above in this study. In addition, it can be possible to examine and develop learners' understanding of quadrilaterals by exploring their properties, connection of them and providing justifications about criteria effectively. Hence, the focus of the current study was on the PMSMT understanding about the quadrilaterals.

Based on these explanations, a learning environment was designed in order to improve the PMSMT understanding of quadrilaterals. The learning environment in the study was guided by the hypothetical learning trajectory (HLT) since it was beneficial to design and implement lessons effectively. HLT is "The teacher's prediction as to the path by which learning might proceed. It is hypothetical because the actual learning trajectory is not knowable in advance and it characterizes an expected tendency" (Simon, 1995, p. 135). The HLT can be used as a framework for encouraging learning of the concepts. By connecting the instructor's content knowledge with their knowledge about the learners, the HLT can provide predictions about the instructional sequence. Hence, by the HLT, learning environment can be designed and implemented effectively.

In design of the HLT, Cobb (2003) explains the preparation process under four interrelated elements; *the instructional tasks, the tools students use, the nature of the classroom discourse and the classroom activity structure*. The first element, *instructional tasks*, includes the geometrical activities encouraging students' engagement and their development of conceptual understanding. The second element, *the tools students use*, refers to imagery, materials, figures, pictures, schemes, tables, invented tools by the students and (non)standard symbols (Gravemeijer, 2004; Stephan, 2003; Thompson, 1996). These tools have important place in the design of the HLT since they help the students reorganize their development of conceptual knowledge and understanding (Cobb, 2003). The third element, *the nature of the classroom discourse*, is about the examination of social and sociomathematical norms. These norms focus on the structure of the participation of the students in tasks and learning process. By these norms, the social learning process can be established and mathematical practices can emerge (Cobb et al., 1997; Cobb & Yackel, 1996; Stephan & Cobb, 2003). The last element, *the activity structure of the classroom*, refers to how the students take place and roles in the instructional sequence. These four elements are formed and expressed by five groups; *learning goals, content, tools, task/imagery, and possible discourse* (see Table 1). In designing the HLT of the current study, the groups of these elements formed by considering the characteristics of problem-based learning, quadrilaterals and argumentations. With this view, the goal of the study was to design and analyze the effects of the HLT prepared for teaching quadrilaterals to the PMSMT by the mathematical practices.

Mathematical Practices

To analyze the effects of the HLT in the current study, mathematical practices, formed in an instructional sequence guided by the HLT, were examined so the PMSMT's understanding about quadrilaterals were reported clearly and systematically. Mathematical practices are taken-as-shared ways of reasoning and discussing about ideas mathematically. By the term of taken-as-shared, it is focused on an environment including discussions related to problems and ideas using mathematical formulization system (Cobb et al., 1997). In the process of emergence of mathematical practices, collective and individual learning processes are used together. In other words, "students actively contribute to the evolution of classroom mathematical practices as they reorganize their individual mathematical activities, and conversely, that these reorganizations are enabled and constrained by the students' participation in the mathematical practices" (Cobb & Yackel, 1996, p. 180). The students make modifications on their mathematical reasoning directly related to their individual learning using and criticizing collective learning in local social situations. Therefore, mathematical practices illustrate the development of students' mathematical understanding, reasoning, and ability of producing justifications and making mathematical explanations in mathematical classroom community based on particular contexts, tasks and ideas (Cobb et al., 1997). This classroom community provides the emergence of mathematical practices in a collaborative learning environment including the discussions localized into the classroom (Stephan & Cobb, 2003).

In that respect, there has been research focusing on the emergence of mathematical practices in different mathematical concepts at different grade levels. The mathematical practices were formed in the mathematical concepts such as differential equations, measurement, circles, integers, and triangles by analyzing the nature of and occurrence in instructional sequences and they emphasized the role and impact of the practices (Akyuz; 2014; Rasmussen & Stephan, 2008; Stephan & Akyuz, 2012; Stephan & Rasmussen, 2002; Uygun & Akyuz, 2019). By these mathematical practices, the role of mathematical tasks and learning contexts could be clearly examined in these research.

With this motivation, the present teaching experiment study was focused on identification of mathematical practices about quadrilaterals referring to understanding and learning of the PMSMT in a learning environment guided by the designed HLT. The emergence of these practices could provide determining the effects of the HLT. Also, in the current study, it was aimed to encourage the PMSMT's understanding of quadrilaterals in a social learning environment including mathematical practices in which they share, understand, criticize and justify their ideas.

In line with this aim, it could be provided that the PMSMT construct their ideas and knowledge about quadrilaterals using their own invented strategies and constructed knowledge systems by the current study instead of taking ready-made definitions, properties and classification systems as suggested by de Villiers (1994). Given these explanations, argumentations are useful to extract mathematical practices since argumentation forms in mathematical discourses including mathematical justifications, explanations of ideas and production and representation of understanding about conceptual mathematics (Lampert, 1990). Hence, in the current study, mathematical practices about quadrilaterals were determined by argumentation. In line with this view, argumentation is explained to be useful for identifying mathematical practices (Cobb et al., 2011) so some previous research (Akyuz; 2014; Rasmussen & Stephan, 2008; Stephan & Akyuz, 2012; Stephan & Rasmussen, 2002; Uygun & Akyuz, 2019) about mathematical practices have used argumentation to perform their studies.

To conclude, the present study was focused on the effects of HLT, and these effects were effectively analyzed by mathematical practices identified through argumentations representing the learners' conceptual understanding.

Method

This teaching experiment study was carried out using case study research design. It was focused on the PMSMT's understanding of quadrilaterals using classroom mathematical practices in a collective learning environment guided by Hypothetical Learning Trajectory (HLT). In other words, by the case study research design, this collective learning environment performed by the designed HLT was explored in detail in order to extract mathematical practices. Moreover, the designing, enacting, testing and evaluating the impact of the HLT based on the emergence of the mathematical practices and the PMSMT's understanding of quadrilaterals were performed by the teaching experiment methodology.

The teaching experiment methodology is useful to make connection between students' development of understanding and mathematical communications; theory and practice about teaching and learning. In teaching experiments, the researchers can directly engage in data by relating practice and theory because "students' mathematics is indicated by what they say and do as they engage in mathematical activity, and a basic goal of the researchers in a teaching experiment is to construct models of students' mathematics" (Steffe & Thompson, 2000, p. 269). With the help of teaching experiments, it is possible to design instructional sequence, to test the effects of this sequence in a classroom, to analyze the effects and process, and to make modifications through retrospective analysis in an iterative process (Gravemeijer, 2004).

To conclude, based on the teaching experiment methodology, the HLT was designed to teach quadrilaterals to the PMSMT, and tested by reporting their understanding through mathematical practices in a learning environment guided by it. Also, the case study research design was used through enacting the HLT in instructional sequence, and analyzing and implementing the PMSMT's learning about quadrilaterals and emergence of mathematical practices. By teaching experiment, theory and practice are used together. In the theory part of teaching experiment, design of the HLT was performed including problem-based learning, nature and properties of quadrilaterals and argumentations. The practice part of the study focused on testing and enacting instructional sequence, and mathematical communications of the learning community was performed and analyzed. In line with this view, this study could make effective connection between theory and practice and contribute to the literature.

Hypothetical Learning Trajectory for Teaching Quadrilaterals

Five-week instructional sequence was enacted by the HLT design based on problem-based learning (PBL) strategy and argumentations. While designing the HLT, supporting tasks as in Table 1 were prepared considering the nature and properties of PBL. The activities and activity sheets were prepared in a way that they were composed of problems having multiple solutions and being viewed in different perspectives. Then, tools and imagery were selected in this way. Afterwards, part of possible discourses of the HLT were organized based on the nature and formation process of mathematical arguments. Finally, the whole HLT was organized into five parts as suggested by Gravemeijer, Bowers and Stephan (2003) as in Table 1. This HLT was designed based on the literature and previous research about quadrilaterals such Aslan-Tutak (2009), Athanasopoulou (2008), Miller (2013), and

Oztoprakci (2014), Turkish middle school mathematics curriculum and the suggestions made by the studies of Zazkis and Leiken (2008), Jones (2000/2001) and Wares (2004).

Hence, it was aimed to form the categories of the HLT related to classification of types of quadrilaterals, the relationship among them, production of new classifications instead of typical ones, and roles of main and auxiliary elements into three phases as in Table 1. Moreover, these phases were determined considering the previous research about quadrilaterals in the literature as explained above. By the HLT, the PMSMT could analyse and acquire deep understanding by classifying, defining and analysing the properties of quadrilaterals.

Table 1
A Hypothetical Learning Trajectory for Quadrilaterals

Phase	Learning Goals	Concepts	Supporting tasks	Tools/Imagery	Possible Discourse
1	Reasoning of the definition of quadrilaterals	Definitions of the quadrilaterals Examples and non-examples of the types of quadrilaterals	Define the figure True/ False and Explain problems	Drawings Worksheets Colourful shapes	Definitions Examples and non-examples Types of quadrilaterals
2	Reasoning of the classification of quadrilaterals	Relationship among the types of quadrilaterals	Concept diagram	Diagrams Drawings	Inclusion and hierarchical relationship among the types of quadrilaterals
3	Reasoning of the properties of quadrilaterals	Properties, main and auxiliary elements of quadrilaterals	Properties Elements	Drawings Charts Diagrams	Main and auxiliary elements Properties of related to main and auxiliary elements

The five-week instructional sequence included twelve activity sheets and different tasks guided by three-phase HLT. The first phase of the HLT occurred in the first week of the instructional sequence, and the PMSMT engaged in the tasks of determining examples and non-examples of different types of quadrilaterals by explaining the reasons behind their identifications. Then, the PMSMT produced the formal definitions for different types of quadrilaterals by identifying the critical attributes of them. Within this phase, the instructor guided the discussions and asked questions to help the PMSMT analyse and comprehend the critical attributes of quadrilaterals and connecting these to produce the definitions intellectually.

In the second phase in the second week, the PMSMT studied on the tasks of classification of quadrilaterals and inclusion among them by diagrams and tables. For example, with respect to middle school mathematics curriculum, the PMSMT are expected to know that square is a kind of rectangle or parallelogram or rhombus and rectangle is a kind of parallelogram with right angles. They analysed this kind of relationships in all types of quadrilaterals.

In the third phase in the third week, the PMSMT focused on the tasks of properties of quadrilaterals by their main and auxiliary elements. They were expected to analyse and attain deep knowledge about these relationships, so the instructor guided the discussions and asked the questions to make the PMSMT realize the relationship among the types of quadrilaterals by analysing them in detail. Through argumentations, the PMSMT could form and manipulate quadrilaterals considering their properties including length, angle and diagonal. For example, the PMSMT attained the knowledge that a rhombus was a parallelogram having perpendicular diagonals. This task was also designed in a way proposed by Van de Walle (2004) to draw and understand the properties of quadrilaterals considering angle, side length, diagonals, symmetry axes and altitude. The task was to list the properties of quadrilaterals by preparing the property list.

For this phase in the fourth week, all types of quadrilaterals were examined based on their common and non-common properties. In the third phase, the instructor guided the discussions and asked questions to make the PMSMT realize the properties of the quadrilaterals based on their main and auxiliary elements by considering their definitions and relationship among them. In the activities of the last week related to all of the phases, the PMSMT engaged in the tasks of solving problems related to the objectives of previous lessons. For example, they formed the Venn-diagrams and sets with and without the elements as the quadrilaterals having diagonals bisecting each other. Also, real life problems about quadrilaterals were solved in the last week. The instructional sequence was performed based on the HLT including these tasks, activities and objectives.

Data Collection and Data Analysis

The participants of the present study were composed of ten junior preservice middle school mathematics teachers (PMSMT). Criterion sampling was used to select PMSMT participants who had previously taken necessary courses such as Geometry; therefore the participants were familiar with the concept of quadrilaterals, their properties, and main theorems about it. Of these participants, 6 of the PMSMT were female and 4 were male.
The qualitative data were gathered through video recordings of whole class and peer group discussions, field notes of the instructor and the PMSMT's written worksheets. Group meetings and interviews were also made each week after enacting the lessons. The participants were asked about the events in the lessons and their views related to the activities and their learning process. Hence, they evaluated the activities, tasks and process emerged by the HLT regularly. The instructional sequence lasted five weeks and three hours in each week. By the analysis of the data, the mathematical practices were determined so that understanding of the PMSMT were examined and the HLT was tested.

The qualitative data of this teaching experiment collected through different sources were analysed by three-phase methodology of Rasmussen and Stephan (2008) based on Toulmin's argumentation model (1969) with constant comparative data analysis method of grounded theory. By this way, the classroom mathematical practices were determined using mathematical ideas emerging in taken-as-shared way. Through these phases, initially, the elements such as claim, data, warrant, backing and rebuttal of the argumentation model were determined using whole class discussion transcripts. The claim can be explained as true ideas, answers of problems or conclusion statements of discussions. The data represent the statements encouraging truth of claims and provide evidence. The warrant illustrates the expressions relating the data with the claim. The backing is the statement providing the reason to accept the truth of an argument. The rebuttal refers to the expressions criticizing the truth of the claim. Then, argumentation logs were produced by these elements and taken-as-shared mathematical ideas were determined based on two criteria of disappearance of backings/warrants because of understanding and not challenging ideas, and usage of claims as other elements of the model in further argumentations. Each argumentation log represented a mathematical idea about quadrilaterals and accepted as the category formed through content analysis process of the study. Lastly, taken-as-shared mathematical ideas were reported under common titles as the mathematical practices. These mathematical ideas were determined based on the themes of the study. The categories by argumentation logs and the themes by mathematical practices are illustrated in Table 2. By these criteria, it was aimed to provide evidence for the case that the mathematical practice became self-evident and understood. Trustworthiness was provided with strategies of data triangulation by collecting data through different sources and member checking.

Table 2
Argumentation logs and mathematical practices

Mathematical practices (Themes)	Argumentation logs (Categories)
Reasoning on definitions of quadrilaterals	Critical attributes of a quadrilateral and types of quadrilaterals (Trapezoid, parallelogram, rhombus, square, rectangle).
	Formal definitions of a quadrilateral and types of quadrilaterals.
Reasoning on relationship among quadrilaterals	Trapezoids include parallelograms, rhombi, square and rectangles.
	Parallelograms include rhombi, squares and rectangles.
	Rhombi include squares.
	Rectangles include squares.
	Reasons of the relationship among types of quadrilaterals.
Reasoning on properties of quadrilaterals	Main elements (angle and side).
	Auxiliary elements (diagonal, symmetry line).
	Properties of main and auxiliary elements.
	Similarities among properties of different types of quadrilaterals.

	Differences among properties of different types of quadrilaterals.

Each PMSMT's expression, reported by using direct quotations from the transcripts and argumentation logs, was illustrated by S_n (where n is a number and each number is belonged to a particular PMSMT). The explanations of the instructor were illustrated by using the letter of T.

Findings

In the study, arguments formed about the mathematical activities were analysed and the classroom mathematical practices were identified in a taken-as-shared way. The mathematical practices as the mathematical ideas becoming taken-as-shared and self-evident, and taking different roles in different argumentation cores were identified. By doing so, three mathematical practices emerged in the current study; *reasoning on definitions of quadrilaterals*, *reasoning on relationship among quadrilaterals* and *reasoning on properties of quadrilaterals*. The emergence of these practices was documented in detail under the following titles.

Practice 1: Reasoning on definitions of quadrilaterals

In this mathematical practice, the PMSMT engaged in the activities and discussed about the definitions of quadrilaterals, trapezoid, kite, parallelogram, rhombus, rectangle and square respectively. The PMSMT wrote their definitions by studying individually while the instructor was observing them. The instructor focused on their writing and identified inaccurate/accurate and incomplete/complete definitions produced by them. Then, the whole class discussion period was initiated. In this process, through argumentation, the instructor used incomplete and inaccurate parts of the definitions in order to analyse the PMSMT's concept definitions and understanding effectively, and to help them develop accurate definition and understanding. The argumentation was started by the instructor's question as follows:

T: What is quadrilateral?

S_1: …quadrilaterals need to have four sides [DATA]. By connecting them, they are formed [WARRANT]. So, they are closed geometric shapes with four sides. [CLAIM].

S_2: With respect to this definition, this figure is expected to be a quadrilateral. However, this figure does not have linear and closed lines and edges [REBUTTAL].

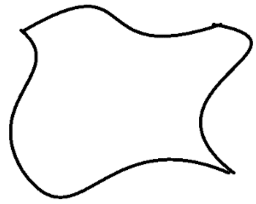

Figure 1. Shape by the definition of S_1

At this part of the argumentation, S_1 provided data focusing on four sides of a quadrilateral and warrant by having connected sides and being closed. Based on them, she explained the claim by quadrilaterals as four sided-closed shapes. This definition included the critical attributes of having four sides and being closed. These attributes were necessary but not sufficient because the geometric shapes encouraging these attributes were not always quadrilaterals. In this view, S_2 provided rebuttal for this argumentation by explaining a geometric shape which was an example for this definition but not a quadrilateral. Then, the discussion period was continued in order to form correct definition of quadrilateral and another argumentation core was produced as follows:

> S_3: Ok. We need line segments for sides [DATA] which are connected to each other [WARRANT]. The quadrilaterals are closed geometric shapes composed of four line segments [CLAIM].
>
> S_2: Based on your definition, this figure is quadrilateral, but it is not in real. [REBUTTAL]

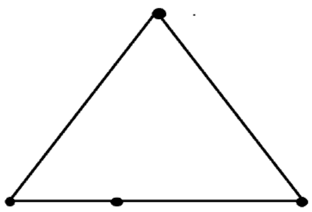

Figure 2. Shape by the definition of S_3

At this part of the argumentation, S_3 provided data by four line segments representing sides and warrant by being connected. S_3 tried to correct the previous definition by adding the attribute of line segments as sides. Hence, the claim was stated by explaining quadrilaterals as closed geometric shapes with four sides formed by line segments. This claim representing the definition of quadrilaterals included the critical attributes of having four line segments as sides and being closed. These attributes were necessary but not sufficient because the geometric shapes encouraging these attributes were not always quadrilaterals. In this view, S_2 again provided rebuttal for this argumentation by representing an example appropriate to this definition but not a quadrilateral so that they realized the necessity of four edges and four angles. Then, the discussion period was continued in order to form correct definition of quadrilaterals and another argumentation core was produced as follows:

> S_4: … quadrilaterals represent closed geometric shapes having four edges and angles [DATA]. In other words, they are formed through connecting four non-linear points with four line segments [WARRANT].
>
> T: Geometric shapes with the vertices or angles taking place on different planes are quadrilaterals based on your explanation but not in real. All of the points forming quadrilaterals must be on the same plane [DATA].
>
> T: Right. So, what is the correct definition of quadrilaterals?
>
> S_5: …quadrilaterals are closed geometric shapes connecting four non-linear points on the same plane by line segments [CLAIM].

S_4 provided data by four edges and angles, and warrant by explaining that these main elements could be connected by four non-linear points. These attributes were necessary to produce definition but not sufficient so the instructor provided the remaining attribute of being on the same plane for these non-linear points as the remaining part of the data for this core of the argumentation. Based on these explanations, S_5 stated the claim representing the accurate and sufficient definition of quadrilateral. All of the critical attributes were used so the expected definition was provided as the claim at the end of the argumentation by S_5. At this point, since the PMSMT did not challenge this claim, the discussion about the definition of quadrilaterals finished. Moreover, this idea produced in this part of the argumentation and represented as the claim was used as data and warrant in the argumentations about the definitions of different types of quadrilaterals in the following discussion periods of the instructional sequence. In this way, this mathematical idea became self-evident and taken-as-shared and placed under the title of the first mathematical practice. At this part of the discussion, the argumentation was initiated and encouraged by the instructor. The instructor identified the incomplete definitions and then, made the PMSMT criticize them. While they were not challenging, the instructor produced explanations such as data for the argument and asked other questions. Hence, the PMSMT learned collectively by criticizing their own knowledge.

After providing accurate and sufficient definition of the quadrilateral, the PMSMT discussed about the definitions of different types of quadrilaterals. The argumentation about the definition of rectangle as a type of quadrilateral was started by the instructor's question as follows:

T: Can you define rectangle?

S_1: A rectangle as a type of quadrilateral whose opposite sides are equal in length... [DATA]

S_2: …the property of having opposite sides with equal lengths does not refer to perpendicularly intersecting sides. This property is about opposite parallel edges [DATA]

S_6: Rectangles are quadrilaterals whose opposite edges are perpendicular and parallel to each other, and equal in length.

S_5: This definition is not valid since two lines cannot be parallel and perpendicular at the same time. Moreover, in the rectangles, opposite sides are parallel so they cannot intersect perpendicularly in the formation of quadrilaterals [WARRANT].

S_6: Hence, we can state that opposite sides are parallel and equal in length and adjacent sides intersect with each other perpendicularly [WARRANT]. …rectangles are quadrilaterals whose opposite sides are parallel and equal in length, and adjacent sides intersect perpendicularly [CLAIM].

DATA

S_1: A rectangle as a type of quadrilateral whose opposite sides are equal in length...

S_2: …the property of having opposite sides with equal lengths does not refer to perpendicularly intersecting sides. This property is about opposite parallel edges.

CLAIM

S_6: …rectangles are quadrilaterals whose opposite sides are parallel and equal in length, and adjacent sides intersect perpendicularly

WARRANT

S_5: This definition is not valid since two lines cannot be parallel and perpendicular at the same time. Moreover, in the rectangles, opposite sides are parallel so they cannot intersect perpendicularly in the formation of quadrilaterals.
S_6: Hence, we can state that opposite sides are parallel and equal in length and adjacent sides intersect with each other perpendicularly.

Figure 3. Argumentation log about the definition of a rectangle by Toulmin's model

In this argumentation, as illustrated in Figure 3, S_1 and S_2 provided data explaining the necessary attributes of having opposite sides in equal length and perpendicular edges. S_6 explained that opposite sides were perpendicular inaccurately and S_5 corrected this wrong explanation by the knowledge that parallel opposite edges did not intersect. Then, S_5 and S_6 provided warrant accurately. At the end of the argumentation, S_6 explained the claim by providing the sufficient and accurate definition of rectangles. Moreover, the definition of quadrilaterals was used as data including the critical attributes of having four non-linear points on the same plane connected by the line segments, four sides, and four edges. This definition was sufficient and necessary since it included necessary attributes of being a closed geometric shape composed of line segments, having four sides intersecting perpendicularly, having four right angles, having opposite sides in equal length. At the end of the argumentation, expected accurate definition was produced for rectangles. Then, the PMSMT did not challenge the claim since it was understood effectively. For example, the definition of a rectangle explained in this claim was used as data and warrant of the argumentation about the definition of a square. In this way, the instructional sequence related to the first phase of the HLT continued to be enacted by the formation of the argumentations about remaining types of quadrilaterals. The discussion processes were performed by focusing on the identification of the critical attributes of the types of quadrilaterals and usage of them to define these types of quadrilaterals as happened in the argumentation about the rectangle. For example, the PMSMT formed the definition of "squares are quadrilaterals all of whose angle measures are 90° and equal, and all edges are equal in length. Therefore, squares are rectangles whose sides are equal in length" as the claim of another argumentation. For other types of quadrilaterals, similar argumentation and taken-as-shared periods were observed in the instructional sequence. In this process, the instructor encouraged the PMSMT discuss the definitions of other types of quadrilaterals and criticize the explanations to produce correct definitions; therefore the PMSMT collectively removed barriers and incorrect understandings.

Practice 2: Reasoning on relationship among quadrilaterals

In this mathematical practice, the PMSMT engaged in the activities and discussed about the relationship among the types of quadrilaterals. While the PMSMT were studying individually on identification of groups and sub-groups among types of quadrilaterals, the instructor observed their studies. The activities included yes/no questions asking the reasons behind the answer, and grouping and sub-grouping of different types of quadrilaterals. The instructor focused on and identified inaccurate/accurate explanations and their drawings. Then, the whole class discussion period was initiated. In this process, the instructor used incomplete and inaccurate parts of the expressions in order to analyse the PMSMT's understanding about hierarchical relationship among the types of quadrilaterals. The relationships were discussed and reported under the title of this mathematical practice. These relationships can be exemplified by the argumentation about the relationship between rhombus and parallelogram taking place as follows:

> S_1: The explanation that rhombi are sometimes parallelograms is not true. The correct one is that rhombi are always parallelograms [CLAIM].
>
> S_2: In rhombi and parallelograms, there are two pairs of parallel opposite sides. However, the lengths of all sides of a rhombus are equal whereas they do not need to be equal for parallelograms since their definitions are … [DATA].
>
> S_3: No, we cannot explain in this way. Previous explanations made for rhombi are true but parallelograms are quadrilaterals whose opposite sides are parallel and equal in length. However, it does not mean that the lengths of two adjacent sides cannot be equal [WARANT].

In this argumentation, initially, S_1 provided the claim accurately by the knowledge that rhombi were a subset of parallelograms. The data were provided by the definitions of rhombi and parallelograms. Then, the warrant was explained by the knowledge about the number of edges in equal length accurately and the PMSMT did not challenge this idea because it was understood. This claim produced in this argumentation became taken-as-shared by taking the role of data or warrant in further argumentations. For example, they produced the argumentation that "squares are always parallelograms. Square as a special kind of rhombus is at the same time a parallelogram since all types of rhombi are parallelograms". In this argumentation, the relationship between squares and parallelograms was used in the claim. The data by square as a type of rhombus were explained and the claim of the above argumentation took the role of the warrant of the argumentation emphasizing the relationship between squares and parallelograms.

Other discussion period was reported and exemplified by reporting the argumentation about the relationship between trapezoids and parallelograms as follows:

> S_5: The explanation that the trapezoids are sometimes parallelograms is true [CLAIM].
>
> S_6: Trapezoids have one pair of opposite parallel edges and these edges do not need to be equal in length since their definitions are … [DATA] so the other pair of opposite sides cannot be parallel to each other [WARRANT].
>
> S_7: For trapezoids, this explanation can be true but it does not mean that it is always true. Trapezoids are quadrilaterals whose at least one pair of opposite sides are parallel among two pairs of opposite sides. Hence, both pairs of parallel sides can be parallel with their pairs in trapezoids [WARRANT].

In this argumentation, initially, S_5 provided the accurate claim by the knowledge that parallelograms were a subset of trapezoids. The data were provided using the definitions of trapezoids and parallelograms by S_6. Then, the warrant was explained by the knowledge about the number of edges in equal length and parallel sides accurately and the PMSMT did not challenge this idea as it was understood. This claim produced in this argumentation became taken-as-shared by taking the role of data or warrant in other argumentations. This mathematical idea was used as data and warrant for the argumentations formed through the classroom discussions about the connection of trapezoids with squares, rectangles and rhombi as special kinds of parallelograms.

Another discussion period was reported and exemplified by reporting the argumentation about the relationship between squares and rhombi as follows:

S_3: The explanation that squares are sometimes rhombi is not true since squares are always rhombi [CLAIM].

S_4: For squares and rhombi, the diagonals bisect each other. However, the sides intersect perpendicularly in squares. Also, the diagonals intersect perpendicularly at middle points in squares so this is not valid for rhombi. In this respect, the cases of squares as rhombi are not considered [DATA, WARRANT].

S_5: Based on their definitions, all of the sides are equal in length and parallel to each other for squares as happened for rhombi. However, squares are quadrilaterals whose sides' lengths are equal and angle measures are equal and 90° so the opposite sides of squares are parallel. This helps me make connection with the definition of rhombus [DATA, WARRANT].

In this argumentation, initially, S_3 provided the accurate claim by the knowledge that squares were a subset of rhombi. The data and warrant were provided by the definitions of squares and rhombi, and the similarities and differences between them. The differences about the cases of intersection of diagonals and the number of opposite parallel edges were explained and provided as data and warrant for the argumentation. This claim produced in this argumentation became taken-as-shared by taking the role of data or warrant in further argumentations and became taken-as-shared. This mathematical idea formed at this episode of the argumentation was used as data for the argumentations formed through classroom discussions about the relationship between rectangles and squares; and rectangles and rhombus.

Practice 3: Reasoning on properties of quadrilaterals

In this mathematical practice, the PMSMT engaged in the activities and discussed about the main and auxiliary elements of quadrilaterals, and their properties based on the relationship among types of quadrilaterals. The PMSMT engaged in the activities of the properties of quadrilaterals based on auxiliary elements such as diagonals and symmetry lines by studying individually as the instructor observed their studies. The instructor focused on and identified inaccurate/accurate explanations and drawings of diagonal and symmetry lines of quadrilaterals. Then, the whole class discussion period was initiated. In this process, the instructor used incomplete and inaccurate parts of the expressions in order to help the PMSMT analyse the main and auxiliary elements, and the properties of quadrilaterals effectively. These properties were discussed about and some of these properties were reported under the title of this mathematical practice. For example, the argumentation about the diagonals of quadrilaterals took place as follows:

T: What can we say about the diagonals of different types of quadrilaterals?

S_1: In parallelograms, the diagonals bisect each other. Also, in rectangles, squares, and rhombi, the diagonals bisect each other since they are special kinds of parallelograms [CLAIM].

T: What can we say about the intersection of diagonals on quadrilaterals?

S_2: …the diagonals are the line segments connecting corresponding vertices of the quadrilaterals… [DATA] On rhombus and squares, the diagonals bisect each other perpendicularly because a diagonal separates these quadrilaterals into two congruent isosceles triangles. When the other diagonal is drawn, they bisect each other since the medians of congruent triangles are formed. These medians are also the altitudes of these isosceles triangles. Therefore, the diagonals bisect each other perpendicularly [WARRANT].

S_3: In this case, we can also say that the lengths of the diagonals are equal since we form congruent isosceles triangles.

T: Right. Can we state that the lengths of the diagonals are equal for these two kinds of quadrilaterals?

S_5: No, we cannot. It is valid for squares but not for rhombi [CLAIM].

T: Why?

S_5: For example, in the triangle of ABC, we can examine the length of the diagonal, d_1, by the Cosine Theorem and we can also find the length of the diagonal, d_2, in the triangle of ABD in the same way. In the equations formed by this way, the lengths of the diagonals are different since the lengths of the sides used in the theorems are equal and $\cos\alpha \neq \cos\beta$. Hence, the lengths of the diagonals, d_1 and d_2, are not equal [WARRANT] (in Figure 4).

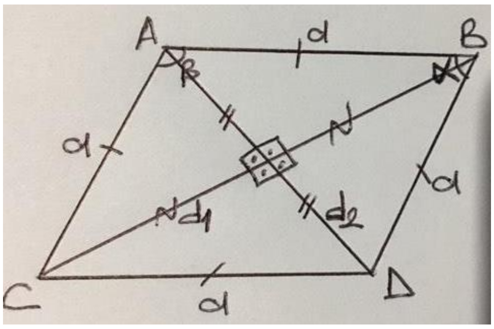

Figure 4. Rhombi's diagonals bisect each other perpendicularly

In this argumentation, the instructor asked the questions to help the PMSMT form the accurate claim. Initially, S_1 and S_5 provided the accurate claim that the diagonals of all types of parallelograms bisected each other and the lengths of diagonals of rectangles and squares were always equal while this was not valid for rhombi and parallelograms. Also, they added that the diagonals bisected each other perpendicularly in rhombus and square for the claim. The data were explained by the definition of diagonal. The warrant was provided into two parts: the intersection of them by bisecting each other and the lengths of them. The case of intersection of diagonals was explained using the congruent triangles formed by the diagonals of quadrilaterals by S_2. Another part of the warrant about the lengths of the diagonals was stated by using the Cosine Formula by S_5. The differences on types of parallelograms about these cases of intersection of diagonals and the lengths of them were explained and provided as the warrant for the argumentation accurately. The claim produced in this argumentation became taken-as-shared by taking the role of data or warrant for other argumentations and became taken-as-shared. In other words, the mathematical idea formed through this argumentation was used as data and warrant for the argumentations about the rectangles so that this idea became taken-as-shared. The process that this argumentation became taken-as-shared took place as follows:

T: We know that the diagonals of rectangles bisect each other but they do not intersect each other perpendicularly [CLAIM]. What can we say about the lengths of diagonals on rectangles?

S_2: When we consider about the way in which we examine the diagonals of rhombus, we find that the lengths of diagonals of rectangles are equal by following the similar steps to the ones for rhombus [DATA]. A diagonal forms two congruent right triangles. Diagonals are the hypotenuse of these triangles. By doing so, when we use Pythagorean Theorem, we find that the lengths of diagonals are equal since the lengths of the edges used in the formula of the theorem are equal [WARRANT].

In this argumentation, the instructor initiated the discussion by using the claim produced previously so that she encouraged the PMSMT to acquire relational understanding. The previous idea was used as warrant of this argumentation. Also, this idea was used as warrant in order to show that the diagonals of a kite were not equal in length although they intersected each other perpendicularly. In this argumentation, the instructor provided the accurate claim that the diagonals of rectangles bisected each other but they did not intersect perpendicularly. S_2 explained the data by the lengths of diagonals of rhombi and the warrant by the formation of congruent triangles and Pythagorean Theorem accurately. At this point, the previous mathematical argumentation about the diagonals of a parallelogram and types of parallelograms was used as data and warrant for this argumentation about the diagonal of rectangles. In this way, the previous claim produced in the argumentation became taken-as-shared by taking the role of data or warrant in this argumentation. Moreover, the mathematical ideas produced in this argumentation was used as data and warrant for the discussion of the activity about the classification of quadrilaterals by producing the definitions based on their properties and elements such as diagonals and symmetry lines using Venn-diagrams.

After providing accurate and sufficient knowledge about diagonals of quadrilaterals, the PMSMT discussed about the symmetry lines as another auxiliary element of quadrilaterals. The argumentation about the symmetry lines was started by the instructor's question as follows:

T: How many symmetry lines do squares, rectangles, parallelograms and rhombi have? What are these lines?

S_3: Squares have four symmetry lines [CLAIM]. When the square is folded on the symmetry line, the shapes coincide [DATA]. Two of them are the diagonals since as we have discussed, a diagonal of a square separates the square into two congruent isosceles right triangles. Also, they bisect each other perpendicularly [WARRANT].

At this part of the argumentation, the mathematical idea about diagonals became taken-as-shared. This was the second instance that the mathematical idea about the diagonals was observed. S_3 produced the claim about the diagonals as symmetry lines. This was accurate but not sufficient since the squares had two more symmetry lines. The definition of symmetry lines was explained as the data of the argumentation. The warrant about the formation of congruent triangles was added. Then, the remaining parts of the claim and the argumentation were supported as follows:

S_2: …The other symmetry lines are the lines passing through the midpoints of the edges as it is seen in the figure [CLAIM].

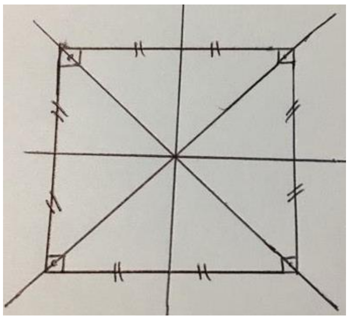

Figure 5. The symmetry lines of a square

S_5: A square as a special kind of rectangle, rhombus and parallelogram [WARRANT] based on their definitions [DATA] also have four symmetry lines [CLAIM].

At this episode of the argumentation initiated by the instructor, the PMSMT used the connection of squares with rectangle, rhombus and parallelogram as data by providing warrant for another claim. Hence, this mathematical idea became taken-as-shared. The data and warrant of the argumentation were accurately explained but the claim could not be explained appropriately. The revision of the claim as the rebuttal of the argumentation was performed by S_3 as follows:

S_3: Squares are special kinds of rectangles, rhombi and parallelograms. They have two symmetry lines, which are their diagonals as it is in the figure [REBUTTAL].

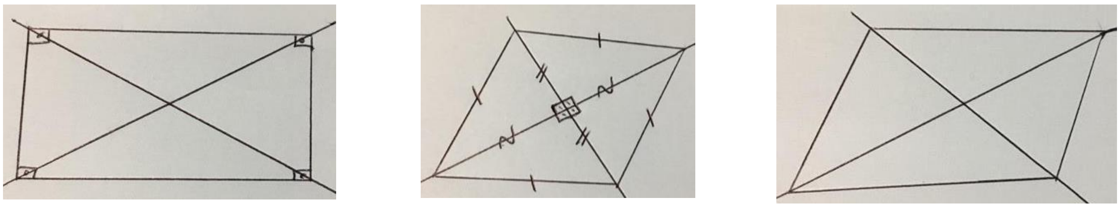

Figure 6. The symmetry lines of rectangle, trapezoid and parallelogram

S_7: It is clearly observed that a rhombus has two symmetry lines, which are its diagonals. However, this case is not valid for other two types of quadrilaterals [CLAIM]. Based on the definition of symmetry line… [DATA], we know that a diagonal separates a rhombus into two congruent triangles. Symmetry lines are diagonals since the creation and position of these triangles take place in a way that one of them is the image of the other one through reflection with respect to their common side which is diagonal [WARRANT].

In this episode of the argumentation, S_7 explained the accurate claim of the argumentation. Then, the data by the definitions of symmetry line and the warrant by the formation of congruent triangles and their positions were provided. This was the other instance that the mathematical idea about diagonals was observed in the instructional sequence. Afterwards, the argumentation about the symmetry lines of rectangles took place as follows:

S_6: …the diagonals are not symmetry lines of rectangles [CLAIM].

T: Why?

S_6: … Based on the definitions of symmetry lines… [DATA] In rectangles, a diagonal separates the rectangles into two congruent triangles but their position and orientation is not appropriate for reflection with respect to their common side which is diagonal [WARRANT].

T: Right. So, we can say that the rectangles do not have symmetry lines, can't we?

S_1: The rectangles have symmetry lines.

T: What are they?

S_1: I do not know.

T: Ok. Let's think about the definition of reflection. How can we fold the rectangle so both hand sides of the folding line match exactly?

S_9: By the reflection… [WARRANT]. So, the lines passing through the middle points of the edges are symmetry lines for rectangles. In this respect, rectangles have two symmetry lines [CLAIM].

In this argumentation, the instructor made the PMSMT criticize and reason the explanations and ideas by asking questions. A part of the claim that the diagonals were not symmetry lines of rectangles was provided at the beginning of the argumentation by S_6 and the other part of the claim about the symmetry lines was done at the end of the argumentation by S_9. The data and warrant for the initial part of the claim were provided by S_6 based on the knowledge about the formation of congruent triangles with different orientations that did not match when they were folded on the diagonals accurately. Also, the warrant of the later claim of the argumentation was explained by the definition of reflection appropriately. This claim produced in this argumentation became taken-as-shared by taking the role of data or warrant in further argumentations. In other words, the mathematical idea formed through this argumentation was used as data and warrant of the argumentations about the symmetry lines of parallelograms so that this idea became taken-as-shared. The process that this argumentation became taken-as-shared took place as follows:

T: Well. Let's talk about parallelograms. What can we say about them?

S_4: The diagonals of them are not symmetry lines. Also, the lines passing through midpoints of the edges are not symmetry lines [CLAIM].

T: Why?

S_4: In the similar way followed through examining rectangles…[DATA], a diagonal separates a parallelogram into two congruent triangles but the position and orientation of these triangles do not match with respect to reflection based on the diagonal as the triangles' common side [WARRANT].

T: What about the lines passing through the midpoints of the sides?

S_2: …For example, think about a parallelogram as in the figure (Figure 6). When we fold the parallelogram with respect to d_1 line, both sides of this line do not match through reflection. For example, the angle on the vertex of A is an obtuse angle while the angle on the vertex of B is an acute angle. Similar explanations and mathematical ideas can be made for the d_2 line [WARRANT].

S_4: …Therefore, parallelograms do not have symmetry lines [CLAIM].

In this part of the discussion, the instructor helped the PMSMT produce the elements of argumentation. S_4 provided the claim that the diagonals and the lines passing through the midpoints of the sides was necessary; thus building upon the previous mathematical ideas since they were the symmetry lines of different types of quadrilaterals such as squares. The data were provided by the mathematical idea produced for the rectangles. Also, the warrant was explained using the formation of congruent triangles, their orientation and matching the sides of the shape by reflecting based on the diagonal as the symmetry line. Hence, this mathematical idea was produced and used as data and warrant for different argumentations and became taken-as-shared.

The mathematical idea produced for the symmetry lines of rhombus became taken-as-shared in the discussions about the definition of rhombus based on their elements and properties by producing the claim "… a rhombus is a quadrilateral having two perpendicularly intersecting symmetry lines. Moreover, the squares can be defined in this way by the same property".

The mathematical idea about the symmetry lines of quadrilaterals was used as the data and warrant for the discussions made related to the classification of quadrilaterals by producing their definitions based on their properties and elements such as diagonals and symmetry lines using a Venn-diagram in the activity.

Discussion and Conclusion

In the current study, it was possible to provide contribution and insight into the literature about learning quadrilaterals with the help of a social analysis. In this social analysis, the development process of the PMSMT through collective learning was documented using mathematical practices. These practices also illustrated the impacts of instructions, discussions, tasks and in-the-moment decisions of the instructor on the learning of the PMSMT. This current teaching experiment represented the PMSMT's understanding of quadrilaterals focusing on three mathematical practices; *reasoning on definitions of quadrilaterals*, *reasoning on relationship among quadrilaterals* and *reasoning on properties of quadrilaterals*. They understood the formal definitions of quadrilaterals, the relationships and inclusions among them accurately through an instructional sequence guided by the HLT. The mathematical practices and the process of the PMSMT's development of understanding about quadrilaterals were reported based on the whole class discussions, their justifications and counter explanations. Through whole class discussions, various mathematical ideas were explained and criticized by the PMSMT. These discussion processes could be analysed by Toulmin's model of argumentation effectively as suggested by Krummheuer (1995) and Knipping (2008). Similarly, the model was used in order to determine mathematical practices about different mathematical concepts in previous research as well (Stephan & Akyuz, 2012; Stephan & Rasmussen, 2002; Uygun & Akyuz, 2019). This model enhances the identification of ideas, reasons for the truth of the ideas, counter-explanations and encouraging statements. The researcher could clearly see the whole picture of the discussions process including connected ideas in a mathematical way. In that respect, it can be stated that the process of formation and revision of mathematical ideas, emergence of mathematical practices and development of understanding of the PMSMT about quadrilaterals could be analysed and reported usefully by Toulmin's model of argumentation.

In the current study, the PMSMT engaged in the activities and improved understanding about quadrilaterals with the help of the argumentation. By the argumentation, they participated in cognitive processes related to the concept of quadrilaterals using definitions, analysed and organized their properties and attributes, identified examples and non-examples by reasoning, conceptualized attributes and non-attributes of quadrilaterals, properties of them, and understood the classifications of quadrilaterals based on the inclusions and hierarchies among them. In this respect, it can be stated that argumentation enhanced understanding about concepts, attaining knowledge and critical thinking (Akyuz, 2014; Jim´enez-Aleixandre, Bugallo, & Duschl, 2000; Jonassen & Kim, 2010; Uygun & Akyuz, 2019). For example, in the argumentation about the definitions of quadrilaterals, the PMSMT examined the wrong and incomplete parts of the definitions formed by the participants. They made explanations for the reasons of these parts so that the participants made corrections on the participants' thoughts related to wrong and missing definitions.

Also, by correcting ideas, they strengthened their knowledge about the definitions. Hence, they produced correct definitions for quadrilaterals by justifying, correcting and empowering their ideas. Through the emergence of three mathematical practices, the PMSMT could understand the quadrilaterals and make reasoning on them effectively. By this teaching experiment, social learning processes were represented with the help of the argumentation as happened in some research in the literature (Stephan & Akyuz, 2012; Stephan & Rasmussen, 2002; Uygun & Akyuz, 2019). Moreover, the instructor guided and provided the formation of the arguments by asking effective questions. By questioning, the instructor helped the PMSMT criticize their ideas and explanations to construct their knowledge effectively. In this respect, the instructor and the questions asked through the discussions can be important for the formation of the arguments and understanding of the concept as emphasized by the study of Yackel (2002). By the tasks on the HLT, and the guidance and questions of the instructor, the PMSMT analysed their ideas and explanations through the argumentation. With the help of the argumentation, the PMSMT revised their incomplete understanding, reorganized their knowledge and constructed new knowledge studying collectively and contributing to the social learning environment as suggested by Cobb and Yackel (1996), and Stephan (2003).

Through the argumentation, the PMSMT focused on examples and non-examples of different types of quadrilaterals using their definitions. These argumentations showed that they could effectively attain the knowledge about the definitions of quadrilaterals and their main properties. By using examples and non-examples, the PMSMT could effectively analyse the critical attributes of these types. By the knowledge attained through this process, they could acquire the understanding needed to produce the definitions of them. The usage of examples and non-examples enhanced the analysis and producing the definitions of quadrilaterals as suggested by previous research (Joyce & Weil, 1996; Skemp, 1987; Sowder, 1980). When the argumentations were examined, it was observed that the PMSMT initially could not identify the critical attributes but they could produce accurate formal definitions of quadrilaterals at the end of the discussions by analysing and attaining sufficient level of understanding so that they could learn the concept effectively. In other words, it was found that the PMSMT's knowledge related to definitions of quadrilaterals was not well-connected but they developed their understanding, produced their definitions and used them in order to illustrate their understanding related to the properties of them and the relationship among them through the argumentation. In this respect, it could be stated that the PMSMT should be provided with opportunities to reach full understanding of the concept and then, the formal definitions could be formed. Hence, the tasks about the definitions of quadrilaterals, and examples and non-examples of quadrilaterals on the HLT and the instructor's guidance improved their argumentation and understanding. This finding was parallel to the explanations and results of the previous studies (De Villiers, Govender & Patterson, 2009; Vinner 1991).

Based on the emergence of mathematical practices, the PMSMT usually developed their understanding of quadrilaterals and their properties using their various definitions with the help of the tasks on the HLT. By determining the critical attributes, the connection of these attributes, their roles for the concept and producing the definition by these attributes, the PMSMT could form the formal definitions by analysing and understanding the concept effectively. In this respect, it could be stated that the process of formation of formal definitions enhanced their understanding of quadrilaterals and reasoning about them. They produced argumentation about various definitions of the quadrilaterals in order to analyse and understand the properties of them through these definitions. This finding is parallel to the previous research in the literature (Leikin & Winicki-Landman, 2001). Also, the mathematical practices emerged in taken-as-shared ways in the study as the PMSMT used the mathematical ideas in further argumentations to produce new ideas. Therefore, by the argumentations formed based on the tasks on the HLT, PMSMT could develop their understanding of quadrilaterals by connecting them using their inclusion and hierarchical connections. This finding can be encouraged by the study of Van Dormolen and Zaslavsky (2003). Because a concept's definition is particular form of a more general one based on hierarchy criterion, they state that production of definition of a concept can be performed by other concepts. Moreover, the PMSMT focused on inclusive definitions of quadrilaterals to examine their properties and improve the understanding about them. The usage of these definitions has been emphasized in many research (Fujita & Jones, 2008; Schwarz & Hershkowitz, 1999).

Earlier studies focusing on the design and planning of lessons emphasize the importance of planning for effective teaching. In spite of this view, these studies explain that most teachers pay attention to the activities that students engage in, rather than the implementation of these activities based on the aims of the activities by providing students' interaction in the process of planning lessons (Clark & Yinger, 1987; Kilpatrick, Swafford, & Findell, 2001). In this respect, the planning process should be performed considering the goals of the activities and discourses that can be observed in the classroom. Moreover, when teachers use inflexible lesson plans they can have problems in solving unpredicted problems, providing classroom interaction and adapting indispensable events (Akyuz, Dixon & Stephan, 2013). In accordance with these necessities, HLT can be useful to be formed during lesson planning process. While designing lessons, the instructional tasks, the tools students use, the nature of the classroom discourse and the classroom activity structure are considered rather than focusing on classroom activities. Also, by determining these elements of the HLT, it is known that this HLT is performed flexibly. While applying the HLT in the classroom, the instructor can make revisions in any part of the HLT based on implications made from the instructional process. To line with this view, in the current study, an HLT about quadrilaterals was designed and tested. Through the planning process, the elements of the HLT were prepared in order to form argumentations through classroom discussions for understanding quadrilaterals. Based on the findings, it could be stated that the HLT used in the current study was beneficial since it could enhance the production of argumentations and the development of the PMSMT's understanding about quadrilaterals. Hence, the social learning environment providing the opportunities for the PMSMT to criticize their ideas and construct their knowledge accurately could be supported with the help of the designed HLT. By the property of HLT, which is being flexible and enhancing to adapt indispensable events, formation of mathematical discourses including argumentations could be encouraged. For example, in the current study, the literature review revealed that preservice mathematics teachers often have difficulty in defining quadrilaterals, so the first phase of the HLT was considered based on the learning goal about defining them. Through this process, the instructor identified the wrong and incomplete definitions of the PMSMT while they were studying individually on the related activity sheet. Then, the instructor used these definitions in the whole class discussions in order to remove the deficiencies and misconceptions about the definitions of quadrilaterals through argumentations. As observed in the emergence of the first mathematical practice, the PMSMT formed the correct definitions by criticizing these deficiency and misconceptions, correcting ideas and forming accurate definitions of quadrilaterals under the guidance of the instructor. Hence, being flexible in the planning process could enhance the formation of argumentations in the current study so that the usage of the HLT could encourage observing the argumentations about mathematical ideas related to quadrilaterals. Moreover, the third mathematical practice about the properties of quadrilaterals, the PMSMT discussed the symmetry lines of quadrilaterals. Through discussion, the instructor realized that the PMSMT could not identify the symmetry lines of a rectangle as an unexpected event. Then, the instructor guided the discussion to the definition of symmetry and symmetry line in order to help them understand and identify the symmetry lines of a rectangle.

Based on the findings of the study, some revisions can be made on the HLT. For example, the first phase can be extended to include the connection of quadrilaterals with other concepts. The activities including concept maps and different diagrams about the classification of quadrilaterals can be added to the second phase of the HLT. The number and types of problems related to all phases of HLT should be increased. By making these revisions on this HLT, effective learning environments encouraging the PMSMT's learning of quadrilaterals can be provided. Moreover, by using the revised form of the HLT according to these suggestions, an instructional sequence can be conducted to PMSMT in further studies. The mathematical practices emerged in the further studies can be compared by the mathematical practices of the current study. The HLT can be tested and revised again by this way so that a beneficial HLT can be suggested effectively for teaching about quadrilaterals for PMSMT.

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