A study of geometric understanding via logical reasoning in Hong Kong[1]

Kin-Keung Poon
kkpoon@eduhk.hk

Chi-Keung Leung
ckleung@eduhk.hk

Department of Mathematics and Information Technology
The Education University of Hong Kong

Abstract

The purposes of the study reported herein were to identify the common mistakes in geometry made by junior secondary school students in Hong Kong, and to compare the students' performance in geometry with their results in a logic test. A geometry test and a logic test were developed and administered to a sample of 554 students aged between 13 and 14. The students in schools with a higher level of academic achievement were found to obtain higher scores in the geometry test. In addition, a strong correlation was found between students' achievement in geometry and their fundamental logical reasoning ability. The findings offer grounds for reflection on the geometry curriculum currently in place in junior secondary schools in Hong Kong.

Keywords: Learning difficulty, Geometric understanding, Logical reasoning

Introduction

Geometry has a very long history as an area of mathematical study. It is a fundamental component of mathematics, and plays a crucial role in bridging the gap between mathematics and science. For instance, both Einstein and Hawking (world-renowned physicists of the 20th and 21st centuries) attributed their success to geometry (Clements & Sarama, 2011). Learning geometry requires an understanding of geometric objects and spatial concepts and properties; proportional, inductive and deductive reasoning skills; the ability to apply knowledge judiciously; the ability to manage and process data; and an understanding of relevant variables (see, e.g., Berthelot & Salin, 1998; Clements & Sarama, 2011).

The teaching and learning of geometry are topics of considerable international interest, and appropriate teaching methods and curriculum designs have been widely debated (Gal, Lin &

[1] The research was supported by the Internal Research Grant, The Education University of Hong Kong

Ying, 2009; Guven, 2012; Hao-hao, 2009; Mammana & Villani, 1998; Royal Society, 2001). Students' attainment in geometry is also a part of the Trends in International Mathematics and Science Study (TIMSS). In TIMSS 1999, the average geometry attainment of Hong Kong secondary students was lower than those of its nearby countries of Japan, Republic of Korea, Singapore and Chinese Taipei (Mullis, Martin, Gonzalez, Gregory, Garden, O'Connor, Chrostowski, & Smith, 2000). TIMSS 2007 reported that the average geometry attainment of Hong Kong secondary students was again behind these four countries (Mullis, Martin, & Foy, 2008).

In the Hong Kong territory-wide system assessment (TSA) 2015, it was reported that Secondary 3 (aged around 14-15) students could find areas and volumes in 2-D and 3-D figures, as well as angles related with lines and rectilinear figures. However, more improvement could be shown in items related to definitions of common terms in geometry as well as deductive geometry (Hong Kong Examinations and Assessment Authority, 2015). This study focuses on identifying some common mistakes in geometry made by junior secondary school students in Hong Kong, and comparing the students' performance in geometry with their results in a logic test.

Previous Studies in Geometry Learning

In some studies, researchers have discussed the proficiency of geometry teachers and the problems that arise when teachers are unaware of their students' difficulties in learning geometry (Gal, 2011; Gal, Lin & Ying, 2009; Kilic, 2011; Kuzniak & Rauscher, 2011). According to the van Hiele model of geometric thought, the development of students' understanding of geometry depends not on age or biological maturity but on the form of instruction received (for details, see Aydin & Halar, 2009; Clements & Battista, 1992; Fuys, Geddes & Tischler, 1988; Patsiomitou & Emvalotis, 2010; van Hiele, 1984). The model suggests that geometric thinking develops sequentially, and that progress through these levels of development is aided by particular teaching techniques. However, little research has been conducted to date on the relationship between instructional methods and students' understanding of geometry (Clements & Battista, 1992; Fuys et al., 1988). Most researchers have focused on the difficulties faced by students in developing an understanding of geometric theory and making the transition to formal proofs (Gal & Linchevski, 2010; Hellmich & Reiss, 2002; Herbst, 2002; Jones, 2000; Rowlands, 2010). For instance, Fujita and Jones (2002) proposed a strategy for linking practical and deductive geometry by developing students' geometric and spatial awareness. Their findings suggest that intuitive and visual teaching approaches play a critical role in developing students' geometric thinking, particularly during the transition from practical to theoretical and deductive geometry.

In recent research, proving geometry theorems has been found to be one of the greatest challenges for secondary school mathematics students (Noboru & Van Lehn, 2005; Wong, Yin, Yan & Cheng, 2011). When a proof is required, the difficulty of a geometry task increases dramatically, perhaps because the choice of construction is an ill-structured problem.

Studies have shown that the learning problems encountered by mathematics students are associated with a variety of factors, such as problematic prior learning experiences, a failure to see the connections between mathematics and other experiences, and difficulties in understanding fundamental mathematical concepts (Jensen, 1998). Poon and Leung (2010) found that students' logical reasoning is highly related to their performance in algebra. Brousseau (1997; cited in Kuzniak and Rauscher, 2011) categorised difficulties in learning geometry by origin, as either ontogenic, epistemological or didactic (the first two relating to students' understanding or ability, and the latter to teachers' use of methods suited to their students' cognitive capacities). Similarly, Gal and Linchevski (2010) discussed the difficulties faced by geometry students in terms of three successive phases of visual perception and knowledge representation (VPR): organisation, recognition and representation. Students' learning difficulties in mathematics, especially in geometry, which constitutes the largest proportion of junior school mathematics curriculums, generally result in poor progress and the development of a negative attitude towards the subject.

Research has also indicated that students' difficulties in learning geometry are related to an inability to understand and accurately interpret geometric concepts, and weaknesses in deductive reasoning (Gal & Linchevski, 2010; Miyazaki, Kimiho, Katoh, Arai, Ogihara, Oguchi, Morozumi et al., 2012; White, 1985; Senk, 1985). However, opinion is divided on the capacity of teaching methods to overcome these difficulties. Some researchers have argued that appropriate teaching methods can help to mitigate the difficulties faced by learners of geometry. For example, Gal, Lin and Yin (2009) argued that the use of a heterogeneous teaching approach, especially one designed to suit individuals' thinking styles and meet their particular needs, can improve students' geometry learning.

In general, students' misconceptions of geometry fall into the following five categories: an inability to understand the real-world applications of geometric concepts; errors in using basic geometric concepts to solve complex problems; an inability to recognise different forms (symbolic, visual, etc.) of the same geometric concepts; a lack of concrete understanding of the geometric concepts underlying particular models; and an inability to recall major geometric principles when using nested geometric concepts (Berthelot & Salin, 1998). We focus on the last three categories. There have been few systematic studies to date of the relationship between students' logical reasoning and performance in geometry, as many researchers believe that the errors and misconceptions of geometry students are generally rooted in a lack of spatial awareness.

This Study

No systematic research has been conducted in Hong Kong on the difficulties faced by students learning mathematics, especially geometry. This is unfortunate, as an empirical knowledge base is necessary to improve students' mathematics learning. To fill this crucial gap in the research, this project is designed to answer the following research questions.

RQ 1. Does students′ attainment in geometry differ with school banding?

RQ 2. What misconceptions of/errors in geometry are common among junior secondary school students?

RQ 3. Is students' logical-reasoning capacity correlated with their performance in geometry?

Methodology

In this project, the difficulties experienced by Secondary 3 (Grade 9 equivalent) students in learning geometry were analysed. Six secondary schools participated in the project, and the sample size (*N*; number of students) was 554. Of the six schools, two were in Band 1[2] (highest level of academic attainment), two were in Band 2 (medium level of academic attainment) and two were in Band 3 (lowest level of academic attainment). A geometry test and a simple logic test were administered to the targeted students. The geometry questions were based on Hong Kong's core (common) junior secondary level curriculum to give every student an equal opportunity for success.

The geometry questions tested students' geometry knowledge up to junior secondary school level. Our investigation had the following two main objectives.

1. To identify students' difficulties in learning geometry and classify their weaknesses by analysing their responses in a diagnostic test comprising a set of mathematics questions developed by the research team.
2. To determine statistically whether students' logical-thinking ability is correlated with their success in learning geometry.

Theoretically, our sample size was large enough to ensure statistically reliable results.

---

[2] In Hong Kong, all the public secondary schools are divided into three bandings. In general, Band 1 schools are the top 1/3 of schools according to the academic background of new intakes at Secondary 1; Band 2 schools are the next 1/3 of the schools and Band 3 schools are the rest.

Instrument

Two instruments were used in the study: a geometry test based on the content of the secondary school mathematics curriculum developed in line with Hong Kong's Territory-wide System Assessment guidelines, and a simple logic test proposed by Poon and Leung (2010). To ensure that both the students' understanding of basic geometric concepts and their deductive proof writing skills were tested, the geometry test consisted of 12 multiple-choice questions, 2 additional explanation-based questions and 2 long questions. The questions were carefully prepared to ensure that all of the essential skills developed at the junior secondary level were tested.

Analysis of Results

Table 1. Mean geometry test scores across school bands.

School band	*N*	Mean
1	188	18.64
2	174	7.30
3	192	7.18
Total	554	11.11

Table 1 shows the geometry-test scores across school bands. The students in schools with a higher level of academic attainment were found to obtain better results in the geometry test.

Responses to RQ 1: Attainment in geometry at different banding

As shown in Table 1, the academic attainment of Band 1 students is much higher than those in the two lower bands. Apparently the mean score of Band 2 students is higher than the one for Band 3 students. Nevertheless, the difference is statistically insignificant ($p > 0.05$).

Responses to RQ 2: Common errors in geometry

To answer RQ2, we collated overall statistical information on the common misconceptions and error types (see Appendix 1) exhibited by the students in the geometry test, as shown in Table 2.

Table 2. Error-type statistics.

Number of mistakes by error type

	E1	E2	E3	E4	E5	E6	E7	E8	E9	E10	E11
N	554	554	554	554	554	554	554	554	554	554	554
Sum	0	21	310	5	115	0	222	538	50	8	299
Percentage	0	3.8	55.9	0.9	20.8	0	40.1	97.1	9.0	1.4	54.0

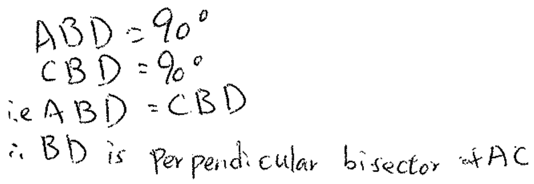
	E12	E13	E14	E15	E16	E17	E18	E19	E20	E21	E22
N	554	554	554	554	554	554	554	554	554	554	554
Sum	148	306	277	0	404	416	118	359	211	5	5
Percentage	26.7	55.2	50.0	0	72.9	75.1	21.3	64.8	38.1	0.9	0.9

We focused on the error types exhibited by more than 50% of the students. Errors in the categories E3, E8, E11, E13, E16 and E17 were made most frequently by the students.

Examples of the most common error types are provided below.

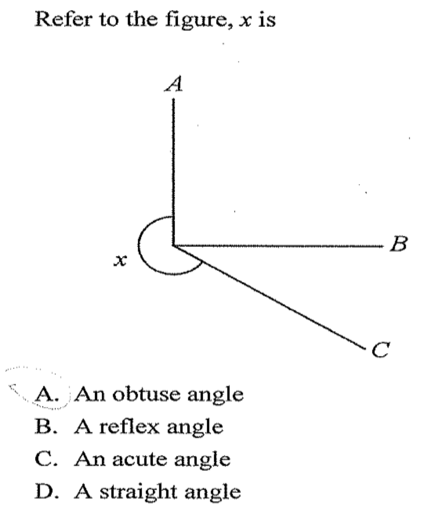
E3 (incomplete proof) and E11 (notation errors)

Example: no conclusive proof is provided that *BD* is a perpendicular bisector of *AC* (E3), and the wrong notation is used for the angles (E11).



E8: Failure to understand definitions and/or properties of mathematical concepts

Example 1: Inability to define a reflex angle, and confusion of reflex angle with obtuse angle.



Example 2: Inability to understand the properties of a regular polygon

Which of the following statements must be **wrong**?

A. Any rhombus must be a regular polygon.
B. All sides of any regular polygon must be equal in length.
C. Rectangle is a kind of parallelogram.
D. Square is a kind of rectangle.

Example 3: failure to understand the term ‘common side’, which is incorrectly interpreted as a side of the same length.

BD = AD (common sides)

∠ABD+∠BDA+∠BAD = 180° (common side)

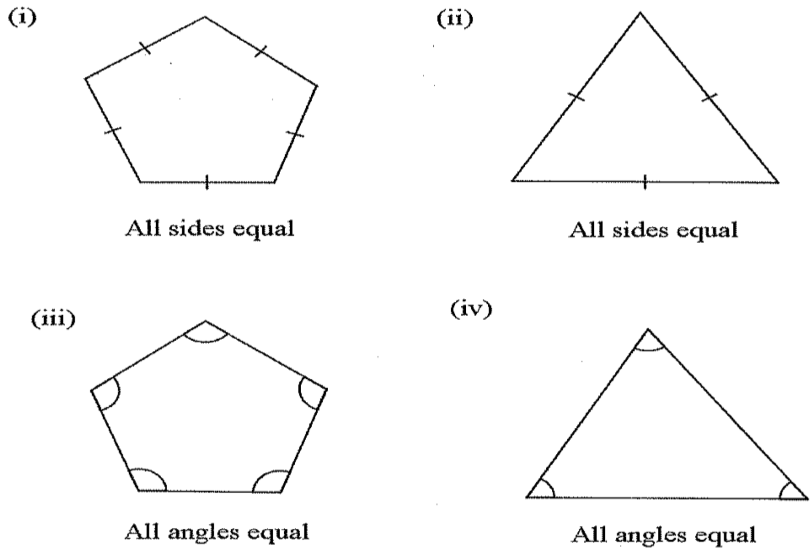
∠ABD = 60

∠BDA = 60

∠BAD = 60

Example 4: failure to understand the properties of a regular polygon (66% chose option D).

Which of the following figures must be regular polygons?
(The figures are not drawn in scale)



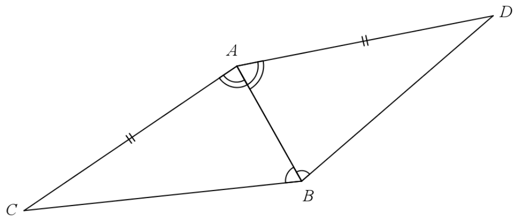
A. Only (ii) and (iv)
B. Only (i), (ii) and (iii)
C. Only (ii), (iii) and (iv)
D. All are correct

E16: Confusion of conditions for congruent and similar triangles

<u>Example</u>:

E17: Misuse of the side-side-angle (SSA) condition or the angle-side-side (ASS) condition to prove that triangles are congruent.

<u>Example</u>: option C was chosen by 54% of the students in response to question 9 in the geometry test.



C. $AC = AD$ (given)

$AB = AB$ (common side)

$\angle ABC = \angle ABD$ (given)

$\therefore \Delta ABC \cong \Delta ABD$ (SSA)

Responses to RQ 3: correlation between geometry-test and logic-test performance

The potential effect of school band on the correlation between geometry-test scores and logic-test scores was investigated. Table 3 shows the Pearson correlation coefficients for the geometry-test and logic-test scores across school bands. The correlation coefficients between these scores across all of the groups were positive, with *p*-values close to 0. This strong association between the geometry-test and logic-test scores indicates that the students who were good at logic were also good at geometry.

Table 3. Correlation coefficients for geometry-test and logic-test scores across school bands.

		Geometry-test score (Band 1)	Logic-test score (Band 1)	Geometry-test score (Band 2)	Logic-test score (Band 2)	Geometry-test score (Band 3)	Logic-test score (Band 3)
Geometry-test score	Pearson's correlation coefficient	1	.494**	1	.535**	1	.415**
	Sig. (2-tailed)		.000		.000		.000
	N	186	186	174	174	192	192

Table 3. Correlation coefficients for geometry-test and logic-test scores across school bands.

		Geometry-test score (Band 1)	Logic-test score (Band 1)	Geometry-test score (Band 2)	Logic-test score (Band 2)	Geometry-test score (Band 3)	Logic-test score (Band 3)
Geometry-test score	Pearson's correlation coefficient	1	.494**	1	.535**	1	.415**
	Sig. (2-tailed)		.000		.000		.000
	N	186	186	174	174	192	192

**. Correlation significant at the 0.01 level (2-tailed).

To further investigate the relationship between the geometry- and logic-test scores, we divided the logic questions into four categories, as shown in Table 4. The questions in the category of deductive reasoning had the highest correlation coefficients (0.451) with the geometry-test scores obtained by the Band 1 students (see Table 5). As the mean score in the geometry test achieved by the Band 1 students was far higher than that of the students in the other bands (as shown in Table 1), this finding indicates that the relationship between performance in geometry and deductive-reasoning ability is particularly strong.

Table 4. Four categories of logic-test items.

Category	Items	Nature
1	Q1-Q5	Numerical patterns
2	Q6-Q10	Symbolic relations
3	Q11-Q15	Spatial relations
4	Q16-Q20	Deductive reasoning

Table 5. Correlation coefficients for geometry-test scores and four categories of logic-test scores across school bands.

School band			Geometry test score	Logic test category1	category 2	category 3	category 4	Logic test score
1	Geometry score	Pearson Correlation	1	.296	.336	.206	.451	.494
		Sig. (2-tailed)		.000	.000	.000	.000	.000
		N	186	186	186	186	186	186
2	Geometry score	Pearson Correlation	1	.468	.340	.410	.334	.535
		Sig. (2-tailed)		.000	.000	.000	.000	.000
		N	174	174	174	174	174	174
3	Geometry score	Pearson Correlation	1	.330	.373	.263	.269	.415
		Sig. (2-tailed)		.000	.000	.005	.000	.000
		N	192	192	192	192	192	192

Discussions and Conclusion

Geometry is an important area of mathematical study. It plays a crucial role in bridging the gap between mathematics and other disciplines such as science. Students' attainment in geometry is of considerable international interest. In TIMSS 1999 and TIMSS 2007, Secondary students in Hong Kong were reported to perform poorly in comparison to those in nearby countries, namely Japan, Republic of Korea, Singapore and Chinese Taipei. Our research aims to locate and identify the problems faced by students in Hong Kong and make some suggestions to improve the learning of geometry.

Our findings indicate that one of the fundamental learning difficulties in geometry is rooted in students' weakness in the understanding of the definitions and properties of mathematical concepts (such as E8, E11), for examples, reflex angles, regular polygons, and common side in a figure. These difficulties might be caused by the lack of seamless connection between primary mathematics and secondary mathematics curriculums. In primary schools, students usually learn geometric concepts via informal approaches by means of folding, cutting or measuring exercises, and thus certain key geometric terms are not clearly defined in the primary-school curriculum. For example, no conceptual distinction is made between a line and the mathematical object known as a 'line segment'. In addition, primary-school pupils are taught only that an angle larger than a right angle is called an obtuse angle; they are not aware of the concept of a reflex angle. These difficulties persist into secondary school, as secondary-school curriculums cover just a little more than the basic definitions of elementary geometric objects. The lack of continuous development through primary and secondary curriculums seems to be one reason for students' ineffective learning at junior secondary schools. This motivates us to suggest that the junior form curriculum could spend more time on such basic topics and develop more teaching models to help students understand and make good use of geometric properties.

The other fundamental learning difficulties are rooted in students' weakness in developing proof, this included poor reasoning skills, misinterpretation of given conditions and misuse of theorems (such as E3 and E17). Secondary-school geometry curriculums require students to exhibit a thorough understanding of geometric objects and good deductive-reasoning skills, as they must use logical reasoning to derive new geometric facts from previously learned principles. Our findings suggest that students doing poorly in deductive proof is highly related to the lack of good logical reasoning. The strong correlation between the results in the logic test and those in the geometry test suggests that further work could be carried out in at least two directions. The first is to design a suitable logic test to assess students' potential before they begin secondary schooling. This would allow an understanding of students' general ability before teaching begins and would allow teachers to refer to the test results to design different teaching methods for different types of students, so that students could learn geometry through different paths and achieve common outcomes. Secondly, teachers would also be able to identify the outlier students with high logic test scores but relatively low geometry test scores. These students are likely to return to the normal stream if they are provided with appropriate instruction.

Finally, in order to ensure students' understanding, teaching shall be based on instruction rather than the memorisation of facts, rules and procedures. Every concept could be comprehensively taught, and teachers could ensure that their students thoroughly understand the content. It would not be easy to learn a new concept. Therefore, conceptual teaching shall be a focus, and every concept shall be taught in a range of modes: verbal and symbolic, for example, as well as visual. As the ability to make connections between geometry and real life is crucial to students' understanding of geometric concepts, teachers could describe real-world applications of geometric concepts and pose realistic problems in the classroom. In addition, teachers could provide a range of examples of correct and incorrect uses of each concept; explain the critical elements of the concept; take into account students' existing knowledge of the concept; establish connections between the concept and students' immediate environment and personal experiences; and construct models to represent the concept. For example, to tackle common errors of E16 and E17, teachers could encourage each student to construct their own figures to illustrate why the conditions of AAA and SSA could not be used in the proof of congruent triangles. With solid individual constructive experience, students would develop better understanding of the use and misuse of geometric concepts and keep the knowledge in long-term memory.

We also suggest that teachers would be responsible for identifying and correcting students' misconceptions. To be competent in doing so, their own knowledge of the relevant concepts shall be concrete and comprehensive. Teachers who understand students' misconceptions could help them to accurately comprehend geometric concepts and avoid future errors. Researchers have shown that the use of dynamic geometry software significantly affects students' academic achievement (Guven, 2012). Bittista (2007; cited in Ubuz, Ustun and Erbas, 2009) showed qualitatively that the creation of dynamic geometry environments improves students' geometry learning. In Wang's (2009) survey of the development of inquiry-based teaching/learning, inquiry-based teaching was found to improve learning, especially students' ability to write proofs. The results of this study might urge the need for a new paradigm of instructions for enhancing students' capability of understanding of basic geometric terms and properties as well as writing proofs.

References

Aydin, N. & Halar, E. (2009). The impacts of undergraduate mathematics reasoning stages. *The Montana Mathematics Enthusiast,* 6(1&2), 151-164.

Berthelot, R. & Salin, M. H. (1998). The role of pupils' spatial knowledge in the elementary teaching of geometry. In C. Mammana & V. Villani (Eds.), *Perspectives on the teaching of geometry for the 21st century*. Dordrecht, NL: Kluwer Academic Publishers.

Clements, D. H. & Battista, M.T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*. NY: Macmillan.

Clements, D. H. & Sarama, J. (2011). Early childhood teacher education: the case of geometry. *Journal of Mathematics Teacher Education*, 14, 133-148.

Fujita, T. & Jones, K. (2002). The bridge between practical and deductive geometry. In A.D. Cockburn & E. Nardi, (Eds.), *Proceedings of PME* 26(2), 384- 391.

Fuys, D., Geddes, D. & Tischler, R. (1998). *The van Hiele model of thinking in geometry among adolescents*. Monograph Number 3. Reston VA: NCTM.

Gal, H. (2011). From another perspective – training teachers to cope with problematic learning situations in geometry. *Educational Studies in Mathematics*, 78, 183- 203.

Gal, H. Lin, F. & Ying, J. (2009). Listen to the silence: the left-behind phenomenon as seen through classroom videos and teachers' reflection. *International Journal of Science and Mathematics Education*, 7, 405-429.

Gal, H. & Linchevski, L. (2010). To see or not to see: analyzing perspectives of visual perception. *Educational Studies in Mathematics*, 74, 163-183.

Guven, B. (2012). Using dynamic geometry software to improve eighth-grade students' understanding of transformation geometry. *Australasian Journal of Educational Technology,* 28(2), 364-382.

Hao-hao, W.(2009). Teaching and learning how to write proofs in concepts of geometry. *US-China Review*, 6(12), 74-80.

Herbst, P. G. (2002). Engaging students in proving: a double bind on the teacher. *Journal for Research in Mathematics Education,* 33(3), 176-203.

Hong Kong Examinations and Assessment Authority (2015). *Results of Secondary 3 Mathematics in TSA 2015*. Hong Kong: HKEAA. Retrieved March 15, 2016.

Jensen, A. R. (1998). *The G factor: The science of mental ability*. Westport, CT: Praeger.

Jones, K. (2000). Providing a foundation for deductive reasoning: students' interpretations when using dynamics geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics,* 44(1-2), 55-85.

Keating, D. P., & Bobbitt, B. L. (1978). Individual and developmental differences in cognitive-processing components of mental ability. *Child Development*, 49, 155-167.

Kilic, H. (2011). Pre-service secondary teachers' knowledge of students'. *Turkish Online Journal of Qualitative Inquiry*, 2(2), 17-85.

Kuzniak, A & Rauscher, J. (2011). How do teachers' approach to geometric work relate to geometry students' learning difficulties? *Educational Studies in Mathematics*, 77, 129-147.

Mammana, C. & Villani, V. (1998) (Eds.). *Perspective on the teaching of geometry for the 21st century*. Dordrecht: Kluwer.

Marrades, R. and Gutiérrez, A. (2004). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics*, 44, 87-125.

Miyazaki, M. Kimiho, C., Katoh, R., Arai, H., Ogihara, F., Oguchi,Y., Morozumi, T., Kon, M & Komatsu, K. (2012). Potentials for spatial geometry curriculum development with three-dimensional dynamic geometry software in lower secondary mathematics. *International Journal for Technology in Mathematics Education*, 19(2), 73- 79.

Mullis, I. V. S., Martin, M. O., Gonzalez, E. J., Gregory, K. D., Garden, R. A., O'Connor, K. M., Chrostowski, S. J., & Smith, T. A. (2000). *TIMSS 1999 International Mathematics Report*. Boston: International Study Center.

Mullis, I. V. S., Martin, M. O., & Foy, P. (2008). *TIMSS 2007 International Mathematics Report*. Boston: TIMSS & PIRLS International Study Center.

Noboru, M. & Vanlehn, K. (2004). GRAMY: A geometry theorem prover capable of construction. *Journal of Automated Reasoning*, 32(1), 3- 33.

Noboru M. & Vanlehn (2005), Advanced Geometry Tutor: An intelligent tutor that teaches proof-writing with construction. *Artificial Intelligence in Education*, 443-450.

Patsiomitou, S. & Emvalotis, A. (2010). Students' movement through Van Hiele levels in a dynamic geometry guided reinvention process. *Journal of Mathematics and Technology,* 18-48.

Poon, K. K. & Leung, C. K. (2010). Pilot study on algebra learning among junior secondary students. *International Journal of Mathematical Education in Science and Technology*, 41(1), 49-62.

Reiss, K. Hellmich, F. & Reiss, M. (2002). Reasoning and proof in geometry in secondary school students, *PME*, 26, 4-114.

Rowlands, S. (2010). A pilot study of a cultural-historical approach to teaching geometry. *Science and Education*, 19, 55-73.

Royal Society/Joint Mathematical Council (2001). *Teaching and learning geometry*. 11-19, London, Royal Society/Joint Mathematical Council.

Senk, S. L. (1985). How well do students write geometry proofs? *Mathematics Teacher*, 78(6), 448-456.

Ubuz, B. Ustun, I. & Erbas, A. K. (2009). Effect of dynamic geometry environment on immediate and retention level achievement of seventh grade students. *Eurasian Journal of Educational Research,* 35, 147-164.

Van Hiele, P. M. (1984). A child’s thought and geometry. In D. Fuys, D. Geddes and R. W. Tischler (Eds.) (1959/1985). *English translation of selected writings of Dina van Hiele-Geldof and Pierre M. van Hiele*. Pp. 242-252. Brooklyn: Brooklyn College.

White, J. M. (1985). Developmental delay and cognitive processes in mathematics. *Focus on Learning Problems in Mathematics*, 7(2), 11-21.

Wong, W. Yin, S. Yan, H. & Cheng, Y. (2011). Using computer-assisted multiple representations in learning geometry proofs. *Educational Technology and Society*. 14(3), 43-54.

Appendix 1: Error types in geometry

Error type	Description of error
E1	Failure to understand question
E2	Incomplete answer
E3	Incomplete proof
E4	Unclear or incorrect logical proof
E5	Inability to state reasons
E6	Use of incorrect fact
E7	Inability to demonstrate correct reasoning
E8	Failure to understand definitions and/or properties of mathematical concepts
E9	Reliance on visual representation of a figure
E10	Measurement errors
E11	Notation errors
E12	Failure to understand the sum of exterior angles
E13	Confusion of concept of a parallel line with its converse
E14	Confusion of Pythagoras’ theorem with its converse
E15	Inability to draw diagram
E16	Confusion of conditions for congruent and similar triangles
E17	Misuse of SSA or ASS to prove that triangles are congruent
E18	Confusion of ASA and AAS when proving

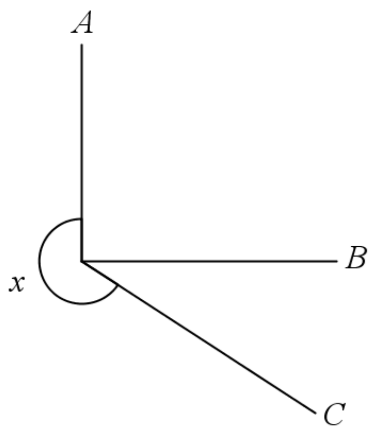
	congruence of triangles
E19	Failure to understand the properties of parallelograms
E20	Failure to understand quadrangle relations
E21	Failure to understand concept of parallel-line angles
E22	Error in formula for summing angles of triangles

Appendix 2 Geometry Test

Please write your answers on the answer sheet.

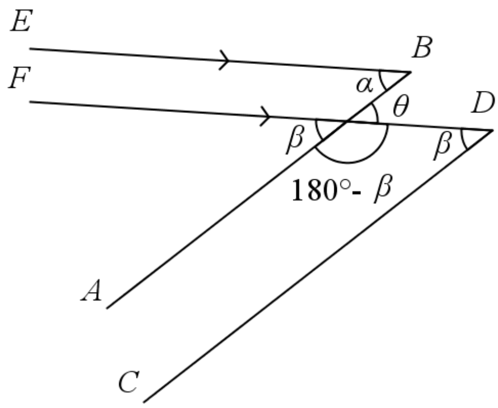
Session A – Multiple Choice (one mark each; choose only one answer to each question)

1. Which of the following definitions describes *x* in the figure below?



A. An obtuse angle
B. A reflex angle
C. An acute angle
D. A straight angle

2. Which of the following statements about this figure is **correct**?



A. $\alpha = \theta$ (alternate angles are equal)

B. *AB* // *CD* (corresponding angles, *AB* // *CD*)

C. $\alpha = \theta$ (interior angles, *BE* // *DF*)

D. *AB* // *CD* (interior angles are supplementary)

3. Which of the following statements is **incorrect**?

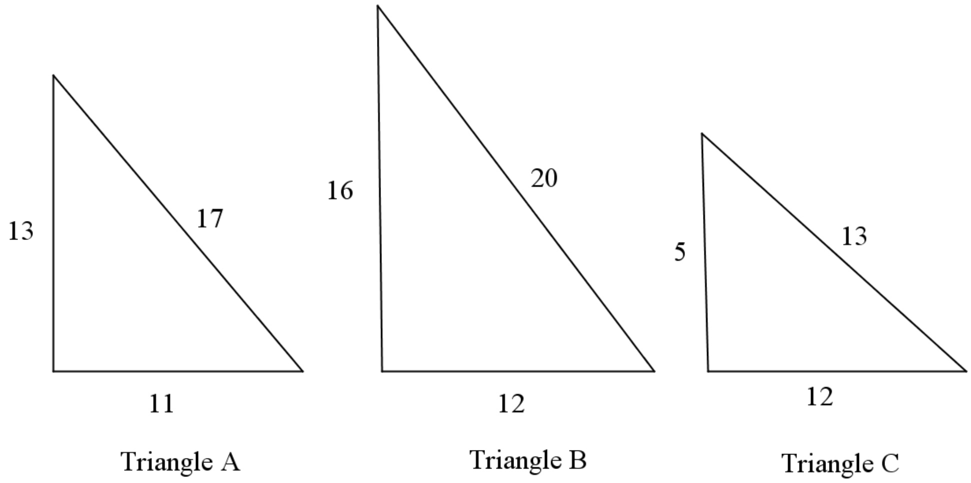
A. Every rhombus is a regular polygon

B. All of the sides of a regular polygon are equal in length

C. A rectangle is a type of parallelogram

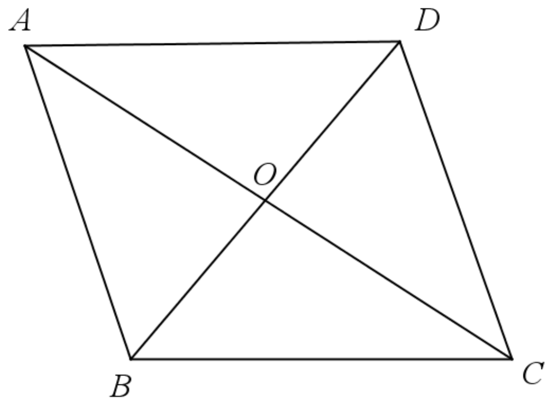
D. A square is a type of rectangle

4. Which of the following combinations of statements and reasons is **correct**?



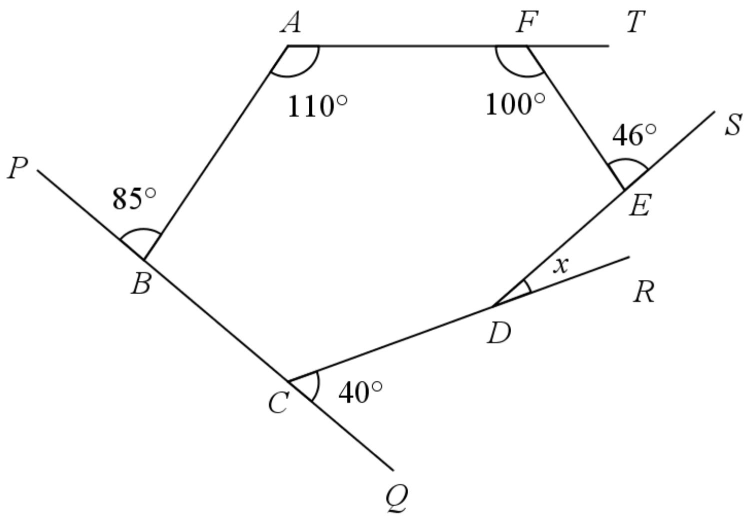
A. Only triangles A and B are right-angled triangles. (Reason: Pythagoras' theorem.)
B. Only triangles A and B are right-angled triangles. (Reason: the converse of Pythagoras' theorem.)
C. Only triangles B and C are right-angled triangles. (Reason: the converse of Pythagoras' theorem.)
D. Only triangles B and C are right-angled triangles. (Reason: Pythagoras' theorem.)

5. Which of the following statements about the parallelogram below is **correct**?
(The figure is not drawn to scale)



A. $BO = OD$
B. $BD = AC$
C. BO is a perpendicular bisector of AC
D. BD is an angle bisector of $\angle ABC$

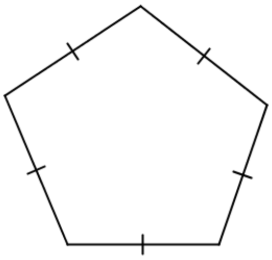
6. Find the unknown value x.



A. 9°
B. 19°
C. 29°
D. 39°

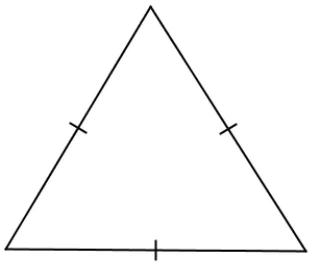
7. Which of the following figures depict regular polygons?
(The figures are not drawn to scale.)

(i)



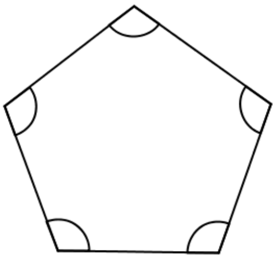
All sides equal

(ii)



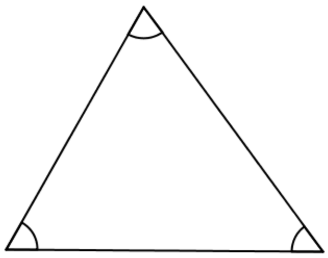
All sides equal

(iii)



All angles equal

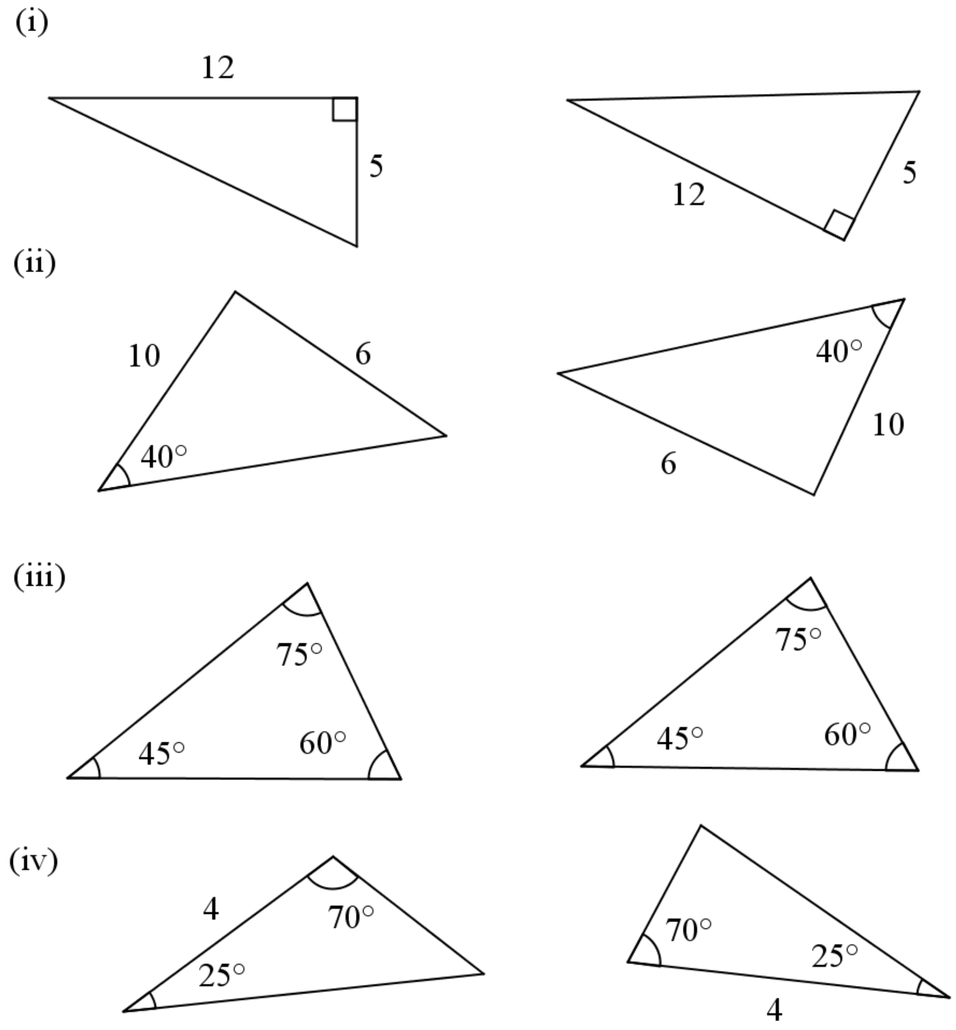
(iv)



All angles equal

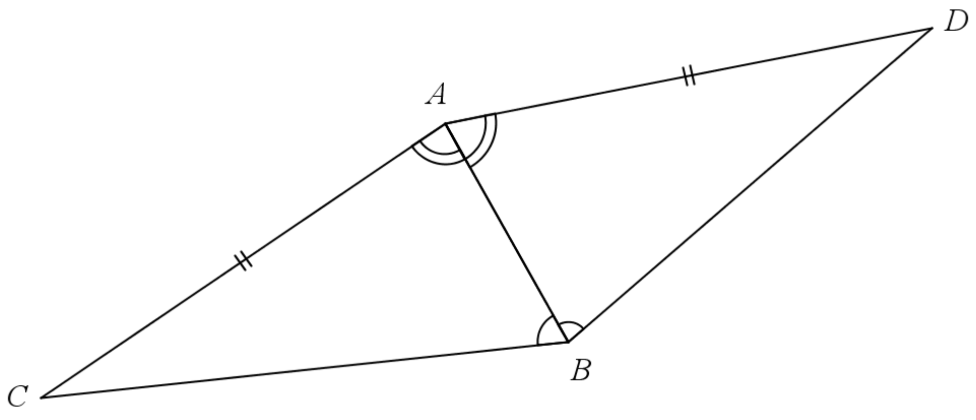
A. (ii) and (iv) only
B. (i), (ii) and (iii) only
C. (ii), (iii) and (iv) only
D. All of the figures

8. Which of the following pairs of triangles are **congruent**?



A. (i) and (iv) only
B. (i), (ii) and (iii) only
C. (ii), (iii) and (iv) only
D. All of the pairs

9. Which of the following statements about this figure is **correct**?



A. $\angle ABC = \angle ABD$ (given)

$AB = AB$ (common sides)

$AC = AD$ (given)

$\therefore \Delta ABC \cong \Delta ABD$ (ASS)

B. $AB = AB$ (common sides)

$\angle BAC = \angle BAD$ (given)

$AC = AD$ (given)

$\therefore \Delta ABC \cong \Delta ABD$ (SAS)

C. $AC = AD$ (given)

$AB = AB$ (common sides)

$\angle ABC = \angle ABD$ (given)

$\therefore \Delta ABC \cong \Delta ABD$ (SSA)

D. $\angle CAB = \angle DAB$ (given)

$\angle ABC = \angle ABD$ (given)

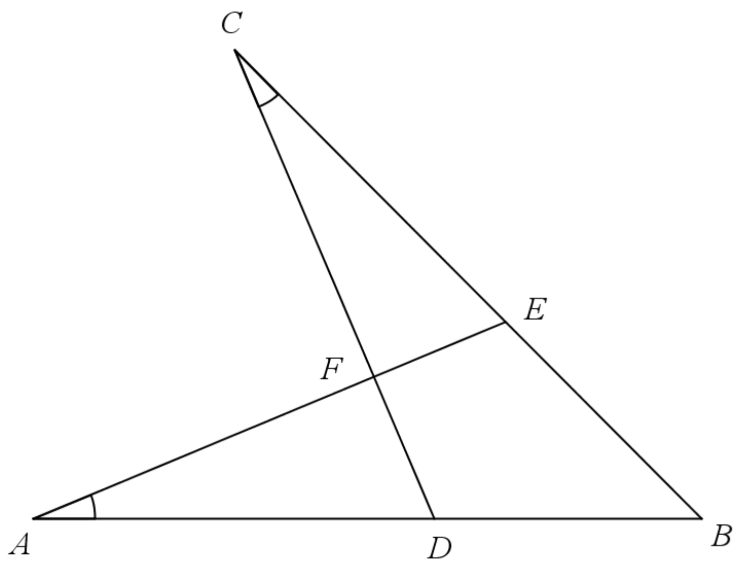
$\angle BCA = 180° - \angle CAB - \angle DAB$ ($\angle$ sum of Δ)

$\angle BDA = 180°$ - $\angle ABC - \angle ABD$ ($\angle$ sum of Δ)

$\angle BCA = \angle BDA$

$\therefore \Delta ABC \cong \Delta ABD$ (AAA)

10. In the figure below, $AB = BC$. Which of the following is **correct**?



A. $\angle EAB = \angle DCB$ (given)

$\angle ABE = \angle CBD$ (common angles)

$AB = BC$ (given)

$\therefore \Delta ABE \cong \Delta CBD$ (AAS)

B. $\angle EAB = \angle DCB$ (given)

$AB = BC$ (given)

$\angle ABE = \angle CBD$ (common angles)

$\therefore \Delta ABE \cong \Delta BCD$ (ASA)

C. $\angle EAB = \angle DCB$ (given)

$\angle ABE = \angle CBD$ (common angles)

$\angle BEA = 180° - \angle EAB - \angle ABE$ ($\angle$ sum of Δ)

$\angle BDA = 180° - \angle DCB - \angle CBD$ ($\angle$ sum of Δ)

$\angle BEA = \angle BDA$

$\therefore \Delta ABE \sim \Delta CBD$ (AAA)

D. $AB = BC$ (given)

$\angle ABE = \angle CBD$ (common angles)

$\angle BEA = 180° - \angle EAB - \angle ABE$ ($\angle$ sum of Δ)

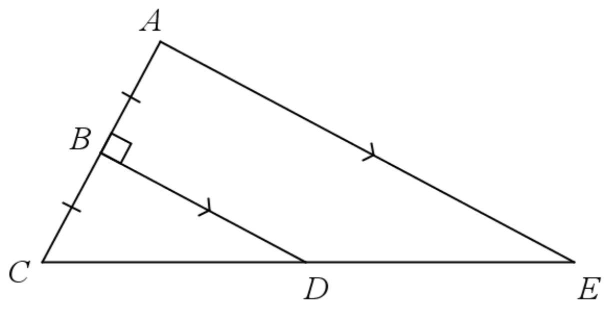
$\angle BDA = 180° - \angle DCB - \angle CBD$ ($\angle$ sum of Δ)

$\angle BEA = \angle BDA$

$\therefore \Delta ABE \cong \Delta BCD$ (SAA)

11. In the figure above, $AB = BC$ and AE // BD. Which of the following statements are

correct, and **why**? (The figure is not drawn to scale)



(i) $CD = DE$

(ii) BD is a perpendicular bisector of AC

(iii) $BD = AB$

(iv) $BD = \frac{1}{2}AE$

A. (i) and (iv) only

B. (ii) and (iii) only

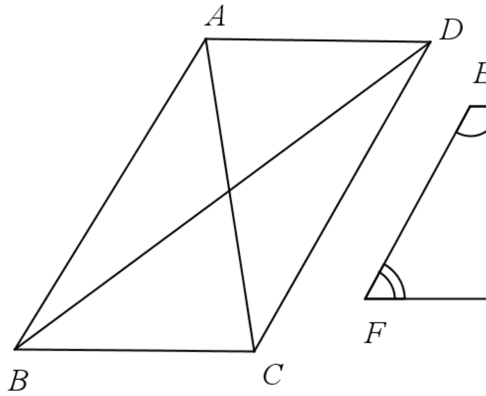
C. (i), (ii) and (iii) only

D. (i), (ii) and (iv) only

Explain your choice of each statement (four marks).

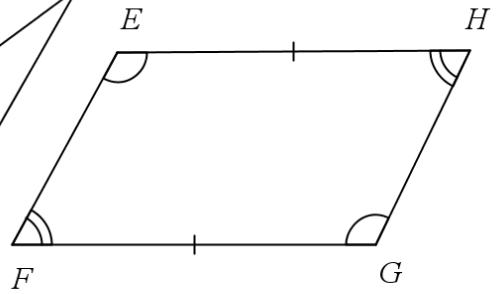
(Please write your answers on the answer sheet.)

12. Which of the following figures **must be** parallelograms, and **why**?
(The figures are not drawn to scale)



$AC = BD$

Figure A

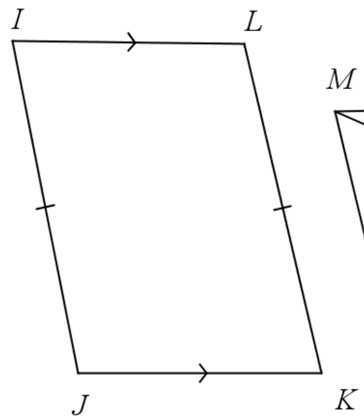


$EH = FG$

$\angle HEF = \angle FGH$

$\angle EFG = \angle GHE$

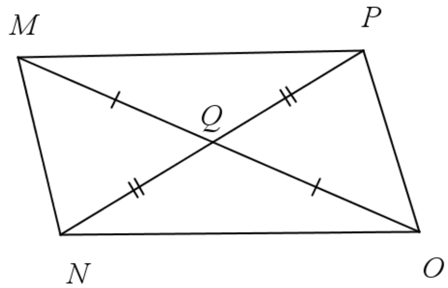
Figure B



$IL \parallel JK$

$IJ = LK$

Figure C



$MQ = QO$

$NQ = QP$

Figure D

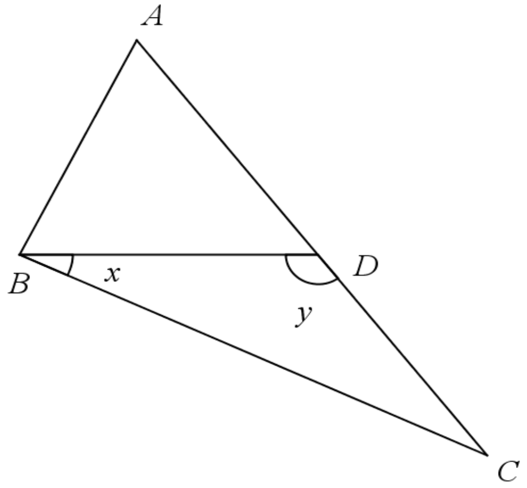
A. Figures A and B only
B. Figures A and C only
C. Figures B and D only
D. Figures C and D only

Explanation: prove that the selected figures are parallelograms and show why the remaining figures are not parallelograms (four marks). (Please write your answers on the answer sheet.)

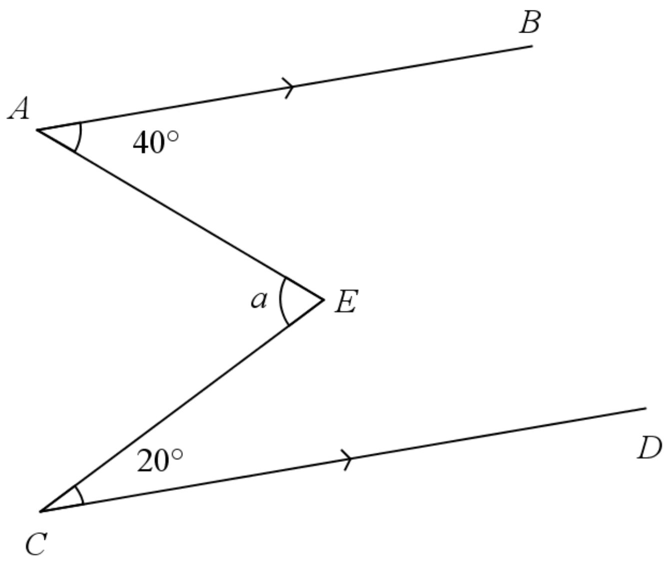
Session B - Long Questions (four marks each)

(Please write down your steps and state the reasons for and/or theorems used at every step where appropriate.)

13. In the figure below, *ADC* is a straight line, and *AB* = *BD* = *DC* = *AD*. Find *x* and *y*.



14. In the figure below, $\angle BAE = 40°$, $\angle ECD = 20°$ and *AB* // *CD*. Find the unknown angle *a*.

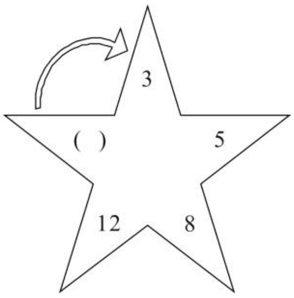


End of paper

Appendix B Logic Test

Please give your answers in the boxes provided.

1. Which of the following numbers fits in the brackets?



A. -1
B. 0
C. 1
D. 2
E. 3

2. Which of the following numbers is missing from the sequence (indicated by brackets)?

16 18 22 ( ) 36

A. 24
B. 26
C. 28
D. 30
E. 32

3. Which of the following numbers is missing from the sequence (indicated by brackets)?

19 17 ( ) 20 15 23 13 26

A. 18
B. 17
C. 16
D. 15
E. 14

4. Which of the following numbers is missing from the sequence (indicated by brackets)?

10 40 90 ( ) 250 360

A. 100
B. 120
C. 140
D. 160
E. 200

5. Which of the following numbers is missing from the grid below (indicated by brackets)?

1	2	3	4
3	7	13	21
7	22	53	( )
15	67	213	531

A. 94
B. 98
C. 102
D. 106
E. 110

☐

6. If $23 \Delta 10 \otimes 2 = 3$, then $26 \Delta 2 \otimes 4 = ?$

A. 10
B. 12
C. 16
D. 18
E. 22

☐

7. If $\Delta + \Delta + \Delta = 15$ and $\nabla + \nabla = 20$, then

A. $\Delta = \nabla$
B. $\Delta + \Delta = \nabla$
C. $\nabla + \nabla = \Delta$
D. $\Delta + \Delta + \Delta = \nabla + \nabla$
E. $\nabla + \nabla + \nabla = \Delta + \Delta$

☐

8. Which of the following letters fits in the grid below (indicated by brackets)?

Z	3	( )	10
1	X	6	Q

A. S
B. U
C. V
D. W
E. Y

☐

9. If $\Delta 25 \otimes 10 \nabla 2 = 248$, then $\Delta 3 \otimes 10 \nabla 20 = ?$

A. 2

B. 5

C. 7

D. 10

E. 20

10. If $\Delta + \Delta + \Delta = 15$, then $\Delta - \Delta + \Delta = ?$

A. 5

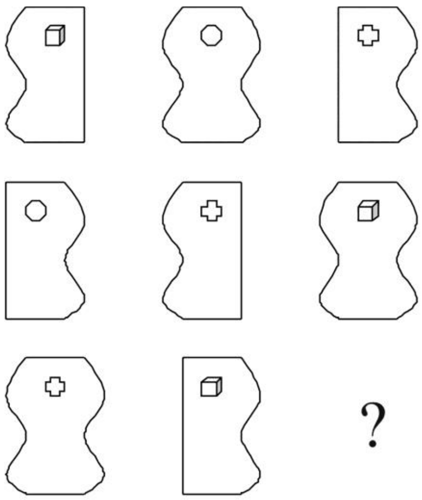
B. 4

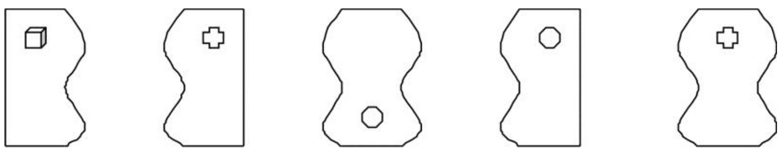
C. 3

D. 2

E. 0

11. Which of the following pictures does the question mark represent?





1 2 3 4 5

A. 1

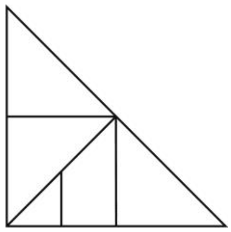
B. 2

C. 3

D. 4

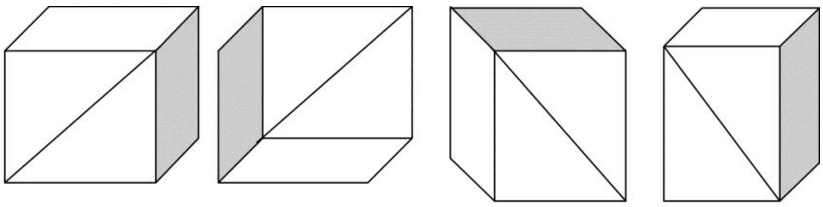
E. 5

12. How many triangles does the following figure contain?



A. 5
B. 7
C. 8
D. 9
E. 11

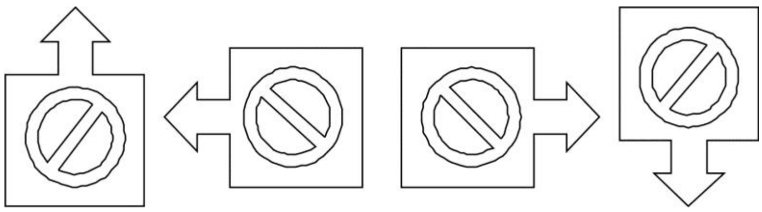
13. Which of the following figures is different from the others?



1 2 3 4

A. 1
B. 2
C. 3
D. 4
E. 1, 2, 3 and 4 are the same

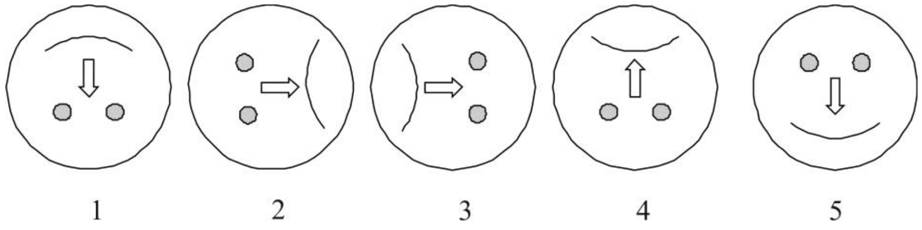
14. Which of the following figures is different from the others?



1 2 3 4

A. 1
B. 2
C. 3
D. 4
E. 1, 2, 3 and 4 are the same

15. Which pair of faces in the following five figures has the same pattern?.



A. 3 and 5
B. 1 and 2
C. 3 and 4
D. 2 and 4
E. None of them are the same

16. If 1 June is a Sunday, what day will it be 10 days later?

A. Monday
B. Tuesday
C. Wednesday
D. Thursday
E. Friday

Use the following information to answer questions 17 and 18.

KK Bookstore sells only mathematics books and music books. There are 100 students in the bookstore. Forty of the students buy only mathematics books; 20 buy only music books; and 10 buy no books.

17 How many students buy both mathematics and music books?

A. 10
B. 20
C. 30
D. 40
E. 60

18. How many students buy either mathematics books or music books, but not both?

A. 10
B. 20
C. 30
D. 40
E. 60

Use the following information to answer questions 19 and 20.

Eight motorcyclists (*M, N, O, P, Q, R, S* and *T*) took part in a 1500-cc speedway. Their positions were recorded after finishing a lap. The following statements describe their positions.

1. **No drivers occupied the same positions.**
2. ***M* was in front of *T*.**
3. ***Q* was first.**
4. ***S* was immediately behind *O*.**
5. **Both *P* and *S* were in front of *M*.**
6. **There was a car between *N* and *R*.**

19. Which of the following describes the ranking of the eight motorcyclists?

A. *Q O M P S N T R*

B. *Q O S P M N T R*

C. *Q O S M N P T R*

D. *Q O S T P N M R*

E. *Q P M N S T O R*

☐

20 If *S* came fourth and *R* came fifth, which of the following statements is correct?

A. *M* came second

B. *M* came seventh

C. *N* came third

D. *P* came third

E. *T* came eighth

☐

- End -