Some methods for composing mathematical problems

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ABSTRACT. The article sustains the idea that the mathematical educations should be performed as a continuous research and discovery, not just as a simple transmission of already known ideas.

An essential contribution to this activity would be the invention of new mathematical problems.
Here are some methods: paraphrasing, changing of statement's data, analogy, generalization, combination.

The contemporary school has a well determined role in developing of the huge intellectual potential, represented by intelligence and creativity, that, being capitalized properly, can provide an uninterrupted social-human progress. In the relation between the pupil and the learning process, knowing the level of intellectual advancement of each student is very important for the utilization of adequate methods, that would allow the individualization of the education, in order to maximize each pupil's creative capacities and abilities.

From the creativity point of view, it is necessary to revise the traditional teaching methods used in school, by assimilation of creative strategies, as well as by promoting new methods.

At the second Congress for the mathematical education (Ecster City, 1972), the famous French mathematician R.Tom considered that it was necessary to move to the "creative" heuristic method of teaching and learning. Also, many European mathematician scientists pleaded for a new concept of mathematical instruction (1982), according to which:

- Mathematics is to be considered as a learning activity for people and not a finished studying object;
- Mathematics is to be studied by making it interesting and not by imposed memorization;
- Mathematical instruction is to be performed as a process of continuous research and discovery and not as a simple conveyance of already known ideas.

The systematic solving of mathematical problems contributes to the conscientious assimilation of knowledge and, specifically to the accumulation by pupils of the experience of creative activities, and to the developing of creative skills.

On the correlation line between creativity and solving mathematical problems we were interested in a few aspects:

- Pupils' training in creative activities requires a system of cognitive problems of research nature;
- Problem solving represents a favorable frame for creativity development;
- Defining for creativity is the part of problems' composition (wording) and not that of their solving (G. Polya, 1965; A.N. Kolmogorov, 1970; P.C.Wason, J. Laird, 1986; J.T.Dillon, 1988 etc.).

For the composition of mathematical problems we should have general guidance and concrete methods that would guide our actions in the direction of achieving the wanted result.

The general methods for mathematical problems composing can be the following:

-Establish experimental statements, with as many numerical verifications as possible (probable truths);

- In geometry, create the auxiliary constructions that are deleted after formulating problem’s statements;

- Starting from a problem (theorem), formulate another problem (theorem) using elements of mathematical logic;

- Starting from a problem (theorem), formulate a new problem (theorem) using generalizations, analogies etc.

- Establishment of connections between different domains (combinations);

- The solving of a known problem, using another method.

The composing of mathematical problem activity should start with their controlled composing. Then, applying methods as paraphrasing, changing of statement's data, analogy, combination, that has an advanced degree of independence, we can obtain different original problems.

Controlled composing of mathematic problems. The achieving of new knowledge by self-instruction is a creative process. An essential help is this direction are the creative tasks and more specifically the controlled composition of mathematical problems. The later means the problems’ composing activity is initiated and unfolded under the permanent teacher's supervision. In this case the mechanism of composing problems for demonstration could be described in as follows:

- the selection of research objects and objectives (professor);
- the analysis of problem's situation (pupil);
- the obtaining of new information about the object (pupil);
- problem's statements wording based on the ascertained fact (pupil);
- distinguish the theme applied in problem's solving (pupil-professor);
- obtaining the solution of the formulated problem (pupil-professor);
- conclusions (pupil-professor).

The analysis of problem's situation depends on the selected objects and objectives, and it could be realized:

- based on calculus, constructions and measures (induction);
- based on deduction of logical consequences from the selected conditions (deduction);
- based on passing information from on object to another one (analogy, generalization).

We do not exclude the existence of other modalities.

The mechanism of controlled composing of demonstration problems determines the organizational technology of pupils' activity for tasks fulfillment which should contain a problem situation and its object of research.

Below we will present a concise description of the technique of demonstration problem's composing based on calculus, constructions and measures (the inductive method of obtaining knowledge).

The teacher presents to the pupils a task that contains objects and research objectives. Then every pupil executes the operations for the given objects, corresponding to the presented objectives (for geometry, for example, construct and measure). The obtained results are written in a general table whose analysis will permit the observation of the regularity and the formulation of an hypothesis. The next step is the formulation of the problem's statement, emphasizing the applied theme for solving the problem and finally problem's solution.

Paraphrasing. By paraphrasing we mean a reformulation of the problem. We distinguish the next tips of paraphrasing:

- the paraphrasing that doesn't change the essence of the problem;
- the paraphrasing that changes the problem partially;
- the paraphrasing by which we obtain a new problem;

The paraphrasing can be made by:

- reformulation of the problem under a different aspect;
- the change of a figure with another one (in geometry);
- reformulation of the problem in another domain of mathematics (or in another activity domain described in the statement);
- reformulation of the problem, applying elements of mathematical logic.

We don't exclude the existence of other methods.

Example (*the author is the known Russian mathematician – I.M. Ghelifand*). *A pawn is situated in a corner of a chess table. Two players move the pawn one by one (in the figure 1 these moves are drawn with arrows). The player that will place the pawn on the opposite field of the table wins. Who wins in a fair play?*

This is a reformulation of this problem in another kind of activity:

There're two clusters with seven stones each. Two players make consecutive moves: take a stone from a cluster, or take a stone from each cluster, or moves a stone from a cluster to another. The player that takes the last stone wins. Can the player that starts the game win?

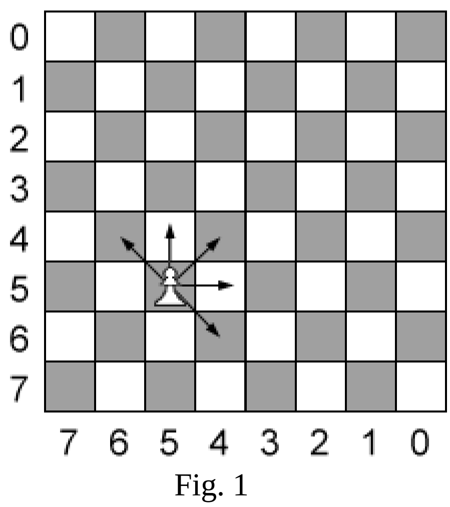

Fig. 1

In order to formulate problems using elements of mathematical logic we can apply the properties:

1) $(p_1 \wedge p_2 \wedge \ldots \wedge p_{i-1} \wedge p_i \wedge p_{i+1} \wedge \ldots \wedge p_n) \to q \equiv \left(p_1 \wedge p_2 \wedge \ldots \wedge p_{i-1} \wedge \left(\overline{q}\right) \wedge p_{i+1} \wedge \ldots \wedge p_n\right) \to \overline{p_i}$.

2) $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \to (q_1 \vee q_2) \equiv \left(p_1 \wedge p_2 \wedge \ldots \wedge p_n \wedge \overline{q_1}\right) \to q_2$.

3) $(p_1 \wedge \ldots \wedge p_i \wedge \ldots \wedge p_n) \to q \equiv (p_1 \wedge \ldots \wedge p_{i-1} \wedge p_{i+1} \wedge \ldots \wedge p_n) \to \left(q \vee \overline{p_i}\right)$.

Change of the data in the statement. Among the possible changes of problem's statement are:

- Replacing one notion with another;
- Replacing several notions with other ones;
- Replacing one relationship from the statement with another one;
- Adding new conditions;
- Changing numerical values of a constant;
- Combining some of the above mentioned methods.

We assume there are also other possibilities.

As a result, the problem can be changed partially or a new one can be obtained.

Replacing one definition with another one and replacing several definitions with other ones is made through concretization and specialization. Replacing one relationship from the statement, for example, an equality with an inequality is possible.

The addition of some restrictive conditions is made through particularization (concretization). The change of a numerical value of a constant can lead to generalization.

Example: *Let's consider a quadrilateral regular pyramid. We need to find its volume, if the height is equal to h, and the angle between a lateral edge and the basis plane is* a.

Replacing the term "quadrilateral" with "triangle" or "hexagon" etc.; the term "height" with "lateral edge" or "apothem" or "basis side" or "radius of the circumcised sphere" etc; angle a with angle b, or g, or d, we obtain a series of new problems (b – the dihedral angle between a side and the basis plane, g – the plane angle from the peak of the pyramid, d – the dihedral angle, whose edge is a lateral edge).

The analogy. Mathematics has its own particularities when applying analogy. These particularities differ from a compartment to other. Geometry offers a very vast field of activity in which new affirmations can be obtained through analogy. A simple example: the analogy between the geometry of the triangle and the geometry of the tetrahedron.

A simple mechanism of creation of geometrical affirmations (planimetry – stereometry) through analogy can be:

- Select an affirmation from planimetry;
- Make a replacement of notions after a well-determined scheme: point – straight line, straight line – plane, angle between two straights – dihedral angle between two planes, the length of the segment – the area of the polygonal surface, the area of the polygonal surface – the volume of the polyhedron etc.

- We settle the structure of the affirmation from the stereometry which is analogous with the selected affirmation;
- The obtained hypothesis can be confirmed or infirmed by reasoning;
- If possible, the hypothesis is completed until it becomes a true affirmation.

In algebra, an example of analogy for the theory of integer numbers divisibility is the theory of polynomial divisibility. It would be wrong if this analogy would not be taken into consideration by teachers when teaching this subject.

In algebra we can apply the following methods for composing new mathematical problems:

- We select a problem that has a solution that is obtained from important affirmations;
- If possible, we define an analogous affirmation to the affirmation from the solution;
- A new mathematical problem is composed using the defined affirmation.

The confirmation through reasoning of the solution of the new mathematical problem is evident because both affirmations are based on the same method of construction.

The mathematical analysis isn't an exception and contains many examples of analogies. For example: the limit of a function in a certain point and the differential of a function in a certain point. If we replace some expressions in remarkable limits by other expressions analogical to their limits, we can obtain interesting results. Applying also the combination we can obtain original mathematical problems. We can use the same method when calculating the indefinite integral by applying the first method of changing the variable.

It can be concluded that an analogy applied efficiently can lead to a considerable contribution in education.

Generalization. Some of the methods of generalization of a mathematical affirmation are:

- Replacement of numerical data from the problem's statement with parameters;
- Exclusion of some restrictions from the problem's statement;
- Application of the affirmation from the problem to a larger number of objects;
- Transfer of a geometrical property from one object to another;

Probably this is not a complete list.

The generalization starts from a thorough analysis of the demonstration in order to find out which conditions were essential. This method doesn't always lead to a result: it can happen that a demonstration doesn't suggest a generalization, but another demonstration, with another idea, more comprehensive, can suggest it. There are situations when a concrete example serves as a result of a demonstration of a general case.

For example: *The side AD is the diameter of the circumscribed circle to the convex quadrilateral ABCD. The point of intersection of the bisectors of angles B and C belongs to the segment AD. Show that AB + CD = AD.*

The condition that *AD* is the diameter of the circle can be omitted. A generalization of the problem:

ABCD is an inscriptible quadrilateral and the point of intersection of the bisectors of angles B and C belongs to the segment AD. Demonstrate that AB + CD = AD.

The comparison, the abstracting and the induction influence directly the value of the final generalization.

Combination. Combination leads to new mathematical problems.

This method can be realized by logical combination of:

- Two or more results from the same domain;
- Two or more results from different domains.

For example, combining two important results from the theory of symmetrical functions, the theory of algebraic equations and from planimetry (triangle), we can obtain identities, important and original inequalities for the sides of the triangle, for its heights etc. Interesting problems can be obtained by combining important results from geometry (planimetry or stereometry) and from the theory of inequalities.

A successful combination needs a hard work, a strong intuition and a deep knowledge of the theory.

A practical base for this method is the resolving of this kind of mathematical problems, paying attention to their structure and method of combination. In this case it is necessary to explain thoroughly the scheme of the studied problem.

The list of the methods of composing mathematical problems doesn’t end here. This activity also contains other methods.

Here is an example for the last method. Our object of study will be the symmetrical polynomials with three unknown variables and some sizes of a triangle: three sides, or three heights, or the radiuses of escribed circles etc. A polynomial with three unknown variables x,y,z is called symmetrical if its form doesn’t change at any permutation of its unknown variables. For example : $P(x,y,z)=(x-y)^2 + (y-z)^2 +(z-x)^2$ is a symmetrical polynomial, $Q(x,y,z)=x-y-z$ is not a symmetrical polynomial. The most simple symmetrical polynomials are considered $P_1= x+y+z$, $P_2= xy + yz + zx$, $P_3= xyz$. It can be shown that every symmetrical polynomial with the unknown variables x,y,z can be expressed using the polynomials P_1, P_2, P_3. For exemple : $x^2 + y^2 + z^2 =(P_1)^2- 2P_2$, $x^3 + y^3 + z^3 = (P_1)^3 – 3P_1P_2 + 3P_3$, $(x + y)(y + z)(z + x)=P_1P_2 – P_3$, etc.

The Vietes’ formulas for the third rank equations : if x_1,x_2,x_3 are the solutions for the third rank equation $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$ ($a_0 \neq 0$), then $x_1 + x_2 + x_3 = - a_1/a_0$, $x_1x_2 + x_2x_3 + x_3x_1 = a_2/a_0$, $x_1x_2x_3 = - a_3/a_0$.

Are given *a*, *b* and *c* the lengths of the sides of the triangle *ABC*. It can be proved that these values are the solutions of the equation:

$$x^3 – 2px^2 + (p^2 + r^2 + 4Rr)x – 4pRr=0.$$

Applying Vietes' formulas we obtain identities for values of a, b and c:

$a + b + c = 2p$, $ab + bc + ca = p^2 + r^2 + 4Rr$, $abc = 4pRr$. From the last relation we obtain a known formula for the area of the triangle : $S=(abc)/4R$. If we continue we obtain : $a^2 + b^2 + c^2 = 2(p^2 - r^2 - 4Rr)$, $a^3 + b^3 + c^3 = 2p(p^2 - 3r^2 - 6Rr)$, $(a + b)(b + c)(c + a) = 2p(p^2 + r^2 + 2Rr)$.

We combined the results from the theory of symmetrical polynomials (functions) with the results from the domain of algebraic equations and from planimetry.

The composition of mathematical problems has a considerable contribution in developing the creativity by developing such mental capacities as observation, comparison, description, analysis, synthesis, induction, deduction, analogy and such imagination proceeding as substitution, modification, adaptation etc. Once formed these capacities improve the divergent and the convergent way of thinking, of the abilities for mathematics and of the motivation for studying.

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