Beautiful Mathematics and Beautiful Instruction:
Aesthetics within the NCTM *Standards*

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Michael J. Bossé
Department of Mathematics and Science Education
College of Education
East Carolina University
Greenville, NC 27858
bossem@ecu.edu
(252) 328-9367

Michael J. Bossé has a PhD in Curriculum and Instruction from the University of Connecticut and is an Associate Professor of Mathematics Education in the Department of Mathematic and Science Education at East Carolina University in Greenville, NC, 27858, USA. His research interests include cognition, learning, pedagogy, distance education and instruction through technology in mathematics.

ABSTRACT

Today, research often considers the content and pedagogy associated with the NCTM *Principles and Standards for School Mathematics* (NCTM, 2000). However, philosophic analysis of NCTM's position remains only infrequently investigated. This paper investigates the *Principles and Standards* from an aesthetic perspective, asking the question, "What does NCTM believe to be 'Beautiful Mathematics?'" In order for teacher education faculty to fully promote NCTM's *Principles and Standards* to pre-service K-12 mathematics teachers, this aesthetic dimension should be considered. These considerations can bring an added perspective and richness to mathematics education that is infrequently considered.

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> Mathematics is one of the greatest cultural and intellectual achievements of human-kind, and citizens should develop an appreciation and understanding of that achievement, including its **aesthetic** and even recreational aspects. (NCTM, 2000, p. 4) [Bold Added]

Aesthetics is the study of beauty. Whether in art, nature, or music, debates abound regarding the beauty of human creation and the natural realm. All too often, however, discussions of beauty are unfortunately absent from more scientific fields. Within mathematics, we occasionally recognize a "clever" solution or an "elegant" proof. These are content-based, subjective judgments regarding aesthetic value. Within mathematics education we recognize well presented lessons. Although this assessment may include evaluations from the perspectives of mathematical content, epistemological soundness, and pedagogical value, aesthetics may also play some part.

It can be argued that every educational reform movement carries with it some assessment of beauty within its field. Mathematics educational reform efforts also continue to hold aesthetic values. Albeit overly simplified, the New Math Movement (1950s-1970s) perceived the beauty of mathematics to be found within mathematical structure, set theory and logic, the utilization of symbols, the focus on conceptualizing mathematics, and a move toward abstraction. Today, the NCTM *Principles and Standards for School Mathematics* (NCTM, 2000) presents its own interpretation of beautiful mathematics. This paper investigates and reports NCTM's opinions of aesthetics as it relates to K-12 mathematics education. This is accomplished through an analysis of the *Principles and Standards* as NCTM's foundational document representing its reform ideology.

It must be stated that NCTM's documents do not explicitly define beautiful mathematics. Thus, to construct such a definition from NCTM's documents it was necessary to first investigate passages which use words such as "beauty" and "aesthetics" and seek repeated connections to other descriptors or concepts. Herein, therefore, the definition provided is synthesized from passages and concepts which were initially, albeit tangentially, connected to the central concept of beauty. Furthermore, this investigation was deliberately restricted to considering only the *Principles and Standards*; other equally appropriate and revealing documents published by NCTM were not investigated. This was justified based upon the prominent role which the *Principles and Standards* play in mathematics education in the U.S. and throughout the world.

Beautiful Mathematics in History. A spark of ingenuity has long characterized the history of mathematics. Developing new mathematics, in general, and generating proofs of mathematical theorems or solving specific problems, in particular, often exhibit a level of creativity tantamount to more "artistic" fields. Although proofs of theorems and solutions to problems may be attempted in many ways, some attempts may be recognized as more aesthetic and elegant than others. The following example demonstrates two methods of solution for one problem. These two solutions demonstrate varying levels of aesthetic value and elegance. The first is rather conventional; the second, in part due to its simplicity, could be deemed more elegant.

This is a standard problem in introductory calculus. There are two towns located near a river on the same side (points A and B). A pumping station has to be situated on the river bank to pump water to both the towns (point P). The costs of running pipe lines

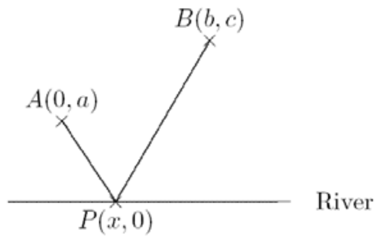

Figure 1

from the station to the towns are very high. Which is the best location for the pumping station on the river so the amount of pipe line is minimum?

As depicted in Figure 1, to solve this problem we can consider the river to be x-axis and the location of the towns as shown. If we let $f(x) = d(A, P) + d(P, B)$ represent the total length of pipeline, we can use calculus to find x which provides a minimum value for $f(x)$.

Another solution (Figure 2) can be developed with simple geometry. One can find the best position for the pumping station by considering the reflection of town B with respect to the river. The pumping station should be positioned at the intersection of the linesegment from A to B' and the river. The elegance of this solution resides in the simplicity of the mathematics through which it can be solved.

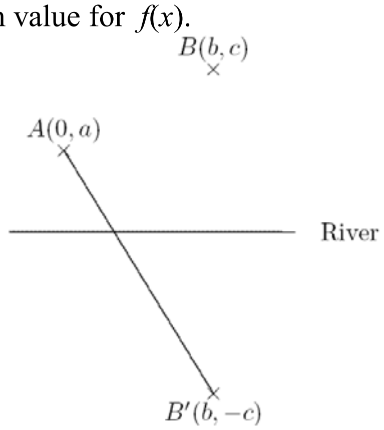

Figure 2

The intersection of mathematics in art and nature has been investigated through the millennia, with a resurgence in the past few decades. Many scholarly and entertaining articles, books and websites are now available to students. While many of these sources are interested in revealing mathematics in art (Coxeter et al, 1986; Field, 1997; Peterson, 2001; Schattschneider, 1987) and nature (Adam, 2003; Jacobs & Andersson, 1998; Stewart, 1997), others focus on using mathematics to create art (Bartashi, 1981; Boehm, & Prautzsch; 1994; Crannell, A. and M. Frantz, 2000; Kappraff, 1990).

Therefore, whether through the creativity of proofs and/or problem-solving or through the recognition of the beauty of mathematical connections in nature and art, discovering examples of beautiful mathematics may be more easily done than defining beautiful mathematics. To a significant extent, this can be credited to the subjective nature of recognizing and assessing beauty in any field. Nevertheless, the following sections will attempt to develop a definition of the beauty of mathematics from NCTM's perspective.

I – NCTM's Vision

In the *Principles and Standards* (2000), NCTM states its vision for school mathematics:

> Imagine a classroom, a school, or a school district where all students have access to **high-quality**, **engaging** mathematics instruction. ... The curriculum is **mathematically rich**, offering students opportunities to learn **important** mathematical concepts and procedures **with understanding**. ... Students confidently **engage** in **complex** mathematical tasks chosen carefully by teachers. They draw on knowledge from a **wide variety** of mathematical topics, sometimes approaching the same problem from **different mathematical perspectives** or representing the mathematics in different ways until they find methods that enable them to make progress. ... Students are flexible and resourceful **problem solvers**. ... They **value** mathematics and **engage actively** in learning it. (NCTM 2000, p. 3)[Bold added]

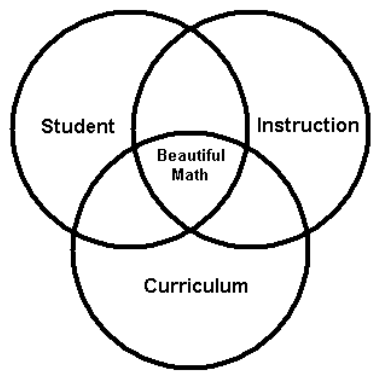

Figure 3

Although the highlighted terms and phrases begin to shed some light on NCTM's aesthetic beliefs, in isolation, this citation is far from definitive. Thus other citations within the *Principles and Standards* will be utilized to further explicate notions within the vision statement. Analysis will reveal that the *Principles and Standards* recognize three interconnected realms within the notion of beautiful mathematics: the student, the curriculum, and the instruction. These three realms are depicted in Figure 3. However, prior to addressing these three realms, it is necessary to demonstrate that NCTM recognizes beauty within mathematics.

II – Beautiful Mathematics: Introduction

NCTM believes that mathematics has its own intrinsic beauty. As the following citations demonstrate, the beauty of mathematics is in its structure, form, connection to the real world, and utility.

> Mathematics is one of humankind's greatest cultural achievements. It is the "language of science," providing a means by which the world around us can be represented and understood. The mathematical representations that high school students learn afford them the opportunity to understand the **power and beauty of mathematics** and equip them to use representations in their personal lives, in the workplace, and in further study. (NCTM 2000, p. 364)

> Many concepts and processes, such as symmetry and generalization, can help students gain insights into the **nature and beauty of mathematics**. (NCTM 2000, p. 15)
> Part of the **beauty of mathematics** is that when interesting things happen, it is usually for good reason. Mathematics students should understand this. (NCTM 2000, p. 56)

Admittedly, these characteristics of mathematics are far from a concrete definition for beautiful mathematics. Nevertheless, they lay the foundation for the argument that a definition, or at least a set of underlying assumptions about mathematical beauty, exists. The fact that a definition of mathematical beauty may be elusive, however, should not come as a surprise; defining beauty alone, or as part of any field of study, has for millennia often befuddled scholars. Two initial questions, therefore, must be investigated: What is beauty in mathematics? and What mathematics is most beautiful?

> When young students use the relationships in and among mathematical content and processes, they advance their knowledge of mathematics and extend their ability to apply concepts and skills more effectively. Understanding connections eliminates the barriers that separate the mathematics learned in school from the mathematics learned elsewhere. It helps students realize the beauty of mathematics and its function as a means of more clearly observing, representing, and interpreting the world around them. (NCTM 2000, p. 132)

Herein, again, hints can be found within the citation directing attention toward the threefold realms of students, curricula, and instruction. These will be addressed separately in the following sections.

Interpreting, Representing, and Interacting with the World. Fundamental in the *Principles and Standards* is the notion of the "mathematical representation" of the real world and real-world experiences. Both explicitly and implicitly, NCTM reasons that a foundational aspect of the nature of mathematics which makes it inherently beautiful is that it can be used to model real-world events and experiences. The representational power of mathematics resides in two realms. First, mathematics is a sense-making paradigm. Through mathematics, children and adults can analyze, organize, and make sense of the world. People can come to a greater understanding of the world by investigating phenomena mathematically. Second, mathematical models or representations can then be developed in order to further analyze phenomena and guide decision-making and planning. This twofold integration of mathematics and the real world gives mathematics power and utility. Summarily, NCTM perceives that the primary beauty in mathematics rests in its utility for mankind to observe, interpret, and modify the world in which we live. In short, mathematics is a tool through which mankind can interact with the natural and social environment. The importance of this perspective in defining beautiful mathematics through NCTM's eyes cannot be understated.

The remainder of this paper will report various characteristics of beautiful mathematics which are applicable to mathematics education. In so doing, the definition of beautiful mathematics will be further constructed on applicable findings within the *Principles and Standards* (2000).

III – Beautiful Mathematics: Student Perspective

> Students ... should have frequent experiences with problems that **interest, challenge, and engage** them in thinking about **important** mathematics. (NCTM 2000, p. 182)

Within the *Principles and Standards* certain adjectives are repeatedly employed to build a definition for beautiful mathematics in the realm of the student. Beautiful mathematics is characterized as important (pp. ix, 3, 4, 13, 14, 15, 66, etc.), significant (pp. 5, 76, etc.), interesting (pp. 14, 18-19, 53, 60, 80, 117, 211, 279, 341, etc.), engaging (pp. 18, 21, 80, 182, 205, etc.), challenging (pp. 12, 15, 18, 21, 53, 117, 182, 205, 211, 275, etc.), worthwhile (pp. 21, 53, 60, 200, 341, etc.) and connected to the real-world experiences and interests of students (pp. 13, 14, 19, 52, 73-75, 200, 205 227, 256, etc.).

The following quotation summarizes the definition of important mathematics within the *Principles and Standards*:

> An effective mathematics curriculum focuses on **important** mathematics—mathematics that will prepare students for continued study and for solving problems in a variety of school, home, and work settings. A well-articulated curriculum challenges students to learn increasingly more sophisticated mathematical ideas as they continue their studies. ...
>
> Mathematics topics can be considered important for different reasons, such as their utility in developing other mathematical ideas, in linking different areas of mathematics, or in deepening students' appreciation of mathematics as a discipline and as a human creation. Ideas may also merit curricular focus because they are useful in representing and solving problems within or outside mathematics. (NCTM 2000, pp. 14-15)

Comments. NCTM recognizes that student interests affect student learning. Students learn most effectively when students are engaged in challenging pursuits of important mathematical ideas and when they see that these mathematical pursuits have an impact upon their personal lives and future livelihoods. Thus, the individual abilities, learning styles, interests, and future needs of students should be considered in the process of selecting curricula and planning instructional materials. Even though this may seem like a daunting task for an educator, the epistemology of constructivism argues that students more efficiently construct knowledge when they are personally interested in the topic and involved in the problem-solving experience.

NCTM's historical documents demonstrate a significant evolution in their published position concerning tracking and individualizing instruction. Although the writers of the 1989-1995 NCTM reform documents stated that equity was a significant concern within education, and that varying student needs required different pedagogical techniques, some opined that the 1989 *Standards* fell short of dogmatically castigating the practice of tracking (Stake & Raizen, 1997). Therefore, NCTM took strides in the *Principles and Standards* (2000) to further articulate the need to individualize instruction for differing student needs and interests.

> *Principles and Standards* call for a common foundation of mathematics to be learned by all students. This approach, however, does not imply that all students are alike. Students exhibit different talents, abilities, achievements, needs, and interests in mathematics. (NCTM, 2000, p. 5)

> Educational equity is a core element of this vision. All students, regardless of their personal characteristics, backgrounds, or physical challenges, must have opportunities to study – and support to learn – mathematics. Equity does not mean that every student should receive identical instruction; instead, it demands that reasonable and appropriate accommodations be made as needed to promote access and attainment for all students. (NCTM, 2000, p. 12)

Without diminishing the advances made by the NCTM *Curriculum and Evaluation Standards* (1989), one could argue that the *Principles and Standards* (2000) are more student-centric and to a greater extent emphasize individuating the curriculum according to student needs and interests. One can readily notice the student-centric nature of the previously mentioned adjectives *important*, *significant*, and *worthwhile*. Unfortunately, due to the unique nature, perspectives, and interests of individual students, definitions for these characteristics may be equally as elusive as that of beauty. Furthermore, due to increased understanding and a more future-oriented perspective of professional educators, *important*, *significant*, and *worthwhile* mathematics may be interpreted somewhat differently by teachers than by students.

Interacting with the World. Students yearn to interact with their world. Their natural inquisitiveness connotes their innate desire to understand the world in which they live (NCTM 2000). Curiosity leads – albeit informally and intuitively – to investigation, hypothesis, experimentation, verification, understanding, and generalization. The sense-making characteristic of mathematics allows for mathematics to be a robust tool at a student's disposal by which to observe, interpret, and modify his or her world.

The ability to both interact with, and modify, the world and make generalizations which simplify complex notions and stimulate problem-solving can significantly motivate students to learn. The beauty and power of mathematics becomes personalized as students recognize that they can manipulate and make decisions about their world and future which would be less possible without mathematics. Mathematical understanding empowers students to take control of their world and lives. Through this perspective, mathematics can become viewed as a trusted ally. This relationship between student and mathematics is fundamental to NCTM's notion of beautiful mathematics.

IV. – Beautiful Mathematics: Curricular Perspective

> School mathematics experiences at all levels should include opportunities to learn about mathematics by working on problems arising in contexts outside of mathematics. These connections can be to other subject areas and disciplines as well as to students' daily lives. Prekindergarten through grade 2 students can learn about mathematics primarily through **connections with the real world**. Students in grades 3–5 should learn to apply **important mathematical ideas in other subject areas**. This set of ideas expands in grades 6–8, and in grades 9–12 students should be confidently using mathematics to explain **complex applications in the outside world**. (NCTM 2000, pp. 65-66)

Student-Centric: Curricular discussions within the *Principles and Standards* also repeatedly utilize adjectives in the development of beautiful mathematics curricula. Some of these curricular suggestions are student centric: interesting (pp. 14, 18-19, 53, 80, 117, 211, 341), engaging (pp. 18, 21, 80, 182, 205, etc.), challenging (pp. 12, 15, 18, 21, 53, 117, 182, 205, 211, 275, etc.), worthwhile (pp. 21, 53, 60, 200, 341) and connected to real-world experiences and interests of students (pp. 13, 14, 19, 52, 73-75, 200, 205 227, 256, etc.).

It is important to realize that an educator's opinion of what mathematics may be *interesting*, *engaging*, *challenging*, and *worthwhile* may differ significantly from the students' opinions of these characteristics. Nevertheless, it is incumbent upon the classroom teacher to try to synchronize these perspectives through curricular selection and classroom instruction.

Content-Centric: Other curricular recommendations are more oriented toward mathematical content: important (pp. ix, 3, 4, 14, 15, 66, etc.), significant (pp. 5, 52, 76, 275, 284, etc.), and interconnected to other areas of mathematics (pp. 14, 15, 17, 64, 73-75, 200, 212, 227, 256, 289, etc.). In addition to the characteristic of *important*, which has previously been discussed, within this list of curricular suggestions, *connected* appears most frequently.

Figure 4 diagrammatically represents the student-centric and content-centric components of the curricular perspective.

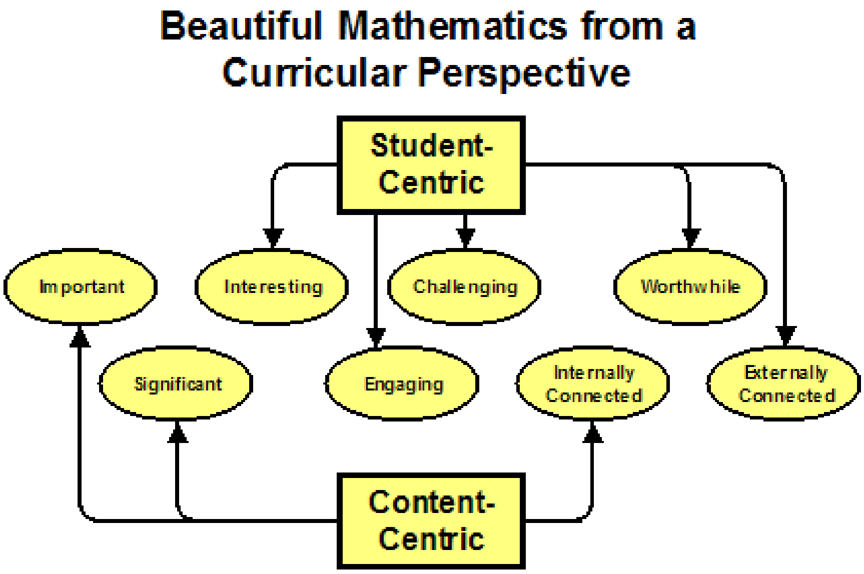

Figure 4

Comments. The *Principles and Standards* promotes the expansion of student mathematical understanding through connections both within and outside of mathematics. The focus upon connections between mathematical topics and topics from other fields and interests outside mathematics – such as science, engineering, economics, the arts, history, and social issues – was one of the foundational proposals within the NCTM *Curriculum and Evaluation Standards* (1989). Previously, one of the central characteristics of the New Math Movement of the 1950s-1970s was the focus upon making connections among concepts and topics within mathematics. The *Principles and Standards* combine the two perspectives of internal and external mathematical connections and advance the argument that beautiful mathematics is that which is linked to many topics both within and outside of mathematics (Bossé, 2003), thereby demonstrating the reach and utility of the study and application of mathematics and providing the understanding that mathematics could address and further motivate the individual interests of all students (pp. 212-213, 227, 277, 279, etc.).

> In a coherent curriculum, mathematical ideas are linked to and build on one another so that students' understanding and knowledge deepens and their ability to apply mathematics expands. (NCTM 2000, p. 14)
>
> Mathematics comprises different topical strands, such as algebra and geometry, but the strands are highly interconnected. The interconnections should be displayed prominently in the curriculum and in instructional materials and lessons. A coherent curriculum effectively organizes and integrates important mathematical ideas so that students can see how the ideas build on, or connect with, other ideas, thus enabling them to develop new understandings and skills. (NCTM 2000, p. 15)
>
> Problem solving in grades 6–8 should promote mathematical learning. Students can learn about, and deepen their understanding of, mathematical concepts by working through carefully selected problems that allow applications of mathematics to other contexts. Many interesting problems can be suggested by everyday experiences, such as reading literature or using cellular telephones, in-line skates, kites, and paper airplanes. (NCTM 2000, p. 256)

Through multiple vantage points, this focus upon connections is consistent with NCTM's epistemological position of constructivism. Internal connections among mathematical topics assist students to integrate multiple mathematical concepts into fewer connected ideas (Barnard & Tall, 1997; Bossé, 2003; Gray and Tall, 1994; Thurston, 1990). In respect to the cognitive domain of learning, these connected ideas are more readily constructed by students into knowledge. External mathematical connections among mathematical concepts and extra-mathematical subjects can also significantly assist students to construct knowledge, as these external investigations can consider topics in which students have significant personal interest. Hence, the affective domain of student learning can also be addressed through connections. Thus, both internal and external mathematical connections assist students to construct knowledge and learn.

Added to the previously mentioned characteristics, the notion of important mathematics also carries a conceptualization of economic rewards. Important mathematics for K-12 education should assist the student in becoming autodidactic and be able to pursue personal interests. Additionally, important mathematics should assist students in finding and succeeding in meaningful future employment which leads to the students becoming productive members within society.

> Our students deserve and need the best mathematics education possible, one that enables them to fulfill personal ambitions and career goals in an ever-changing world. ... In this changing world, those who understand and can do mathematics will have significantly enhanced opportunities and options for shaping their futures. Mathematical competence opens doors to productive futures. A lack of mathematical competence keeps those doors closed. (NCTM 2000, pp. 4-5)

One could argue that providing students with curricula which are *interesting*, *engaging*, *challenging*, *worthwhile*, *connected to the real-world experiences and interests of students*, *important*, *significant*, and *interconnected to other areas of mathematics* is a daunting task for most K-12 mathematics educators. This correlates with NCTM's call for teachers to have mathematical and curricular understanding which has both depth and comprehensiveness and which recognizes internal and external mathematical connections, important concepts of mathematics and applications of mathematics in the real world.

> Teachers need several different kinds of mathematical knowledge—knowledge about the whole domain; deep, flexible knowledge about curriculum goals and about the important

> ideas that are central to their grade level; knowledge about the challenges students are likely to encounter in learning these ideas; knowledge about how the ideas can be represented to teach them effectively; and knowledge about how students' understanding can be assessed. ... Teachers need to understand the big ideas of mathematics and be able to represent mathematics as a coherent and connected enterprise. ... This kind of knowledge is beyond what most teachers experience in standard preservice mathematics courses in the United States. ... Teachers also need to understand the different representations of an idea, the relative strengths and weaknesses of each, and how they are related to one another. (NCTM, 2000, p. 17)

Figure 5 represents the commonalities between the student and curricular perspectives. The extent of the depicted commonalities between student and curricular perspectives should come as no surprise. Indeed NCTM should be commended for the extensive correlation between these two perspectives.

Interaction with the World. Previously it was mentioned that the beauty of mathematics resides in its role as an empowering agent by which students can interpret and interact with their world. Additionally, mathematics, through the eyes of reform effort since the 1980s (CFEE, 1986; MSEB, 1990; NCEE, 1983; NCTM, 1989; NRC, 1989) and currently in the *Principles and Standards*, has the power to bring equity to a nation which is fractured and segregated between the mathematically capable and the mathematically illiterate. The NCTM *Curriculum and Evaluation Standards* (1989) states:

> Mathematics has become a critical filter for employment and full participation in our society. We cannot afford to have the majority of our population mathematically illiterate: Equity has become an economic necessity. (p. 4)

This line of reasoning is repeated in the NCTM *Principles and Standards* (2000).

> In this changing world, those who understand and can do mathematics will have significantly enhanced opportunities and options for shaping their futures. Mathematical competence opens doors to productive futures. A lack of mathematical competence keeps those doors closed. ... *Principles and Standards* calls for a common foundation of mathematics to be learned by all students. ... A society in which only a few have the mathematical knowledge needed to fill crucial economic, political, and scientific roles is not consistent with the values of a just democratic system or its economic needs. (p. 5)

This ability to mitigate historic social and occupational inequities introduces another realm of beauty to mathematics. Mathematics curricula can assist students to become socially adjusted and culturally functioning. Thus, as mathematics knowledge is a tool by which individuals can understand and affect their world, so also is it a more global mechanism by which equity can grow within a culture. Individual mathematical understanding of all students leads to strengthening equity among all within a society. Therefore, the beauty of mathematics resides dually in its empowerment of an individual and its potential for developing equity within a culture or nation.

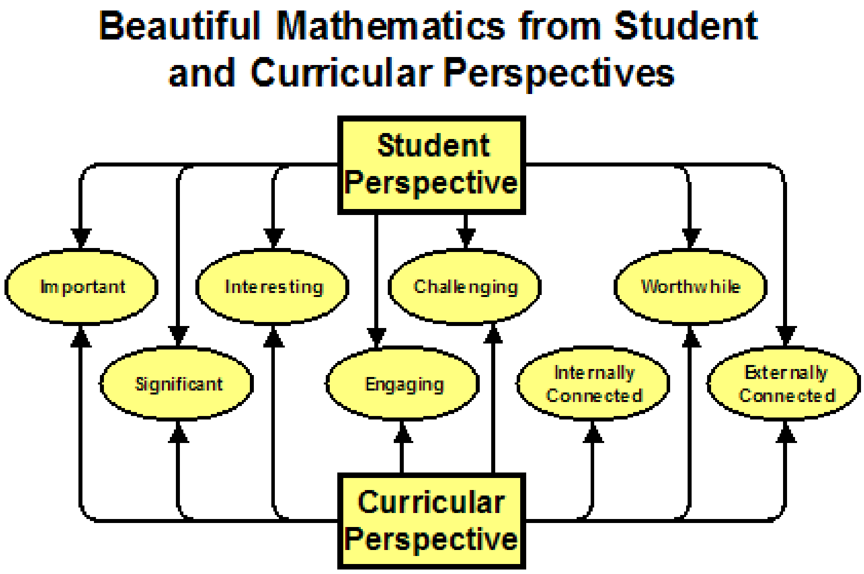

Figure 5

V. – Beautiful Mathematics: Instructional Perspective

> To be effective, teachers must know and understand deeply the mathematics they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks. They need to understand and be committed to their students as learners of mathematics and as human beings and be skillful in choosing from and using a variety of pedagogical and assessment strategies (National Commission on Teaching and America's Future 1996). In addition, effective teaching requires reflection and continual efforts to seek improvement. Teachers must have frequent and ample opportunities and resources to enhance and refresh their knowledge. (NCTM, 2000, p. 17)

> Teaching mathematics well involves creating, enriching, maintaining, and adapting instruction to move toward mathematical goals, capture and sustain interest, and engage students in building mathematical understanding. (NCTM 2000, p. 18)

Some clues regarding NCTM's interpretation of proper instruction have already been mentioned in previous sections. However, in the realm of instruction, additional characteristics of beautiful mathematics are provided as well as responsibilities for the teacher. Figure 6 depicts many of the characteristics in the *Principles and Standards* associated with beautiful mathematics from the instructional perspective.

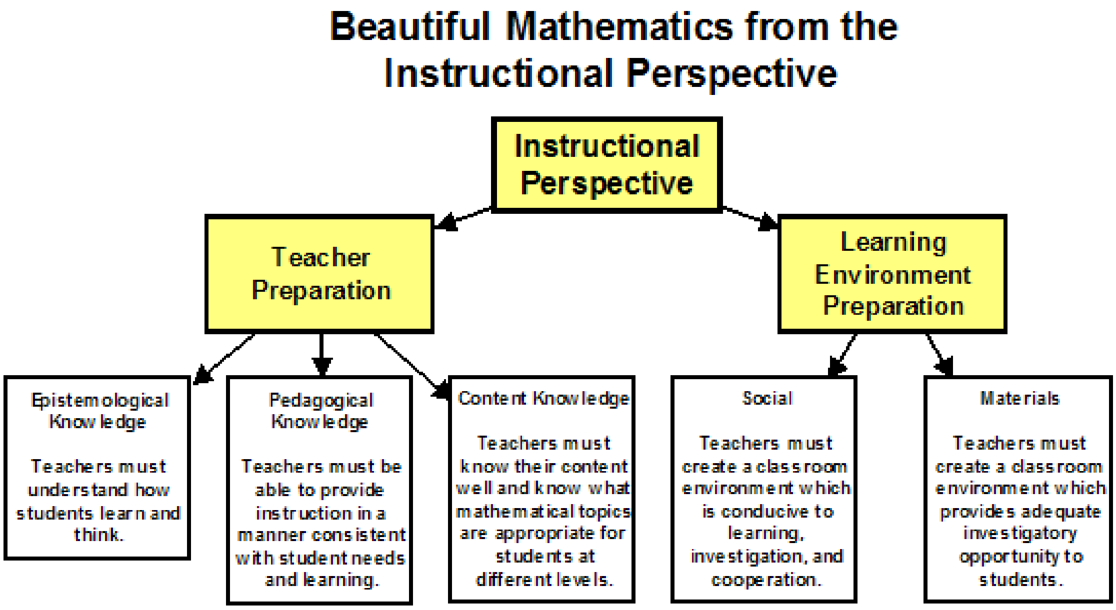

Figure 6

Within the realm of instruction, beautiful mathematics must be beautifully and professionally taught. Educators call for teachers to grow in pedagogical content knowledge (Bromme, R., 1995; Carpenter, T. P., Fennema, E., Peterson, P. L. & Carey, D. A., 1988). This can be understood as the bridging of rich mathematical content with epistemologically sound instructional strategies. Content knowledge alone is insufficient for teacher preparation; mathematical content cannot be divorced from sound pedagogy. Central to the characteristics associated with beautiful mathematics is the notion of adequate professional teacher preparation within the realms of mathematical content, epistemological understanding, pedagogical knowledge and application, and understanding of learning environments. These are addressed independently in this section.

Content Understanding: NCTM promotes the notion that beautiful instruction is prefaced and supported by a teacher's strong content knowledge.

> Teachers need ... mathematical knowledge ... about the whole domain; deep, flexible knowledge about curriculum goals and about the important ideas that are central to their grade level. ... Teachers need to understand the big ideas of mathematics and be able to represent mathematics as a coherent and connected enterprise. ... Such understanding might be characterized as "profound understanding of fundamental mathematics" (Ma 1999). Teachers also need to understand the different representations of an idea, the relative strengths and weaknesses of each, and how they are related to one another (Wilson, Shulman, and Richert 1987). (NCTM, 2000, p. 17)

A teacher's mathematical knowledge should be both deep and wide. TIMSS (1994-1995), the *Third International Mathematics and Science Study*, revealed that mathematics education in the U.S. is fraught with repetition of topics (Schmidt, et al, 1997 Schmidt, McKnight, and Raisen, 1996), and is better recognized for its breadth rather than its depth of topical study (NCES, unpublished 1998; Schmidt, McKnight, and Raisen, 1996; Stigler and Hiebert, 1997). To overcome these weaknesses, interconnections both within mathematics and between mathematics and extra-mathematical topics should be understood. Multiple representations of mathematical concepts should be well known by teachers. However, a thorough knowledge of mathematics is not, in and of itself, adequate for teaching. Teachers must be able to assess *meaningful* and *important* mathematics – meaningful to students and important for its building and connecting of concepts.

Epistemological Knowledge: The *Principles and Standards* repeatedly make epistemological statements which lead to curricular considerations. Throughout the document, NCTM states that inquisitiveness and a desire to learn through play is innate within children.

> Mathematics learning builds on the curiosity and enthusiasm of children and grows naturally from their experiences. Mathematics at this age, if appropriately connected to a child's world, is more than "getting ready" for school or accelerating them into elementary arithmetic. ... Adults can foster children's mathematical development by providing environments rich in language, where thinking is encouraged, uniqueness is valued, and exploration is supported. Play is children's work. Adults support young children's diligence and mathematical development when they direct attention to the mathematics children use in their play, challenge them to solve problems, and encourage their persistence.
>
> Children learn through exploring their world; thus, interests and everyday activities are natural vehicles for developing mathematical thinking. (NCTM 2000, pp. 73-74)

It is imperative that teachers have a solid understanding of how children learn and how they connect and construct mathematical concepts. NCTM believes that epistemological knowledge is necessary for pedagogical planning and curricular modification.

> Effective mathematics teaching requires a serious commitment to the development of students' understanding of mathematics. Because students learn by connecting new ideas to prior knowledge, teachers must understand what their students already know. Effective teachers know how to ask questions and plan lessons that reveal students' prior knowledge; they can then design experiences and lessons that respond to, and build on, this knowledge. (NCTM, 2000, p. 18).

Pedagogical Knowledge: Teachers must be knowledgeable in effective pedagogical techniques. These instructional methodologies must enter the classroom and be competently implemented.

> Teaching mathematics well is a complex endeavor, and there are no easy recipes for helping all students learn or for helping all teachers become effective. Nevertheless, much is known about effective mathematics teaching, and this knowledge should guide professional judgment and activity. (NCTM, 2000, p. 17).

> Teachers have different styles and strategies for helping students learn particular mathematical ideas, and there is no one "right way" to teach. However, effective teachers recognize that the decisions they make shape students' mathematical dispositions and can create rich settings for learning. Selecting and using suitable curricular materials, using appropriate instructional tools and techniques, and engaging in reflective practice and continuous self-improvement are actions good teachers take every day.
>
> One of the complexities of mathematics teaching is that it must balance purposeful, planned classroom lessons with the ongoing decision making that inevitably occurs as teachers and students encounter unanticipated discoveries or difficulties that lead them into uncharted territory. Teaching mathematics well involves creating, enriching, maintaining, and adapting instruction to move toward mathematical goals, capture and sustain interest, and engage students in building mathematical understanding. (NCTM, 2000, p. 18)

The *Principles and Standards* propose that teacher pedagogical knowledge also be deep and broad. It is inadequate for teachers to simply possess pedagogical knowledge to apply in the planning of lessons; well prepared teachers should possess an instantaneous flexibility to implement, evaluate, and modify instructional practices within the classroom.

Learning Environment: The *Principles and Standards* delineates some roles and responsibilities of K-12 mathematics teachers including creating an effective learning environment (pp. 16-17, 73-74, etc.) Two primary components are considered within an effective learning environment: a social environment conducive for learning and an environment with adequate curricular and material resources to promote curiosity and inspire investigation.

> Students learn mathematics through the experiences that teachers provide. Thus, students' understanding of mathematics, their ability to use it to solve problems, and their confidence in, and disposition toward, mathematics are all shaped by the teaching they encounter in school. The improvement of mathematics education for all students requires effective mathematics teaching in all classrooms. (NCTM, 2000, pp. 16-17)

The positive and negative effects of the learning environment cannot be overstated. A positive learning environment requires that the mathematical and teaching disposition presented by the teacher leads to a disposition of learning among the students. Surrounding the classroom with entertaining and interesting mathematical representation, discussions, and bulletin boards is only the beginning. Developing a mathematically positive attitudinal learning environment is also necessary.

> Teachers establish and nurture an environment conducive to learning mathematics through the decisions they make, the conversations they orchestrate, and the physical setting they create. Teachers' actions are what encourage students to think, question, solve problems, and discuss their ideas, strategies, and solutions. The teacher is responsible for creating an intellectual environment where serious mathematical thinking is the norm. More than just a physical setting with desks, bulletin boards, and posters, the classroom environment communicates subtle messages about what is valued in learning and doing mathematics. Are students' discussion and collaboration encouraged? Are students expected to justify their thinking? If students are to learn to make conjectures, experiment with various approaches to solving problems, construct mathematical arguments and respond to others' arguments, then creating an environment that fosters these kinds of activities is essential. (NCTM, 2000, p. 18)

Problem-Solving: The primary mathematical characteristic emphasized within teaching is interconnected with NCTM's focus upon student learning through problem-solving. As previously mentioned, the mathematics investigated by students should be worthwhile. Therefore, both instructional tasks and problem-solving scenarios should be worthwhile endeavors.

> The teacher's role in choosing worthwhile problems and mathematical tasks is crucial. By analyzing and adapting a problem, anticipating the mathematical ideas that can be brought out by working on the problem, and anticipating students' questions, teachers can decide if particular problems will help to further their mathematical goals for the class. There are many, many problems that are interesting and fun but that may not lead to the development of the mathematical ideas that are important for a class at a particular time. Choosing problems wisely, and using and adapting problems from instructional materials, is a difficult part of teaching mathematics. (NCTM, 2000, p. 53)

Comments: NCTM's focus on problem-solving does not exist for its own sake. Problem-solving is inextricably conjoined with constructivism, NCTM's primary epistemological position. Students construct knowledge as they involve and invest themselves in a meaningful problem-solving experience. Therefore, the development and assessment of "worthwhile problems and mathematical tasks" is founded upon a teacher's extensive knowledge of important mathematical content, epistemological factors associated with their respective students, and proven pedagogical practices.

> In effective teaching, worthwhile mathematical tasks are used to introduce important mathematical ideas and to engage and challenge students intellectually. Well-chosen tasks can pique students' curiosity and draw them into mathematics. The tasks may be connected to the real-world experiences of students, or they may arise in contexts that are purely mathematical. Regardless of the context, worthwhile tasks should be intriguing, with a level of challenge that invites speculation and hard work. Such tasks often can be approached in more than one way, such as using an arithmetic counting approach, drawing a geometric diagram and enumerating possibilities, or using algebraic equations, which makes the tasks accessible to students with varied prior knowledge and experience. (NCTM, 2000. pp. 18-19).

In addition to a focus on problem-solving, however, NCTM recognizes beautiful mathematical instruction as that which is content rich and valuable, epistemologically grounded, pedagogically sound, and provided within an environment conducive for learning. Therefore, adequate teacher preparation is paramount to providing quality instruction to students.

Interaction with the World. Central to the student and curricular perspectives of the beauty of mathematics was the notion that mathematics empowered students to interact with their world and empowered cultures toward equity. Beautiful mathematics instruction introduces the student to mathematical experiences which will affect and enhance his or her life. According to NCTM, neither mathematics nor mathematics instruction are static and impersonal. Beautiful mathematics instruction infuses important mathematics into the lives of students in a relevant and interesting way. Beautiful instruction changes students into mathematical participants with the world. Mathematical instruction culminates in students being able to make various representational mathematical models of real world phenomena.

Conclusion

When student, curricular, and instructional perspectives are all adequately addressed, beautiful mathematics culminates into a beautiful mathematics education program. The beauty of mathematics lies in its nature to allow people to interpret, model, analyze, and interact with the world, leading to the ability to conjecture, plan, execute, and evaluate scenarios. This paper has demonstrated that NCTM indeed has specific aesthetic beliefs regarding mathematics and mathematics education. In order for teacher education faculty to fully promote NCTM's *Principles and Standards* to pre-service K-12 mathematics teachers, this aesthetic dimension should be considered. These considerations can bring an added perspective and richness to mathematics education that is infrequently considered.

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