Translations Among Mathematical Representations: Teacher Beliefs and Practices

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Abstract

Student ability, teacher expectations, respective degrees of difficulty, and curriculum and instructional practices all work together to provide students experiences leading to differing levels of success in respect to mathematical translations. Herein, we discuss teacher beliefs and instructional practices, investigate why some translations seem to be more difficult than others, and provide instructional recommendations to assist students and teachers with mathematical translations.
Transitions Among Mathematical Representations: Teacher Beliefs and Practices

Through formal and informal observations of high school Algebra and Pre-Calculus teachers and students, possible links of causality have been observed to explain students’ difficulties in translating among mathematical representations (verbal, symbolic (or algebraic), tabular (or numeric) and graphical). Affecting students’ success are dimensions including: teachers’ expectations of students being able to perform these translations; what teachers directly instruct and model in the classroom regarding translations; the complexity of some of these translations; and the level of the mathematics depicted in some of the representations. These dimensions and roles as causal factors in students translating from one representation to another are discussed in this paper.

Background on Representations and Translations

Representations are external objects (tables, graphs, words, or symbols) whose conventions of denoting conceptual relationships (Cobb, Yackel & Wood, 1992) serve as instruments for coding and describing mathematical relations, communicating mathematical knowledge, and operating with mathematical objects (Zhang, 1997). Most educators and researchers agree that key to understanding, communicating, and effectively operating on mathematical concepts is connected to performing translations among these graphical, tabular, symbolic and verbal representations.

Translation is a process in which constructs of one mathematical representation are mapped onto those of another (Janvier, 1987) (e.g., the relation expressed in a table reinterpreted using algebraic symbols). Though there is general agreement among mathematics educators and researchers on the importance of translations in mathematical comprehension, there is
considerable evidence that students struggle to accurately translate among verbal, tabular, graphical and algebraic representations of mathematical relations (Gagatsis & Shiakalli, 2004; Galbraith & Haines, 2000; Porzio, 1999; Wollman, 1983).

Moreover, although translation difficulties have been extensively documented in the general research literature, much of this research emphasizes whether or not students could complete the translations (e.g., Knuth, 2000) and not the various errors which arise and the timing of such in the process of performing the translation. (See Adu-Gyamfi, Stiff, & Bossé, in press, 2012, for a fuller examination of interpretation, implementation, and preservation errors.) Moreover, fewer studies have focused on the influences of classroom instruction in mitigating these difficulties (e.g., Cunningham, 2005). Additionally, although Janvier (1987) recognizes some translations as direct and some as indirect, others (Duval, 2006; Kaput, 1987; and Leinhardt, Zaslavsky, & Stein, 1990) consider some translations actions to be local and others global, and others (e.g., Clement, 1982) have noted that translations to and from verbal representations are among the most complex, few have analyzed the level of difficulty of each translation as have Bossé, Adu-Gyamfi, and Cheetham (2011). Notably, in this present study, teachers were unaware of much of this research and simply employed their own classroom experiences by which to evaluate which translations students found more difficult.

Although numerous studies have investigated the effects of teacher beliefs on instructional practices and student performance (e.g., Ball, 1996; Beswick, 2005, 2007; Handal & Herrington, 2003; McLeod & McLeod, 2002), little research exists which precisely explore teachers’ beliefs on which translations are more difficult for students. While some authors argue that teacher beliefs and practices are inconsistent (e.g., Cooney, 1985 & Shield, 1999), many more claim significant consistency between teacher beliefs and practices (e.g., Stipek, Givvin,
Salmon, & MacGyvers, 2001; Thompson, 1984). In this study, we interviewed teachers to discern their beliefs regarding which translations students find most difficult and compared these with the teacher’s classroom practices in respect to instructional experiences involving translations. Since teacher assumptions about student understanding can ultimately affect the type of instructional translation opportunities afforded to students – thus ultimately impacting in which translations students are successful, we argue that such studies are needed. However, left uninvestigated in this study is whether or not teacher beliefs affected student success in performing translations.

**Methodology**

Throughout this discussion, translations between representations are coded such that: verbal = V, symbolic = S, tabular = T, and graphical = G, and, for instance, the translation from a graphical to a symbolic representation is denoted G→S. This study investigates student success in translating between mathematical representations through the eyes of the classroom teacher (and not through objective external assessment).

Participants in this study included high school Algebra II (N = 15) and Pre-Calculus (N=15) teachers from a school district in eastern North Carolina. The teaching experience of these teachers ranged from three to twenty-five years, with a relatively even distribution among these experience levels.

To glean teacher beliefs in this area, two facets were used. First, participating teachers completed three copies of the template in Figure 1 on which to provide their opinion on different dimensions (each form of the template had a different title to direct participant responses). In one template, teachers were to weigh how well students generally performed each pair of
translations (V→S, V→T, V→G, S→V, S→T, S→G, T→V, T→S, T→G, G→V, G→S, and G→T). On a second blank template, teachers were to weigh their expectations of students’ abilities to perform each pair of translations. On a third blank template, teachers were to weigh the degree to which they provided direct instruction to their students on precisely how to perform each respective translation.

Figure 1. Blank template.

Teachers were directed to write adjectives or adverbs on each link to respond to the question regarding each translation. They were neither directed as to which terms to use nor provided any additional guidance regarding how to weigh the links. Thus, there was significant inconsistency among the terms used by the teachers. Figure 2 provides an example of one teacher’s responses to the templates.

Figure 2. One teacher’s responses.
During the completion of the templates, teachers were observed as they made comments regarding their selection of terms to complete the templates. Simultaneously, teachers participated in semi-structured interviews regarding their beliefs. While the interviews were not used in the analysis of the completed templates, they were employed to better understand teacher thinking. Synthesizing the completed templates with the recorded observations from the interviews created a more complete picture of the teachers’ thoughts, rationales, and actions.

Each completed template was then analyzed by each individual researcher and then corporately by all three researchers to ensure consistency among the coding, interpretation, and analysis of the templates. Then the composite results were synthesized into one figure for each line of inquiry (provided in following discussions as Figures, 3, 4, and 5). In addition to the completion of the templates, participating teachers were each observed for three instructional class periods to determine their actual (rather than perceived) teaching practices in respect to translations. Each lesson was observed by one of the researchers and records were kept regarding which mathematical translations were observed during the instruction. Additionally, teaching practices were investigated through curricular materials (i.e., lesson plans, assignments, homework, quizzes, and tests) to determine frequency of translation examples which the respective students would experience. These observations were then synthesized with the template denoting the teachers’ instructional practices to modify the teachers’ responses to more accurately depict actual classroom practices.

While discrepancies existed between teachers’ perceptions of the emphases they placed on particular translations and what was recorded through direct observation of classroom instruction and curricular materials, this was not the focus of this research and was not further pursued. Rather, the qualitative and quantitative data from the instructional observations
together with the templates completed by the teachers indicating what they believed that they did, were synthesize to paint a more complete picture of both desired and actual classroom activities.

Summarily, apart from simple numeric tallies of observed instructional practices and curricular materials, all analysis within this study was qualitative in nature. No statistical methodologies were used to study the data.

Notably, a factor which must be considered in this study is that most of the data regarding student success and expected ability to perform translations is based on teacher opinions and beliefs and not on empirical, quantitative evidence. In that classroom teachers are informed and observant practitioners with daily, intimate experience in classrooms and unparalleled expertise in instructing the topics of this study to the students appropriate for this study, teachers’ opinions and beliefs were considered valuable, representative, and informative data for this investigation.

Altogether, the compiled data and analysis provided an interesting picture which may shed some light on elements of causality regarding students’ difficulties in performing some mathematical translations. Herein we investigate some of these dimensions and make some recommendations regarding how to mitigate these.

**Student Ability, Teacher Expectations, and Instructional Practice**

The analysis of the data for this investigation reveals a number of issues regarding student ability with translations from one mathematical representation to another, teacher expectations in respect to whether students should be able to perform particular translations, and instructional practices used by teachers in respect to these translations. We begin by looking at each of these independently.
An initial global finding is immediately interesting. No significant differences were found in instructional practices, curricular materials, of teacher opinions between teachers with more or less years of classroom experience. While this in and of itself may be a significant finding which warrants future investigation, it was not the focus of this study as was not pursued further.

**Student Ability with Translations**

Teachers rated general student ability to perform translation on a blank template. Taking all teacher responses together, the researchers compiled a general template depicted in Figure 3. These results indicate that secondary Algebra II and Pre-Calculus teachers recognize that students generally performed some translations better than others. This agrees with a number of research studies (e.g., Duval, 2006; Knuth, 2000). In Figure 3, heavy, solid arrows indicate teacher recognition that students generally performed some translations relatively well (V→T, S→G, S→T, T→G, and G→T). Lighter, solid arrows denote teacher recognition that students perform some translations with less consistent success (V→S, T→S, and G→S). The dashed arrow depicts that students could generally perform the V→G translation, albeit, teachers regularly qualified such by stating that students began with the verbal representation, used a transitional representation (usually a table), and finally derived an appropriate graph. This is discussed in more detail in following sections.

The dotted arrows in Figure 3 designate that teachers believe that students’ ability to perform some of these translations (S→V, T→V, and G→V) is dependent upon the definitions of these translations. If these translations mean precisely that students could produce a verbal representation fully commensurate with the relationships connoted in a table, graph, or symbolic
expressions, teachers felt that students were generally quite incapable of such. However, if these translations can be interpreted as students verbally (orally or in writing) reporting some of the features or relationships represented in the table, graph, or symbolic expression, teachers found that students could generally perform some of these translations. For instance, given the graph of \( y = x^2 + x - 6 \), some students are able to make statements to the effect that “the graph crosses the \( x \)-axis at (-3) and 2”, “the graph crosses the \( y \)-axis at 6”, and “the graph is a parabola.” While these statements indicate that students can decipher a graph and discuss it verbally, it is far from constructing a verbal representation fully containing independent and dependent relationships which would be consistent with the graph of \( y = x^2 + x - 6 \).

![Student Ability in Transformations diagram](image)

**Figure 2.** Student ability.

It is again important to put these findings in context. It is not stated herein that students actually perform better on some representations than others based on some objective measure. Rather, teachers in this study simply believe that students are more successful at performing some translations over others. Interestingly, however, the opinions of the teachers do mirror both studies which look at student success with these translations (e.g., Cunningham, 2005) and studies which look more specifically at the difficulty of each translation (e.g., Bossé *et al.*, 2011).
**Teacher Expectations**

Teacher expectations regarding which translations students should be able to correctly do is bifurcated on two dimensions. Some teachers respond in respect to their perception of student ability and others respond in respect to what they believe that national, state, and local curricular guidelines indicate that students should be able to do. In fact, most teachers modified the latter in respect to the former to provide their final results. Thus, rather than considering curriculum standards as sacrosanct, teachers tend to use their classroom experience with students to modify what curriculum standards state students should be able to do. Although much consistency is immediately recognizable between the results in teacher perceptions of student ability and teacher expectations of what students should be able to do in respect to translations, the fact that teachers looked at both their students and curriculum standards to set their expectations speaks to why the researchers saw it necessary to investigate both dimensions. The similarities and differences between student ability and teacher expectations is further investigated in following discussions.

Figure 4 indicates teacher expectations regarding students’ ability to be able to perform particular transformations. Heavy, solid arrows indicate teacher recognition that students generally should be able to perform some translations relatively well (V→T, V→S, S→G, S→T, T→G, and G→T). Lighter, solid arrows denote teacher recognition that students should be able to perform some translations with less consistent success (T→S and G→S). The dashed arrow depicts that students are expected to be able to perform the V→G translation, albeit by using a transitional representation (usually a table).
As in the discussion regarding Figure 3, the dotted arrows in Figure 4 designate that teachers’ expectations that students are able to perform some of these translations (S→V, T→V, and G→V) is again dependent upon the definitions of such. In respect to students translating a table, graph, or symbolic representation into a verbal representation fully commensurate with the independent and dependent variable denoted in the source representation, teachers firmly expect that students are generally unable to do so. This agrees with a number of research studies (e.g., Koedinger, 2004; MacGregor & Price, 1999). However, if these translations are interpreted more simply as students verbally (orally or in writing) reporting some of the features or relationships represented in the table, graph, or symbolic expression, teachers expect that students should generally perform some of these translations.

The comparison of teacher beliefs of which translations students are more or less able to do with teacher expectations of the same introduces some interesting dimensions. First, the commonality among the two causes one to wonder if teachers do not simply set their expectations of student performance based on the classroom experiences they have with students and not upon what curricular standards, professional development, and research which may
Instructional Practices

Figure 5 indicates teacher instructional practices in respect to translations among representations. As mentioned in the research methodology, this figure represents teacher completion of the opinion template mitigated by direct observation of their actual classroom practices. Heavy, solid arrows indicate that teachers emphasize these translations in their instructional practices (S→G, S→T, T→G, and G→T). Lighter, solid arrows denote that teachers provide some but significantly fewer examples and experiences of these translations in their instruction (V→S, V→T, T→S, and G→S). The dotted arrows designate that teachers rarely developed examples and experiences regarding these translations in their regular instructional practices (S→V, T→V, and G→V) apart from students verbally defining what they observe in a table, graph, or symbolic expression. The findings of the frequency of teachers covering particular translations in instruction agree with a number of research studies (e.g., Cunningham, 2005; Porzio, 1999).

![Instructional Practices with Transformations](image)

Figure 5. Teacher practices.
The most notable contrast between student ability, teacher expectations, and instructional practices in respect to translations resides in the fact that while teachers often have students experience the $V\rightarrow G$ translation through instruction, they both recognize that students are regularly unable to perform this translation and have lower expectations regarding student success. Interviews with the teachers indicate that, despite poor student performance and low teacher expectations regarding such, teachers were willing to place greater instructional emphasis on the $V\rightarrow G$ translation for two distinct reasons. First, they were hopeful that students would eventually gain facility with this translation if they regularly encountered such in the classroom. Second, teachers claimed that they commonly recognized the $V\rightarrow G$ translation in state curricular guidelines, assessments, and assessment preparation materials and felt the need to emphasize this translation in order to best prepare students for state exams.

On the contrary, however, teachers tended to avoid instructional experiences with translations which they both found students have the least ability to perform and had the lowest expectation for such. So certain were teachers of the seeming impossibility of students consistently performing well on some translations ($S\rightarrow V$, $T\rightarrow V$, and $G\rightarrow V$, and to a lesser extent $G\rightarrow S$), that they infrequently addressed such through instructional experiences. As expressed in teacher interviews as they completed the research templates, hopelessness seemed to deter any possibilities that instructional experiences could help students overcome these deficiencies. This agrees with numerous studies which recognize that students have most difficulties with translations which involve verbal representations (e.g., Adu-Gyamfi, Stiff, & Bossé, 2012; Bossé et al, 2011; Clement, 1982; Roth & Bowmen, 2001).

Causal Effects?
Cursory, and possibly overly trivialized, evaluation of these findings may produce a simple path of causal relationships. As depicted in Figure 6, teachers’ experiences with instruction, curriculum, standards, and students lead to teachers’ expectations of what translations students should be able to regularly and correctly perform. These expectations then lead to teachers’ planning and implementation of instruction which, in turn, lead to student learning experiences. The greater frequency of learning experiences with particular translations then leads to increased student success with these translations.

This simple model of causality, however, must be mitigated by additional dimensions and factors. First, despite the fact that teachers’ expectations regarding students’ ability to perform particular translations is born from, and may be validated by, professional expertise with a myriad of students in secondary mathematics classrooms, these expectations that students will have difficulty or be unable to do some translations may be unavoidably transmitted to students in the classroom. Thus, students struggling with some translations may in part be simply living up to the expectations which they perceive.

Second, teacher expectations of student abilities may affect what teachers do in the classroom. While some teachers in this study argue that it is nonsensical to instruct students in
mathematical techniques which they will not be able to perform, others argue that students should see and experience mathematics beyond what they are required to do in the classroom, even if these experiences are limited. Thus, despite the supporting argument, teacher expectations naturally cause teachers to voluntarily limit instructional practices regarding some of the translations they anticipate as most difficult or of lesser importance. The question then arises, apart from a long history of classroom experience, why teacher perceive some translations as more difficult than others. This dimension is addressed in the following discussion.

Finally, Figure 6 is not validated through any research outside of this study. It is simply what was recognized among the teacher participants through curricular and instructional observation, completion of the research templates, and the semi-structured interviews. Thus, this study cannot state with any reliability that this cycle would not be more complex (with the potential of some bidirectional links) in some other environments.

**Student and Teacher Perception of Difficulty of Translations**

In comments from teachers within this study and other research (e.g., Bossé *et al*, 2011; Duval, 2006; Janvier, 1987; Kaput, 1987; Leinhardt *et al*, 1990), some translations between representations are recognized as having different characteristics than others. These are depicted in Figure 7. First, the thick, solid, black arrows (V→T, S→T, T→G, and G→T) denote that these translations are performed directly from one representation to the other. This set of translations, interestingly, correlates to the set of translations with which teachers reported that students least struggle.
Second, while the thinner, solid, black arrows (T→S and G→S) represent translations with which students find more problematic, these too are generally regarded as direct translations from one representation to the other. Notably, while less often than those previously mentioned, students have some level of success with these translations.

Third, three translations (V→G, S→G, and V→S) have been found to be different from the previous five translations in that they generally involve indirect translations (Janvier, 1987) from one representation, through an intermediary (or transitional) representation (Bossé et al., 2011), to the final representation. Notably, these three translations typically use the tabular representation in the process. Thus, V→G = V→T→G; V→S = V→T→S; and S→G = S→T→G. Recognizing the nature of these multistep translations both reveals why students may have more difficulty with such and informs teachers that different strategies might be necessary for instructing such and leading students to success in these translations.

Fourth, a number of additional dimensions arise in respect to the translations represented with the dotted arrows (G→V, T→V, and S→V). As previously discussed, the success at which students perform these translations is dependent upon the definition of this translation. Since the
verbal restatement of features and characteristics of tables, graphs, and mathematical symbolization is the trivial case of $G\rightarrow V$, $T\rightarrow V$, and $S\rightarrow V$ translations, we will focus upon these translations when they are intended to produce a verbal representation (word problem, verbal example, or full explanation) which captures and depicts the salient components of the initial representations.

Students seem to only be able to master $G\rightarrow V$, $T\rightarrow V$, and $S\rightarrow V$ translations when they are in the context of simple linear functions ($y = ax + b$, where $a$ and $b$ are integers) (see also Adu-Gyamfi et al., 2012). When the functions depicted in the graph, table, or equations are of higher degree polynomials or transcendental functions, students are understandably incapable of translating those situations to a word problem or verbal example which correctly captures the same, precise function. This may speak to the complexity of verbally expressing non-linear relationships more so than students’ ability to perform the translation. Interestingly, however, students have some success verbally representing scenarios which encompass various exponential relationships.

**Teaching Strategies**

While students are exposed to symbols, graphs, and verbal patterns throughout elementary, middle, and high school, most students see this as disconnected, scattered information (Selden & Selden, 1992). Beyond mere exposure to these representations, students need to be guided in translating among representations. This requires that teachers provide students with sufficient instructional and assessment opportunities involving representational translations with increased dedication to translations which are most problematic. The following dimensions and recommendations are provided to assist teachers in emphasizing the translation of representation
within their mathematical classrooms. Many of these recommendations were provided from the teachers who participated in this study and project. Notably, since teachers were communicating with, and presumed a level of mathematical and instructional knowledge of, teacher educators, the recommendations they made were generalized and not accompanied by specific examples.

1. Teachers should come to understand their own beliefs regarding which translation students typically can do, cannot do, and should be able to do and must recognize that these beliefs may impact their instructional plans and decision-making.

2. Real or perceived, teachers should recognize which translations are more difficult for students than others. For instance, some translations are considered easier (V→T, S→G, S→T, T→G, and G→T), some more difficult (V→S, T→S, and G→S), and some most difficult (S→V, T→V, and G→V)

3. Teachers should pay attention to the order in which representations (and therefore translations) are presented to students through instruction and teachers should vary this order to ensure that all translations are seen, experienced, and learned – particularly those which are seemingly more problematic. For instance, teachers should ensure that students experience both V→T and T→V translations.

4. Teachers should consider discourse in their classrooms as representations are being utilized (Lloyd & Wilson, 1998). Teacher questioning techniques play an important part in helping students to balance their use of representations and translations and demonstrate the varied appearance of a particular concept within an equation, table, and graph.

5. Teachers should pay attention to the types of assessments given to students. If all assessments are traditional in nature, emphasizing translations which teachers believe that students can perform, then students will value and develop skills with these translations at
the expense of other equally important translations. Assessments should contain situations which force students to apply other translations.

6. Teachers may consider instructional techniques which: make stories about graphs; match tables or domain and range to graphs; or make a hypothetical table to represent a sketch of a graph.

7. Teachers can use real world contexts with which students are familiar or curious to help students bridge representations, create interest, and give purpose to translations.

8. Teachers can select a rich problem scenario to study over multiple days, providing students a common point of reference which can be considered through all representations and can be further affected by slight modifications in the problem scenario.

**Conclusions**

Student ability, teacher expectations, respective degrees of difficulty, and curriculum and instructional practices all work together to provide students experiences leading to differing levels of success in respect to mathematical translations. Altogether, strong associations exist between the translations which teachers regularly cover in the classroom and those with which students are most successful and vice versa. Additionally, some translations appear to be more complex and difficult than others. However, instructional strategies can enhance student experiences with translations among representations. It is hoped that this paper explains some of these dimensions from the perspective of the high school mathematics teacher, provides some useable instructional strategies, and provides encouragement to continue research and instructional practices in respect to translating mathematical representations.
References


