Understanding how students translate between mathematical representations is of both practical and theoretical importance. Educators and researchers recognize the importance of translations in mathematical comprehension and problem solving success. Unfortunately, some studies have reported only a limited understanding of the exact nature of student abilities with the translation process (Kaput, 1989; Knuth, 2000); the intricacies of this process remain mysterious to both students and teachers (Ainsworth & Van Labeke, 2004; Duval, 1999). Kaput (1987a) summarizes that the research literature provides neither a complete picture of the nature of student activities when translating from one representation to another, nor a coherent account of the intricacies of the translation process itself. In an effort to address student translation difficulties, some have argued the need for research that focuses on translation skills and best practices for teaching (Clement et al., 1981; Janvier, 1987).

The study in this paper takes steps toward achieving such an understanding by analyzing student translations between representations commonly found in instructional situations (i.e., graphical and symbolic representations). The specific purpose of this study is to better understand how students process translations. Duval (2006) argues that, in order to be genuinely able to see the nature of student abilities and difficulties in translations, one must set up a
mechanism of observation that lets actions taken in the translation process manifest. This study attempts to fulfill the characteristics proposed by Duval by systematically examining student processes during translation. In this sense, this study extends research on translations (Knuth, 2000; Schoenfeld, Smith & Arcavi, 1993) by addressing the nature of student processes in translations. This present research was inspired by the question: Do students of different ability levels from the same classroom process translations differently? The answer to this question seemed to be a significant gap in the literature in the field.

This report is presented through a number of stages, beginning with a novel theoretical underpinning constructed upon relevant literature, initial findings, and then some unanticipated additional findings that were both consistent with and complementary to this framework are presented. Finally, to provide explanatory power to some of these additional findings, a second, more complete, theoretical framework delineating the steps involved in the translation process is detailed. Within this framework, additional theoretical constructs are developed and provided. While it is uncommon for new theoretical constructs to be developed late in a research report, their development and presentation connects with the study’s final findings, discussions, and implications.

THEORETICAL FRAMEWORK

The historic and growing body of research in mathematics education focused on representations in general and translations in particular informs this study (e.g., Janvier, 1987; Roth & Bowen, 2001; Duval, 2006). This previous work is rich with robust and extensive frameworks delineating representation interpretation and translation processes. The current study builds upon a number of these models reporting the seminal components (those which are necessary and sufficient to demonstrate consistency among models) from frameworks that are foundational to this study.

The extant literature reveals different interpretations for the term representation dependent on particular perspectives. From the perspective of possible uses, representations are vehicles, artifacts, objects, or devices for making sense of situations (e.g., Olson & Campbell, 1993). The perspective of mode of production leads to the definition of representations as either external or internal objects that stand for something else (e.g., Goldin & Shteingold, 2001). From the perspective of signs and their complex operations, representations are a symbol system (i.e., a set of symbols and rules for combining them) with their field of reference (e.g., Duval, 2004; Kaput, 1987b; Kaput, 1989; Zhang, 1997).

Herein, the term representation is used solely in reference to external objects (e.g., tables, graphs, symbols, or words) which serve as a symbol system for coding, describing mathematical relations or ideas, communicating mathematical knowledge, and operating with mathematical constructs, whose relationship with the mathematical object or concept they signify is connected to fields of reference established by the mathematics community (Cobb, Yackel & Wood, 1992; Kaput, 1987a). Although different authors employ somewhat different nomenclature for such, it is commonly recognized that each representation has associated with it character combinations and syntactic rules or conventions that allow for the depiction and transformation of mathematical relations (Kaput, 1987b; Sims-Knight & Kaput, 1983). For example, Ainsworth (1999) describes format and operators of a representation; Kaput (1987b) considers symbol scheme and its concretely realizable set of characters and rules for identifying and combining them; Zhang (1997) discusses a symbolic coding encapsulating and describing mathematical relations, knowledge, and operations; Goldin (1987) defines primitive elements, characters, or
signs with their collection of permitted configurations; and Duval (1999, 2006) considers signs
and their rules of association. Significant commonalities exist among these frameworks. For the
purpose of simplicity and consistency, these various descriptors for characters, their
configurations, syntactic rules, and conventions associated with a representation are herein
collapsed into a single working term, *micro-concepts*.

We argue that all mathematical representations have uniquely associated micro-concepts
(i.e., character combinations and conventions) that must be interacted with in order to access
encoded relationships or transform mathematical ideas. We contend that, in a translation from
one representation to another, a translator must not only recognize through interacting with
micro-concepts (available in the source) encoded cognitions and relations but must also be able
to relate a subset of those relations to a distinct subset of micro-concepts associated with the
target representation; this is explicated in depth in the subsequent section.

**Dissecting the Translation Process**

Herein, the term *translation* is used in reference to a process in which cognitions and relations of
one mathematical representation (source) are successfully reformulated into a targeted
representation. Different researchers have articulated different meanings for the term translation.
For example, Janvier (1987) describes a translation as a process involved in moving from one
mode of representation to another, and Roth and Bowen (2001) describe a translation as a
movement that requires an interpretation into a different modality of representation. Duval
(2006), arguing from the perspective of registers of semiotic representations, uses the term
*conversion* (in place of translation) and defines this as the process used to change representation
register without changing the objects being denoted. Herein, the term *translation* is used in
reference to a process in which the well-formed constructs or ideas of one mathematical
representation (source) are successfully reformulated into a targeted representation, using
structures and characteristics appropriate to the latter.

Many researchers have identified mechanisms through which cognitions and relations are
articulated across source and target representations in the course of a translation (e.g., Clement,
1982; Duval, 2006; Kaput, 1987b; Knuth, 2000; McGregor & Stacey, 1993; Sims-Knight &
Kaput, 1983). For example, Kaput (1987b) describes: reading and encoding. Lesh, Post and
Behr (1987) delineate: analysis of the source representation; synthesis of meanings of the source
representation; target representation formulation; transfer of the content of the source
representation into the target representation; and restructuring of target representation. Sternberg
(1984) describes: selective encoding, selective combination, and selective comparison. The
dense lists of researchers cited and framework components indicate significant consistency
among these models.

In order to better address the question of how students process translations, we delineate
common themes among these different articulated mechanisms and dissect the translation
process thematically into four novel sub-processes: unpacking the source, preliminary
coordination, constructing the target, and determining equivalence. These four sub-processes
compose the theoretical lens or framework through which student processes in translations are
observed. Using this lens, the question of how students (of different ability levels) process
translations is reframed into a question of how do students (of different ability groupings) unpack
the source, perform preliminary coordination, construct the target, and determine equivalence.
The subsequent paragraphs briefly elaborate on each of these processes.
Unpacking the source: Many agree that representations are associated with characters, configurations, syntactic, and semantic rules. Von Glasersfeld (1987) argues that a representation is simply a set of meaningless characters to the uninitiated and only take on their intended meaning through learning their conventions. Thus, each mathematical representation contains a set of densely packed micro-concepts which a learner must unpack in order to access the broader mathematical idea encoded by the representation. In any representation, Duval (2006) opines that students must distinguish those features that are mathematically relevant. Thus, the process of unpacking the source is prominently identified in the literature. For instance, Lesh et al. (1987) discuss the analysis of the source representation and the synthesis of meanings of the source representation; Kaput (1987b) articulates reading the representation; and Sternberg (1984) considers selective encoding (sifting out relevant information from irrelevant information in order to select information for further processing).

Preliminary coordination: In a translation, one must identify and articulate the same mathematical relations between representations with different characters, configurations, and syntactic rules (micro-concepts). Duval (2006) opines that, in order for students to be able to identify the same object denoted in two different representations, there is a need to progress beyond networking of facts between the representations. During preliminary coordination, the mathematical relations unpacked from the source are initially amorphously conceptualized in light of the target representation. Mechanisms identified in the literature as associated with the process of preliminary coordination unify and expand upon the notions of: target representation formulation and transfer of the content of the source representation into the target representation (Lesh et al., 1987); selective encoding, selective combination (combining selected information in such a way as to render it interpretable), and selective comparison (Sternberg, 1984); and reading and encoding (Kaput, 1987b). Notably, through this process the student evolves from networking ideas to making associations between representations (Duval, 2006).

Constructing the target: Commonalities exist among processes defined within the literature regarding constructing a target representation, including: latter aspect of target representation formulation, transfer of the content of the source representation into the target representation, and restructuring of target representation (Lesh et al., 1987); selective combination (Sternberg, 1984); and encoding (Kaput, 1987b). Duval (2006) argues that, in order for a student to construct the target representation, he must be able to discern the concepts in the source and target representations which can be associated. Ideas recognized across both source and target representations and micro-concepts from the source are used to construct the target representation through appropriate micro-concepts.

Determining equivalence: Others define aspects of this process as restructuring of target representation (Lesh et al., 1987) and selective comparison (rendering newly encoded or combined information meaningful by perceiving its relations to old information previously stored) (Sternberg, 1984). Determining representational equivalence requires considering ideas that are shared between the two representations.

Framework Summary
Integrating associated constructs from previously recognized theoretical frameworks into a novel framework, which recognizes the translation process as an amalgam of four sub-processes, allows for more detailed observation and interpretation of student activity regarding performing translations from one mathematical representation to another. Without wishing to diminish the importance of distinctions among former respected theories, this study’s framework possibly
creates greater coherence among other theories and potentially sheds additional observational and interpretive light on student processes regarding transformations. Thus, this framework is used as a lens through which to observe student communication and work and interpret such. This framework was selected for this study because it was successfully employed in parallel studies performed by the authors and was found to be observationally insightful regarding student activities and understanding in respect to mathematical translations. It is through this framework that the research question is investigated: Do students of different ability levels from the same classroom process translations in the same manner or differently? The following discussion details the research methodology employed to address this question.

METHOD

In order to examine how students of different ability levels process translations, a collective case study design (Stake, 2000) was employed. In general, case studies are used to explore the interpretative and subjective dimensions of a phenomenon (Strauss & Corbin, 1990). Although the researcher’s comprehension of student thinking and understanding is never complete or perfect, this strategy of inquiry offers a viable means through which to make inferences about cognitive and translation processes of students of different ability groupings. In this study, a total of twenty-four students were observed and interviewed in order to develop a combined case for students in each of three ability levels. A constant comparison method was used to focus data collection and analysis, and interpretative case studies on each of the three ability levels were developed using data collected. This method allowed researchers to dissect the data into discrete events and code them into categories born from student behavior and verbal articulations and from those that the researchers deemed most significant and revelatory. Additionally, a cross-case analysis was undertaken to illuminate patterns across individuals within each ability level and across ability levels.

The Participants and Task

Participating students were purposefully selected based on ability level in their respective classrooms: high (n=8), medium (n=8), and low (n=8) ability levels (Strauss & Corbin, 1990). Students, ages 15 to 17 (with no correlation between age and ability level), volunteered from three high school pre-calculus classes taught by the same teacher in a school in the southeastern part of the United States. The researchers had no role in the selection of student participants or the placement of students into a particular ability level grouping. The selection and grouping of students was entirely in the hands of the classroom teacher who, with fifteen years of professional classroom experience, had the most complete knowledge of student ability based on nearly two months’ worth of observation of student classroom activity and recorded performance. The classroom teacher was given no guidance regarding the meaning of high, medium, and low “ability levels.” The researchers relied on the knowledge and experience of the teacher to intuitively understanding these characteristics and to be able to make this selection and determination. Students were unaware of grouping by ability levels and each ability level group had no knowledge that there were other ability level groups being investigated.

Polynomial functions and graphical/algebraic representations were chosen as the content and representations of focus for the task because participants had already investigated and interacted with this concept and these representations in their pre-calculus classes. Notably, all student participants had experienced identical content and instructional practices; they had merely performed at different levels through preceding assessments. The specific tasks were
used because they addressed three of the basic processes identified in Krutetskii’s (1976) model of mathematical abilities (i.e., reversibility, flexibility and generalizability) and served as a means through which student mental processes could be elicited and the nature of their conception characterized. Moreover, the task was solvable through numerous heuristics and required a translation from one representation to another. Additionally, in the context of the classroom experiences and instruction, the classroom teacher of these students considered the task both challenging to most students and doable by all.

The totality of the task included a sequence of three items of differing levels of sophistication. Students were allowed to use the TI-83 graphing calculators they commonly used in the class, and they were only allowed to proceed to the next item after they believed that they had successfully completed the previous item. On the first item, students were to investigate a given graph of a polynomial function and asked to write a polynomial function that would equate to such (Figure 1). This item was chosen because it required that students reverse their thinking (Reversibility) about polynomials and factoring in a direction counter to what they typically experienced during instruction. Expecting that some students would have difficulty with this task, the researchers anticipated that students would most struggle with: (a) should the factors involving $a$ and $b$ be $(x-a)(x-b)$ or $(x+a)(x+b)$? and (b) should exponents on the linear factors be even or odd natural numbers? Since this item assessed student understanding and connections of numerous mathematical concepts in respect to polynomial functions in factored forms and the graphs of polynomial functions, it was considered the most difficult task posed.

The following is a truncated graph of a polynomial. (It is one connected graph in which only the behavior near the x-axis appears.) There is no scale for the y-axis. Write the equation of a polynomial function which would produce this truncated graph.

![Graph of a polynomial function](image)

**Figure 1.** Reversibility item utilized on the task

In Item 2 (Figure 2), students were given five more opportunities to write polynomial equations in factored form which would produce the same x-axis behavior as seen in the graph. This item was chosen because it required that students be able to solve a problem in more than one way and to be able to understand more than one solution (Flexibility). Researchers expected that fewer students would have difficulty with this task, since it primarily involved only one concept: whether the respective odd or even exponent could be any odd or even natural number. Simultaneously, however, the researchers recognized that rarely would a student have been asked such a question through traditional instruction.

The third task (Figure 3) required that students formulate a general algebraic rule for all polynomial functions that would model the behavior of the given graph. This item was chosen because it required that students generalize from specific cases and make deductions from given or known facts (Generalizability). Since this task required little more than inductive reasoning regarding the odd or even nature of the exponents, researchers expected that this would be the easiest task. Again, however, the researchers recognized that rarely would a student have been asked such a question through traditional instruction.
After students were selected and proper requisite permissions granted, each student was observed and videotaped while he or she individually completed given items on a translation task in a task-based interview (Goldin, 2000). Students were given as much time as they needed to complete the first task before being asked to progress to the second or third task. Students were asked to verbalize their thoughts as they worked. When either silence was protracted or students were going in directions unanticipated by the researchers, the researchers reminded the students, “Tell me what you are thinking.” concurrently. Over the course of one week, these sessions lasted between 60 to 90 minutes and were held after school hours in the respective classrooms of these students. Sessions were scheduled in such a manner to ensure that students did not communicate with others regarding the tasks or interviews.

The limitations and delimitations of this study lead to caution regarding generalizability to broader audiences. First, any study with only a limited number of students must be cautious regarding implications beyond its numbers. Second, findings from this study are from a particular class from one particular teacher; this makes application to other audiences tenuous. Third, in this study, a single, three-part task was used to discern whether students of different ability levels performed translations in different ways. While, herein, this proved to be so, it is unclear whether this would be the case for all tasks a student may encounter. Fourth, the use of the TI-83 graphing calculators rather than mathematical software may have some limiting factor in respect to generalizability of findings to audiences using different technologies or no technology at all.

**Data Sources and Analysis**

Data sources for this investigation consisted of: written work and audio and video recordings of individual student interviews. Immediately following each interview, each researcher listened to the audio-tape, watched the video tape, and reviewed the student’s written work to complete a response summary based on the processes outlined in the framework. This summary was organized according to items and accommodated multiple cases. After each interview session,
all three researchers met to compare interview notes. Useful probes were identified and unusual student responses were discussed.

To get a general sense of the information in the data and to reflect on its overall meaning (Creswell, 2003), a process of systematic searching and arrangement of data began (Bogden & Biklen, 2003). Audio and video tapes were transcribed and copies of student work were merged with each transcript. Student transcripts were then independently analyzed to determine strategies they employed, reasoning underlying their strategies, and their success rate on given items of the task. Common themes in reasoning and solution strategies employed by the different ability groupings were characterized and labeled on the basis of patterns discerned in the response summary.

These coding structures were compared and differences reconciled; these processes resulted in the refinement of initial codes. Researchers were able to employ the process of check-coding (Miles & Huberman, 1994) and thereby reach consensus on the analysis of all transcripts. The reconciliation enabled the researchers to clarify their thinking and to sharpen each code’s definition. Student activities were investigated through the lens of the framework and the sub-processes: unpacking the source, preliminary coordination, constructing the target, and determining equivalence. Narrative summaries, including illustrative excerpts from the transcripts of each ability group, described processes students used and difficulties they encountered on the items of the task. These summaries were developed and validated by the researchers against transcripts and student work. Trends across ability groupings were identified and typical and discrepant student responses were described. Descriptive summaries were generated to provide an overall account of student processes and strategies within each ability group and a coherent picture of findings across ability groups. In the next section, we provide the results of our analysis.

RESULTS

The purpose of this study is to elaborate the different ways in which students of different mathematical ability levels translate mathematical representations, and, in particular, how students translate graphical properties into algebraic formulas. The main question is if students of different ability groupings process translations differently. This question is addressed through a theoretical lens or framework, which distinguishes processes of unpacking, preliminary coordination, target construction, and equivalence determination. In respect to students in the different ability groups, Table 1 illustrates the frequency of the framework component and the categories used such as never, rarely, seldom, sometimes, often, or always correctly performed. Unfortunately, the amount of data that would be required to fully provide exemplars for the results in Table 1 would overwhelm this report. Details regarding the specific nature of the understanding, skills, procedural errors, and misunderstandings associated with various ability level groups is reported in the following sections regarding individual groups.

<table>
<thead>
<tr>
<th>Translation</th>
<th>Ability Group</th>
<th>Low: n=8</th>
<th>Medium: n=8</th>
<th>High: n=8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unpacking the Source</td>
<td>Seldom (n=2)</td>
<td></td>
<td></td>
<td>Always</td>
</tr>
<tr>
<td>Preliminary Coordination</td>
<td>Rarely (n=1)</td>
<td>Often (n=6)</td>
<td>Always</td>
<td></td>
</tr>
<tr>
<td>Constructing the Target</td>
<td>Never</td>
<td>Often (n=6)</td>
<td>Always</td>
<td></td>
</tr>
<tr>
<td>Determining Equivalence</td>
<td>Never</td>
<td>Sometimes (n=4)</td>
<td>Always</td>
<td></td>
</tr>
</tbody>
</table>

Table 1. Differentiating student translation activities by ability.
Summarily, using the previously defined theoretical framework, this study found (1) that the higher ability level students were more capable of correctly performing the given mathematical translations and lower ability level students were generally less capable of such and (2) that students at different levels performed or attempted these translations using dissimilar techniques. The results of the analysis are explicated in the subsequent sections. This is initially done in a case-by-case approach according to ability groupings (i.e., low ability, medium ability, and high ability) and then globally according to themes found in the cross-case comparison.

**Findings within Groups**

Through the lens of this study’s framework, the following sections consider and interpret student activity in the translation process in respect to student ability levels. Some of the following findings are compatible with previous research studies and other findings extend upon and complement the findings of others.

In order to examine and explain the activities and understanding of students in each the ability groups, quotes from students (culled from student transcripts) are presented followed by the articulation of the prevalent processes employed by each ability level group. The source of each quote is denoted with a letter and a numeral; for instance, *Student L1, Student M2*, and *Student H3* identifies a student from the low, medium, and high ability levels, respectively. Notably, the transcripts provided are only a small sample of the recorded communication and behavioral observation of the students in this study, and those provided are used as exemplars of student activity and thought. While variations in heuristics and understanding can be observed among the transcripts associated to each individual within each ability level group, the researchers observed sufficient similarity within each group to justify selection of the provided transcripts and more general findings provided later in this analysis.

**Low Ability Group.** The following sampling of transcripts is from students in the low ability group as they complete the given task. Note that these students are later cited in the subsequent paragraphs. In addition to the transcripts provided below, student activity and behaviors evidenced some of the students’ understanding regarding representations and translations. Following these general statements, descriptions of some of student activity are provided within the transcripts.

While low ability students expended time and thought on Task 1, analysis demonstrates that student activity focused on heuristics which would not – and even often could not – prove effective. Most of the time spent by these students was quite evenly split by dejected silence, where students were unable to proceed in any positive manner toward the solution to the problem, and with calculators in hand, where students desperately entered values and polynomial functions (in forms such as $ax^5+bx^4+cx^3+dx^2+ex+f$, with random integer selections for $a$, $b$, $c$, $d$, $e$, and $f$) to see if they could match graphing results with the problem question. This technique might be best described as guess and check. Even when considering the value for the leading coefficient, most of these students had no understanding that $a$ had to be a positive value.

These lower ability level students seemed emotionally connected to using the graphing calculator, albeit in a Balkanized manner. While students had little confidence in their mathematical ability without the calculator, they simultaneously had little idea how to use the device effectively and did not carefully consider the mathematics that they were entering into the device. They wanted the calculator in hand, but were frustrated in recognizing that they did not
sufficiently know what to do with it. They generally wrote little or nothing on paper.

Notably, among students of different ability levels, lower ability level students spent the next to least amount of time working on the problem. Even though they did not get beyond Task 1, on average the students demonstrated exasperation in only 15-30 minutes. Without encouragement from the researchers with leading questions, almost all of these students would have quit attempting to solve Task 1 in little more than ten minutes or so.

**Student L1:** I have no idea how to do this (pressing buttons on a calculator). I don’t know what to plug in. I don’t know what the question means. ( Mostly looking down at the calculator and pressing more buttons.) What does “truncated graph” mean? Don’t we need to know what goes on above and below what we see? I don’t get it. I don’t know how to do it. (Acting defeated and dejected.)

**Student L2:** Some of the xs look like variables and some look like multiplication. Is that right?... I know what polynomial means. It’s something like $x^2 + 2x + 3$. And I think I know what a factor is. 5 is a factor of 10. But I’m not sure that I know what a “polynomial in factored form” means. (Amidst long pauses of simply looking again at the problem question, continues to enter and graph random polynomials into the calculator, seeing if any may match the target graph.)

**Student L3:** Does it matter that $b$ is above the line and the other letters are under the line? I know that the graph and equation are connected and that $b$ has to part of everything. But $b$ is above the line on the graph. I don’t know if this means something else like that. I’m stuck. I don’t think that the location of $b$ is necessary for the equation. (Calculator in hand.) I keep trying to plug numbers in for $a$, $b$, $c$, and $d$ into $y = ax^3 + bx^2 + cx + d$, but nothing seems to work.

**Student L4:** Am I supposed to know how to do this? (Acting hopeless to solve Task 1.) I can graph a function by plugging it into my calculator, but I don’t think that the calculator can take a graph and give you the function. Plus, why would you want to do this? The graph is the answer to a problem. You don’t need to go backwards. We don’t do that for anything but lines.

**Unpacking the source.** For the most part, students in this group shared a common technique through which to unpack the source representation: they used a trial and error method (e.g., L2 and L3) in unpacking the micro-concepts of the graph to uncover information that would be necessary to solve the problem. These students performed little to no selective encoding, as suggested by Sternberg (1984). The prevalent difficulty observed in this group was in interpreting the vocabulary within the directions and the question (e.g., L1 and L2). For example, even in conjunction with the provided picture, students had difficulty discerning the meaning of a “truncated graph.” (The researchers had to assist these students by demonstrating how to connect the visible parts of the truncated graph.) They experienced most difficulty connecting information in the question to the given graph, struggled in differentiating between useful and unnecessary ideas associated with the graph or equation, and were more apt to follow ideas down misleading tangents (e.g., L3).

**Preliminary coordination.** Though students in this ability grouping recognized that a link or connection must exist between the graph and the equation, the majority perceived the two representations as disjoint and connectionless (e.g., L1 and L2). They considered both representations, but were unsure of what ideas from either representation or on which connecting notions they needed to focus (e.g., all). For example, although some students recognized zeros
on the graph and knew something about the correlated \((x - □)\) binomial in the factored form of the polynomial, the issues of the variables \(a\) and \(b\) representing negative values often confused them. In respect to the far-left and far-right behavior of the graph, most students in the low ability group demonstrated little to no recognition of the connections this implied to either the degree or sign of the leading coefficient of the polynomial function. Students in this group, failed in their attempt to integrate ideas across the graph and equation and to discern which features and ideas were more important.

**Constructing the target.** Using techniques paralleling Kaput’s (1987b) syntactic elaboration, many of these students reformulated the given task into a problem of generating in their calculator polynomials of the form \(y = px^3 + qx^2 + rx + s\) that would produce the behavior of the truncated graph. Using this polynomial as their generalized form, they set out on a task of selecting random values for \(p, q, r,\) and \(s\) to create and graph different functions (e.g., \(L3\)) to find specific examples of polynomials which would model the behavior observed in the truncated graph. These students placed much time and effort into graphing randomly selected values for \(a, b, c,\) and \(d;\) the majority of the time was spent either in silence (e.g., \(L1\) and \(L2\)), with no means toward a solution (e.g., \(L1\) and \(L4\)), or with individuals stating that they did not understand and asking for help (e.g., \(L1, L2,\) and \(L4\)).

While they constructed a few equations, they were mostly unable to construct a target which would lead to successfully completing the task. Additionally, while they attempted many of the activities associated with this sub-process, as defined by Lesh et al. (1987), Sternberg (1984), Kaput (1987b), and Duval (2006), these activities did not culminate in progress in correctly answering the questions at hand. Notably, none of these students made significant headway in constructing an appropriate target equation that accurately aligned with the given truncated graph. Even with the hint of polynomials in factored form, students were still unsure of what to do. Even when shown \((x - a), (x - b), (x - c),\) and \((x - d)\) as factors, they: were unsure as to how these were connected to the graph; did not comprehend the effect of exponentiating the linear factors on the graph; were surprised when told that \(y = (x - a)(x - b)^2(x - c)(x - d)^2\) represented one of the numerous possible target representations; and, even after seeing this solutions, tried to rewrite it in the form \(px^n + qx^{n-1} + rx + s\), which they purported to better understand.

**Determining equivalence.** Most students in the low ability grouped checked for equivalence between the representations by utilizing the calculator in a trial and error fashion. They often would create an equation, graph the equation using the calculator, and check to see if the graph generated had the same \(x\)-axis behavior as the given truncated graph. When it did not, they simply attempted to graph a different equation. Notably, they considered equivalence between graphs (comparing the original graph to the graphs of the attempted equations) rather than directly between the original graph and the equation itself. (Rather than comparing graph-\(a\) with equation-\(a,\) they compared graph-\(a\) with the graph-\(b\) derived from equation-\(b.\)) In part, this may have been due to the fact that they had progressed so little toward the development of the required equation that no assessment of equivalence was even possible. However, when asked if they could verify what they had produced, irrespective of its incompleteness, they were unable to articulate how to verify equivalence or even how to start the process, since they could not discern which micro-concepts of the graph and equation were important to consider.

**Medium Ability Group.** The following sampling of transcripts is from students in the medium ability group as they completed the task. Medium ability students in this study commonly
attempted through Task 3. While they demonstrated strong dependency on the graphing calculator to solve the tasks, the heuristics they employed were generally successful. Affectively, they had less trepidation and demonstrated motivation to attack the tasks. Time on task was focused and split among reinvestigating the tasks, considering means to answer the questions, entering well-considered expressions into the calculator, and writing expressions of possible solutions on paper. Moreover, these students continually demonstrated that they recognized the problems to be solvable, even if beyond their individual ability to do so.

These medium ability level students recognized the calculator as a tool complementary to writing expressions of factored polynomials on paper. They generally wrote down expressions (often in a form such as \((x + 2)(x + 1^2)(x - 1)(x - 2^2)\) or \((x + 2)(x + 1)^2(x - 1)(x - 2)^2\), analyzed them to some degree, and then attempted to graph the function.

Among students of different ability levels, medium ability level students spent the most amount of time working on the problems, generally the full allotment of 60-90 minutes or until they were told to end their work. At the conclusion of the problem sessions, these students most often felt angst until they were told solutions to the tasks and could verify their solutions.

**Student M1:** (Writing variations of functions in the forms \(f(x)=ax^5+bx^4+cx^3+dx^2+ex+f\) and \(y=(x-a)(x-b)(x-c)(x-d)\) on paper.) I know that \(a\) and \(b\) represent negative numbers, but I don’t know if they should be written as negative \(a\) and negative \(b\). Can I use my calculator to graph? I’m not sure. I’m just gonna try some things out. (Using graphing calculator, repeatedly attempts to graph polynomials entered in general form and some in factored form. Does not seem to attempt to connect the two forms prior to graphing.) I keep trying to graph different equations, but nothing works. I don’t see how that equation connects to this graph. (Continues attempting to graph functions, but is becoming frustrated with lack of understanding and unsuccessful results.)

**Student M2:** (Pointing to Task 1.) I can see that the line passes through \(a\) and \(c\) and that \(x\) ‘squareds’ hit the axis at \(b\) and \(d\). But I don’t know what that does to the equation. I think that I need to use \(a\), \(b\), \(c\), and \(d\) and graph something like (writing on paper) \(ax^3 + bx^2 + cx + d\). I don’t know if this will work. I don’t know if I can do this without knowing the \(y\)-intercept. (Long pause, shifting gaze from Task 1 to paper on which student writes various linear, quadratic, and cubic equations.) I can graph a function, but if the function is not a straight line, I can’t make it into an equation. (Spends time graphing seemingly random function on the calculator.) I guess that I can do things with equations, but I can’t do much with graphs. These are supposed to be the final answer to our problems.

**Student M3:** I know that we need to put an \(a\), \(a\), \(b\), \(b\), \(c\), and \(d\) in the parentheses. (Writing polynomials in factored form on paper.) I’m not quite sure how they go in, but I think that they do. (Shifting gaze from Task 1 to written polynomials.) I’m not quite sure what to do to connect the parts of the graph. I can graph each part, but putting it all together. I don’t know what happens above or below. Won’t that affect the equation and where it hits the axis? (On calculator, graphs sample polynomials in factored form. Recognizes that most examples go off the screen.) How high and low do these things go? (Looking at features on the calculator.) I know how to use numbers in an equation, but I’m not sure how to use letters other than \(x\), or how to change those into numbers. My calculator won’t let me graph letters. (Pointing to sets of parentheses (e.g., \((x – ___)(x – ___)(x – ___)(x – ___)\) and \((x – ___)(x – ___)(x – ___)\).)
I know that some of these are supposed to be squared somewhere. I don’t know if it is \((x - \text{something}^2)\) or \((x - \text{something})^2\). I think it is \((x + 2)\times(x + 1^2)\times(x - 1)\times(x - 2^2)\). (Graphing this on calculator.) That doesn’t work. I don’t know where I went wrong. (Scratches out work on paper and writes new sets of parentheses to represent factored form.) I’ve got to start over.

**Student M4:** (Writing on paper.) I know a polynomial looks like \(ax^3 + bx^2 + cx + d\), but I don’t know how \((x___)\times(x___)\times(x___)\times(x___)\) is a polynomial. I’m gonna try \((x + a)\times(x + b)\times(x - c)\times(x - d)\). I’m gonna use some numbers instead, like try \((x + 2)\times(x + 1)\times(x - 1)\times(x - 2)\). (Graphs this on a calculator.) That graph doesn’t look right. I need to look at this again. (Writes equations in factored form on paper, and graphs some.) Hey, I think that \((x - a)^{\text{odd}}\times(x - b)^{\text{even}}\times(x - c)^{\text{odd}}\times(x - d)^{\text{even}}\) works. (Tests more examples on the graphing calculator.) I think going through my equations is OK.

**Student M5:** (Writes equations in factored form on paper.) This polynomial has to be raised to an even power to get this going up to the left and right. But, how does that work with the parentheses? (Switching gaze from Task 1, to equations written on paper, to the graphing calculator results.) Tests graphs of sample polynomials in factored form. I think that there is something here about single and double roots. (Pointing to the exponents on linear factors written on paper.) I think that double roots bounce. Or is that single roots? Let me check. (Graphs a function on the calculator.) I think it is \((x + 2)\times(x + 1^2)\times(x - 1)\times(x - 2)^2\). I can graph it. (Considering Task 2.) I can’t think of any other way.

**Student M6:** (Pointing to Task 1.) Since we are looking at the \(x\)-axis, I don’t think that it matters what happens where we can’t see it. Since the \(y\)-intercept is not pictured, I don’t think that it matters. (Writes equations in factored form on paper.) I think that we need exponents outside of each parentheses to make some of these bounce. (Graphs example on the calculator.) I tried \((x + 2)\times(x + 1^2)\times(x - 1)\times(x - 2)^2\) and I think it was right; but it went really high and low, so I had to change the window and only look close to the axis. It took me a while to figure out why \((x + a)\times(x + b^2)\times(x - c)\times(x - d)^2\) wasn’t right; I needed \((x - a)\times(x - b^2)\times(x - c)\times(x - d)^2\).

**Unpacking the source.** Students in the medium ability group in this study put much effort into unpacking the truncated graph. However, when unpacking the graph, they often inadvertently overlooked, or purposely dismissed, necessary features in the graph (e.g., \(M6\)); they decided which aspects of the graph were more important than others before dismissing those of lesser importance (as anticipated by the findings of Duval (2006), Lesh et al. (1987), Kaput (1987b), and Sternberg (1984)). Some students made some connections and gleaned some understanding of the graph before them (e.g., \(M4\) and \(M6\)).

Notably, most in this group recognized a number of aspects of the graph itself including: the fact that it depicted one continuous function rather than four separate functions (e.g., \(M3\), \(M5\), and \(M6\)); the general nature of the graph of a polynomial function (e.g., \(M5\) and \(M6\)); the correlation of zeros, roots, and \(x\)-intercepts from the graph of the function (e.g., \(M5\) and \(M6\)); the far right and far left behavior of graphs of polynomials (e.g., \(M4\), \(M5\), and \(M6\)); and the nature and effects of roots of odd and even multiplicity (e.g., \(M4\) and \(M6\)); and so forth.

**Preliminary Coordination.** Almost all medium ability level students struggled to connect the graph and the equation. They particularly struggled in respect to: the relation of the
factors with the zeros of the polynomial (e.g., \(M_1, M_2, \) and \(M_3\)); the form of the factor when the zero is a negative value (e.g., initially, \(M_4\) and \(M_6\)); and the degree of the polynomial and the multiplicity of roots (e.g., initially all). For instance, using \(a = -3, b = -1, c = 1,\) and \(d = 3\) as examples values, some of these students decided that they would graph \(y = (x + -3)(x + -1)(x − 1)(x − 3)\). After doing so, they realized that it did not produce the correct graph and numerous other equations were attempted to be graphed. Only through protracted trial and error did most medium ability level students come to understand that \(y = (x + 3)(x + 1)(x − 1)(x − 3)\) implied \(y = (x − a)(x − b)(x − c)(x − d)\).

In order to determine which structures from the graph were necessary to develop the equation, they quickly attempted to find a way to map \(a, b, c,\) and \(d\) from the graph to the equation (e.g., the concept of roots on a graph was mapped to roots of a function), often even before they had sufficient recognition of the interconnection (e.g., \(M_2, M_3,\) and \(M_4\), demonstrating Kaput’s (1987b) syntactic elaboration. The activity of trying to create the equation from the graph is consistent with: preliminary aspects of target representation formulation (Lesh et al., 1987); the latter facets of selective encoding, the initial form of selective combination, and selective comparison (Sternberg, 1984); reading and preliminary phases of encoding (Kaput, 1987b); and the attempt to form a network and initial associations between the concepts observed in both representations (Duval, 2006).

Students in this group focused on ancillary structures in both the graph and the target algebraic representation and often missed the most salient features which required addressing. Contrary to processes noted by Duval (2006), these students struggled to distinguish between features in representations which were relevant to the target construction and which were not. For instance, while they understood that the equation for the graph would need to have factors in the form \((x ± □)^2\), they often transferred the notion of an intercept at \(a\) to the factor \((x − a)\) without recognizing the notion of zeros and real roots of graphs and polynomials.

**Constructing the Target.** Unique to this ability level was the consistent development of a temporary, incomplete, working model of the equation (e.g., \(M_4\) and \(M_6\)), which they proposed as inadequately answering the investigation. This temporary model was employed as a mechanism through which ideas from the graph and equation could be integrated and associations between representations could be made (Duval, 2006). This model was later modified to include aspects of both the graph and the equation as they were recognized. Creating this equation, even in its temporary form, resembles: target representation formulation, content transfer from the source into the target representation, and restructuring the target (Lesh et al., 1987); selective combination (Sternberg, 1984); and encoding (Kaput, 1987b).

Realizing that the graph of the temporary equation correctly passed through the desired zeros, some students recognized that the behavior defined by the graph required different multiplicities for some of the factors; but some demonstrated uncertainty as to if a double root at \(b\) was accomplished by \((x − b^2)\) or \((x − b)^2\) (e.g., \(M_3\)). Only a few, after successive trials and random variations, arrived at the conclusion that the correct exponentiation for the equation was \(y = (x − a)(x − b)^2(x − c)(x − d)^2\). However, even for those who came to this conclusion, few were able to generalize the function in respect to odd and even multiplicity of zeros \((i.e., y = (x − a)^{2p−1}(x − b)^{2m}(x − c)^{2q−1}(x − d)^{2n}\), for \(p, m, q, n \in N\) (e.g., \(M_6\)).

While all students in this group noted some ideas shared between the graph and the targeted polynomial, few recognized the association of the odd and even multiplicities of zeros on the graph and their multiplicities with appropriate exponentiating of the factors in the equation. Additionally, as they attempted to form the equation, additional notions regarding
polynomial equations confused them and or hindered their progress (e.g., variables representing positive and negative values, degrees of polynomials in standard and factored form, and exponentiating of linear factors). Thus, most of these students had difficulty integrating ideas shared between the graph and the polynomial function, and constructing the polynomial became hindered by misinterpretations.

**Determining equivalence.** Rather than actually checking for equivalence between the given graph and the finally produced target equation, students in this group utilized equivalence checking as a means to monitor and revise their transitional target representation. While they recognized that checking for representational equivalence was valuable, their assessments were often incomplete and they had difficulty selecting which notions in the equation or graph needed a second review (e.g., $M_3$). Furthermore, when they checked for equivalence by attempting to graph functions, they often did not consider all features of the respective representations. Thus, although these students were apt to make translation errors that lead to nonequivalent representations, they remained unlikely to verify equivalence between the graph and the equation and seemed limited in their understanding of how to verify equivalence (e.g., $M_3$ and $M_5$).

**High Ability Group.** The following sampling of transcripts is from students in the high ability grouping as they completed the given task. Due to greater homogeneity within the actions and understanding of students in the high ability group, transcripts of only three students from this group are reported.

The high ability level students in this study spent the least amount of time working on the tasks and generally solved Task 1 in less than three minutes. They rarely wrote anything on paper (usually only to explain something to the researcher and not for their own needs) and graphed very few sample functions on the calculator. They seemed to never simply guess at answers. Their gaze was often in the air and not particularly on the task, their minimal written work, or even the calculator (which they used only sparingly to check answers).

Since they so quickly solved Tasks 1-3, they were given an additional task that was prepared beforehand. Task 4 prompted: “Fill the blanks in the parentheses to create a polynomial whose graph has the same x-axis behavior as in the truncated graph in Task 1: $y = (x - a)^{\text{odd}}(x - b)^{\text{even}}(x - c)^{\text{odd}}(x - d)^{\text{even}}(x - ___)(x - ___)(x - ___)(x - ___)$.”

Affectively, they simultaneously enjoyed the unusual nature of the task and seemed somewhat annoyed with their perception of its simplicity. They seemed to more enjoy Task 3, because they had never been asked a question in this form before. They were even more intrigued by Task 4, as it involved non-real complex roots and transcended the mathematics that they had studied in class. Nevertheless, they generally solved this in mere minutes as well.

**Student H1:** (Analyzing the graph in Task 1. Without writing any equations or graphing any functions…) The graph shows that this is a polynomial of even degree with a positive leading coefficient. Some of the x-intercepts show that the zeros are of even and odd degree. So, this function must be of degree at least 6. At $a$ and $c$, the function passes through the x-axis, and, at $b$ and $d$ the function bounces off the x-axis. This indicates the odd and even multiplicity of the zeros at those values.

**Student H2:** (Analyzing the graph in Task 1. Without writing any equations or graphing any functions…) $a$, $b$, $c$, and $d$ represent the zeros of the function. (Writes equation in factored form on paper.) Writing $x-a$, $x-b$, $x-c$, and $x-d$ inside the parentheses produces the same zeros. But we need to also consider the
exponent outside of each parentheses. (On paper, compares \((x - b)^2\) with \((x - b^2)\) for a moment, before deciding on the former.) Is it \((x - b)^2\) or \((x - b^2)\)? This is confusing. hmmmmmm I think it is \((x - b^2)\). Well, I think that we can start with \(y = (x - a)(x - b)^2(x - c)(x - d)^2\). (Writes this on paper gazes at Task 1.) But since we are only looking at the x-axis and we don’t care what goes on above and below, \(y = (x - a)(x - b)^2(x - c)^3(x - d)^4\) could also work. In fact, \(y = (x - a)^{odd}(x - b)^{even}(x - c)^{odd}(x - d)^{even}\) could be the answer.

**Student H3**: Most of the time, I can look at a graph and tell if the function is a polynomial. I like it when I get a polynomial in factored form; I always know what it looks like – or at least what its x-axis and far out behaviors look like. I have more trouble picturing what a polynomial equation in general form (like \(ax^3 + bx^2 + cx + d\)) would look like as a graph

**Unpacking the source.** Immediately upon encountering the question, students in the high ability group discerned the meaning of the verbs and phrases associated with the questions, were able to decode salient ideas from the graph, connected the graph and the equation, and operated on the given function through either representation (e.g., \(H1, H2, \) and \(H3\)). Student responses demonstrated that they fluently understood both representations and could communicate such without effort (e.g., \(H1, H2, \) and \(H3\)). They successfully discerned the salient ideas in the truncated graph, recognized that the required target was an even degree polynomial of minimal degree of six with a positive leading coefficient, noted the zeros and factors of the polynomial, and determined that some factors had odd and others even multiplicity. The context of the problem and the hint of polynomials in factored form immediately directed them to the structures which were most important in the graph. Only those structures within the graph, which were necessary to make the translation to the function equation, were considered by this group (e.g., \(H2\)).

**Preliminary Coordination.** Unique to this group was their manner of communication across representations. Most students in this group would routinely mention concepts and attributes associated with one representation in context of another. For instance, they would mention the “roots of the graph” and “multiplicity of the x-intercepts”; rather than discussing the x-intercepts of the graph and the roots of the polynomial in isolation. This communication demonstrated the interconnectedness with which they perceived the two representations. Students in this ability grouping used both their knowledge of the factored form of the polynomial and the context of the problem as a lens through which to do this (e.g., \(H1, H2, \) and \(H3\)); they created a network and association of ideas between the representations, as suggested by Duval (2006). Their ability to recognize structures associated with factored polynomials helped them to integrate ideas across the two representations (e.g., \(H2\)). These students performed a number of activities recognized in the latter stages of the translation process, such as: selective encoding, semantic elaboration, selective combination and selective comparison (Sternberg, 1984); target representation formulation and restructuring of target representation (Lesh et al., 1987); and preliminary encoding and semantic elaboration (Kaput, 1987b).

**Constructing the Target.** After unpacking and coordinating features across representations most students produced the relation \(y = (x - a)(x - b)^2(x - c)(x - d)^2\) (e.g., \(H1, H2, \) and \(H3\)). Even before moving to task Item 3, where they would be prompted to use inductive reasoning to investigate a general case for this solution, students began to mention that the multiplicity of the factors were not limited to 1 or 2, but to all odd or even positive integers and thus leading to the general function \(y = (x - a)^{odd}(x - b)^{even}(x - c)^{odd}(x - d)^{even}\) (e.g., \(H2\)).
Unless required to do so, students in this group did not produce a transitional form of the equation. They simultaneously coordinated between both representations and used whatever they needed from whichever representation most readily provided the information for which they were looking.

**Determining equivalence.** For the most part, students in this ability grouping continually checked for equivalence between the graph and the function equation throughout their translation processes (e.g., $H2$); they checked for equivalence while unpacking the source, performing preliminary coordination, and constructing the target. This seems to be due to the fact that they so well knew each representation that they also knew which actions would possibly lead to the loss of equivalence between the representations, and they avoided such. When they were later asked to confirm how they would determine equivalence between the given graph and their newly formulated equation, they indicated that they already had done so through the process they used and generally perceived the question as an extra, and unnecessary, task in which they had to engage. However, when they were given two different representations developed by others and asked to confirm or deny the equivalence of such, they recognized this as a valuable task. Therefore, they perceived that, when they perform their own translation actions, they do so with at least a tacit understanding of equivalence.

**Global Findings**
The use of this study’s framework provided a novel lens through which to observe and investigate student activity and understanding in respect to the assigned translation tasks. In so doing, a number of findings – some unanticipated – were made regarding student translation processes among mathematical representations. These findings will be discussed in terms of: *confounding micro-concepts*, *fact-mapping*, *concept-mapping*, *target skeleton* formation, and *transitional target* formation.

Notably, these findings and constructs are all consistent with, albeit extensions to, the framework this study used. Two interpretational options immediately present themselves. First, this may speak to the illuminating and explanatory power of the employed framework. The framework lens allowed more activities involved in the translation process to be observed and identified. Second, these additional findings may indicate the insufficient explanatory power of the employed framework. If this is the case, identifying these additional only accentuates the framework for future use. With either interpretation, these new findings and constructs add to the literature’s view of the translation process.

These findings led to the necessary revision of this study’s framework into something more complete and explanatory or student activity. Description of this revised framework follows the immediate discussion of these findings.

**Confounding Micro-concepts:** When students began to unpack a representation, they interacted with *micro-concepts* associated with the representation, albeit often different ones. While this is consistent with unpacking the source representation and preliminary coordination, it was found that students may: be prone to focus too much attention on select micro-concepts (e.g., overly focus on the truncated graph, the symbolic notation of the equation, and the labeling on the $x$-axis, respectively); insufficiently consider micro-concepts associated with the graph and equation (e.g., the multiplicity of the roots in respect to the graph and equation); and be steered away from other necessary micro-concepts (e.g., the far-left and far-right behavior of the graph connoting the even or odd degree of the polynomial) ($L1$, $L2$, and $L3$). Much of this was consistent with the findings of Lesh et al. (1987), Kaput (1987b), and Sternberg (1984), who,
each in their unique way, speaks to the translator considering the representation and, often unsatisfactorily, attempting to discern between valuable information and those which can be ignored.

The results of this study indicate that some micro-concepts associated with a particular representation can cause some students to become more confused than informed. For instance, while graphs typically provide a scaling on the $x$- and $y$-axes, this scaling is both unnecessary for the depiction of the concept of “function” and may even cause the student to focus unduly on this scaling and lose the meaning of function represented in the graph. These distracting micro-concepts are herein noted as **confounding micro-concepts**.

Returning to the constructs of unpacking the source and preliminary coordination, in light of confounding micro-concepts, it can be seen that interpreting a representation’s visible micro-concepts can be more difficult than previously indicated in the literature of the field. It seems that many translation activities are stymied simply by students being unable to correctly interpret a representation’s micro-concepts or by being confused by the micro-concepts they observe.

Interestingly, the data in this study seemingly indicate a connection between student ability grouping and confounding micro-concepts. For instance, no micro-concepts confounded students in the high ability group, some micro-concepts confounded some students in the medium ability group, and virtually all micro-concepts confounded students in the low ability group. For instance, *Student L1* overly focused on the portions of the graph which could not be seen, *Student L2* on symbols in the equation, and *Student M1* on the fact that $a$ and $b$ were on the negative side of the $x$-axis without a negative sign. Hence, it is hypothesized that a connection exists between student understanding of broader mathematical concepts and the existence and effects of confounding micro-concepts and this may have effectively contributed to the results of the study.

**Fact- and Concept-Mapping:** This study recognized that students in the different groups employed different techniques while unpacking and coordinating information across representations. This study’s theoretical framework identifies and distinguishes both micro-concepts and mathematical ideas in the translation process, and expands upon this notion as the translator interacts with, and between, representations; the translator maps micro-concepts and mathematical ideas from the source representation to the target representation. This framework allowed the researchers to parse student activities and communication to recognize a distinction between students mapping micro-concepts and mathematical ideas amongst representations.

Focusing almost singularly on micro-concepts, students in the medium ability group used **fact-mapping** as their means for mapping information across representations. For instance, *Student M3*, attempted to insert $a$, $b$, $c$ and $d$ from the graph into the equation, without referring to the associated mathematical idea of polynomials or functions. Sharing similarities with Kaput’s (1987b) description of syntactic elaboration and the findings of others (e.g., Clement, et al., 1981; Knuth, 2000; McGregor & Stacey, 1993), fact mapping is typically a mechanical, algorithmic process which connotes no deep, conceptual understanding of the mathematics; the transfer of information across representation is syntactic and not based on a recognition of an associated mathematical idea or concept.

High ability level students perform activities more appropriately characterized as **concept-mapping**, in which larger, broader mathematical ideas are recognized and mapped from the source to the target representation. For instance, they recognized and interacted with the idea of function in both graphical and equation form (e.g., $H1$ and $H2$), and utilized this knowledge to integrate information across the two representations. Concept-mapping shares similarities with
Kaput (1987b) notion of semantic elaboration (interacting with a representation based on the features of the ideas represented, rather than the symbols themselves). High ability level students employ trans-representational language; they speak of one representation through the context of another. For instance, in discussing the “roots of the graph” and “multiplicity of the x-intercepts”, they demonstrate that they map concepts, rather than simple facts, between representations.

**Target Skeleton and Transitional Target:** As mentioned in the framework of this study, during preliminary coordination, the ideas decoded from the source are initially amorphously conceptualized into the meta-target. When students of different ability levels gave physical form to the meta-target, they did so in different ways. During preliminary coordination, virtually all students in the low and medium ability groups began by writing “y =” or “f(x) =”. This demonstrates the development of the target skeleton, or the initial, pre-populated form of the target representation. (An additional example of a target skeleton can be seen from parallel research activities. When student were asked to create a table of values for a given function, they immediately constructed a table with two columns: one for x and one for y. Prior to populating the table with data, the table structure is the target skeleton for this representation.) Via syntactic elaboration (Kaput, 1987b), the target skeleton, reveals the most basic level of attempting to network (Duval, 2006) between the source and target representations.

Most students in the low ability level group progressed little further than developing the target skeleton. (Interestingly, however, despite both the defined task and prompting by the researcher, low ability level students assumed the target skeleton, “y =”, led to y = a polynomial in general form rather than y = a polynomial in factored form.) Medium ability level students formed a transitional target representation, or a temporary, working model of the target (in this case the equation) which they proposed and recognized as incomplete, inadequately answering the investigation, and in need of revision. In so doing, and agreeing with the findings of others (e.g., Duval, 2006; Kaput, 1987b; Lesh et al., 1987; and Sternberg, 1984), they exhibited the back and forth comparison of micro-concepts and mathematical ideas between source and target representations depicted in preliminary coordination, constructing the target, and determining equivalence. However, this was accomplished in physical form (not a meta-target) and through an improving iterative process, rather than as a one-time translation event.

Summarily, in respect to the prescribed activity in this study: low ability level students frequently developed correct target skeletons; medium ability level students consistently progressed past the target skeleton, developed a transitional target, repeatedly revised this transitional target, and developed the subsequent final target (whether or not correct); and high ability level students typically combined the development of the target skeleton and the transitional target and made the leap to constructing the final target representation.

**Framework Revision**

As mentioned above, the previously defined additional findings led the researchers to reformulate this study’s theoretical framework. This resulted in an expansion of the description of the four sub-processes associated with the translation process. These are provided below.

While it can be argued that much of this revised framework can be found encrypted in piecemeal fashion in various publications, it is herein found complete, and, thereby, offers value to the literature of the field and to future studies in the realm of translations. Furthermore, the presented framework is multi-representational in nature, combining text with diagrammatic descriptors.
Unpacking the source. As recognized in the previous transcripts and analysis, when students attempted to unpack the source representation, many: did not take into account all of the representation’s visible micro-concepts – some selectively chosen, some purposely avoided, and others inadvertently missed; did not discover all ideas encoded in the representation; and were confused by or misinterpreted some micro-concepts. These factors extend the original construct of unpacking the source in the following way. Figure 4 represents the densely packed micro-concepts (a-o) in the source representation. As they are unpacked, some (m-o) may immediately be discarded as unnecessary and others may be overlooked or not fully recognized. Through investigating and interacting with the micro-concepts of a representation, the learner decodes mathematical ideas (A, B, and C) which were captured in the source representation.

Figure 4. Revised Unpacking the Source.

Preliminary coordination. The data revealed: a distinction between the target skeleton and the transitional target; a differentiated duration of time between unpacking the source and constructing the target; similar interpretive difficulties associated with unpacking the target as with the source; student awareness of distinct micro-concepts encoding similar ideas across representations; student awareness that the target representation encoded additional ideas not necessarily encoded in the source representation; and an iterative process associating the source and target representations. Altogether, this demonstrated that the original construct of preliminary coordination needed to be expanded to be more encompassing.

Extending preliminary coordination (depicted in Figure 5), the ideas decoded from the source are initially amorphously conceptualized in light of the target representation. Micro-concepts, which may be present in the target representation (p, q, r, and s) and what ideas the target typically encodes (A, B, and D), are known and captured in the meta-target. The conceptualization of the meta-target leads to the reinvestigation of the source in order to evaluate whether a different set of micro-concepts are necessary to perform the translation and whether any additional, unnecessary micro-concepts can be discarded. Of the micro-concepts which first existed in the source representation (a-o, in Figure 4), only some were initially considered (a-l) in respect to decoding ideas from the source, and this set was altered after consideration of the meta-target (a-c, e-g, and m-o, in Figure 5). Students can circumnavigate this path any number of times, revise their understanding of the source and meta-target representations, and exit at will.
Constructing the target. The dimensions of fact-mapping, concept-mapping, and transitional target formation revealed that the initial construct of constructing the target may have lacked sufficient explanatory power. Through their articulations, students recognized the target representation as a depiction of already densely packed micro-concepts and that the target contained mathematical ideas both similar and complementary to the source representation.

Through the expansion of constructing the target, as depicted in Figure 6, ideas recognized across both source and target representations and select micro-concepts from the source are used to construct the target representation through appropriate micro-concepts (t-x). Constructing the target involves mapping of facts or concepts from one representation to another, and often entails returning to preliminary coordination and back to constructing the target a number of times in order to ensure the proper associations. Constructing the target includes additional sub-processes to traverse from a meta-target to a target representation. Creating a physical representation of the meta-target often incudes the transition from the target skeleton, to the transitional target, and finally to the target representation.

Determining equivalence. Since students selectively considered some micro-concepts and mathematical ideas in both the source and target representations, not all aspects of each
representation played a part in the translating from the source to the target, a model implying that all aspects of each representation is considered when students are determining equivalence is incomplete. Determining representational equivalence, as depicted in Figure 7, requires considering ideas that are both shared between the two representations (A and B) and unique to each representation (C and D). Micro-concepts must be considered in respect to those which: belong exclusively to each representation; are implicit within the target representation and only discovered after translation steps are taken (y and z); and were previously ignored from the source representation. In essence, all ideas, whether unique to one representation or shared across both, and all micro-concepts must be reconsidered to ensure that the target representation accurately and completely depicts the source without introducing inconsistent ideas. Interestingly, although both representations are in a packed form at this time, the process of verifying equivalence necessitates informally unpacking both representations and comparing micro-concepts and mathematical ideas.

Figure 7. Revised Determining Equivalence

DISCUSSION
The finding that some students are more capable of performing mathematical translations than others is not surprising since other studies have reported similar findings. However, this study found that students in the different ability groupings unpacked, performed preliminary coordination, constructed the target, and determined equivalence using dissimilar techniques. Thus, while some prior research reported student translation difficulties, little had been done in respect to investigating students of different ability levels. If this study had found that high and low ability level students are more and less successful, respectively, at performing translations, this would have been a somewhat trivial finding. However, finding that students of different ability levels actually perform translations actions differently distinguishes the results of this study from others.

In the following sections, generalizations are made in respect to the translation process, first in respect to variations according to student ability levels and second in a diagrammatic form which, together with the previous diagrams and discussions regarding this study’s extended framework, captures the entire translation process in a flowchart. This flowchart can be employed in future research regarding translations between mathematical representations.

Generalizing Student Translation Processes by Ability
In the following discussions, we return to our research question regarding whether students of different ability levels process translations differently? In this section, we outline the processes of the different ability or achievement groups as they transfer information from a graph to an equation via the lens of the revised theoretical framework. It should be noted that the processes
outlined herein only captures the prevalent activities of students within the different groups (low, medium, and high ability groups); some students performed translations using heuristics inconsistent with the majority of students in their ability group.

**Low Ability Level Group.** While unpacking the source, students in the low ability group interact with micro-concepts of the source representation without reference to the larger mathematical context and attempt to discern the individual and disjoint meaning of micro-concepts they observe. During preliminary coordination, they struggle to discern important micro-concepts and ideas from the source representation and to differentiate between ideas that are unique to the source representation and those that are shared among the source and target representations. They are distracted by confounding micro-concepts. In respect to constructing the target, they build target skeletons and employ an iterative process of making and abandoning incorrect targets. When determining representational equivalency, they compare source representation to source-like representations. From target attempts, they create new source representations which they compare with the original source representation in an attempt to verify representational equivalence.

**Medium Ability Level Group.** While unpacking the source, students in the medium ability group interact with micro-concepts of the source representation, attempt to discern larger mathematical ideas encoded in the representation, and determine which mathematical idea are more or less useful in the coming translation activity. During preliminary coordination, they consider micro-concepts and mathematical ideas associated with both the source and target representations, begin to develop a target skeleton, and consider micro-concepts and mathematical ideas shared by the source and target representations. In respect to constructing the target, they: use fact-mapping, to translate necessary micro-concepts from the source representation to a transitional target representation; are apt to chase confounding micro-concepts in numerous unfruitful directions; and, only after doing so, recognize paths as unnecessary, dismiss superfluous information, and reconsider other micro-concepts. Determining representational equivalency is done as a means to revise the transitional target and is done by checking micro-concepts and mathematical ideas in each representation. If equivalency is verified, then the transitional target representation becomes the final target representation. If equivalency is not verified, then the students return to the beginning of the process and look again at both representations to determine where an error or misunderstanding occurred.

**High Ability Level Group.** While unpacking the source, students in the high ability group simultaneously consider both the source and target representations and discern: ideas within, and shared among, the representations; valuable versus unnecessary ideas in respect to the translation process; and broader concepts present within both representations. During preliminary coordination, they recognize mathematical ideas and interconnect these ideas between source and target representations and are rarely confounded by any micro-concepts. When constructing the target, only the micro-concepts they want/need from the target representation are considered through a process of concept mapping – thus bypassing the formation of a target skeleton and a transitional target representation. In respect to verifying equivalence between representations, they do this automatically, informally, and continually throughout the translation process with a recognition of which translation actions potentially violate equivalence.

**Generalizations in Student Translation Techniques**
Altogether, the observations and the generalizations of this study have been diagrammatically encoded into Figure 8. Though Figure 8 was initially designed to capture the translation activities for students in the medium ability group, it was also found to effectively depict and capture translation process when students in the high ability grouping were given additional tasks with significantly greater difficulty. Since students in this ability group correctly completed task Items 1, 2 and 3 in less than the allotted time, more mathematically difficult items (not expounded upon herein) were additionally posed to them. Notably, reasoning and processes articulated when answering these more complex tasks were consistent with the model in Figure 8.

**Figure 8.** Translation Process for Medium Ability Group.

This finding raises questions about task complexity and its mitigating role on student strategies while transferring graphical properties into algebraic formulas. To what extent might the processes observed and strategies employed by students in the high ability group be an artifact of the particular items and content on the task? In order words, were items 1, 2, and 3 merely exercises rather than problems for the high-ability group? Will students in the low ability group resort to processes identified in Figure 8 if the task is appropriately aligned to their level of mathematical sophistication? Moreover since the task assessed reversibility, flexibility and generalizability, (which have being identified by Krutetskii (1976) as a common problem solving process for student of high mathematical ability) is the task as currently constructed more favorable to students in the higher ability group?
Unfortunately answers to these questions are beyond the scope of the data collected for this study and may represent a possible limitation of the study. Thus, though Figure 8 is currently hypothesized to denote the flow of student translation activities whenever the task is of sufficient difficulty so that students cannot instantaneously perform the translation, its generality cannot be assessed singularly using data from this study. In order to assess its generality, it would have been not only interesting to confront high achievers with additional more difficult tasks, but also to have low achievers work on easier tasks. If similar processes are observed in both cases, then it could be concluded that the model represented in Figure 8 is general for different levels, as long as the task level is appropriate for the student. Unfortunately, this left for further research.

CONCLUSION

This study reveals that students translate graphical properties into algebraic formulas by traversing a similar translation path (i.e., unpacking, preliminary coordination, constructing the target, determining equivalency). Although they all traverse a similar translation path, results of this study suggests that students of different ability level groupings traverse these paths using different strategies and heuristics. For example, in preliminary coordination, low and medium ability level students utilize fact mapping and high ability level students employed concept mapping when integrating information across the graph and the equation. Similarly, in respect to constructing the target, students in the low ability grouping form the target skeleton, medium ability students transition from target skeleton to developing transitional target and final target representation, high ability level students made the cognitive jump from the source to the final target representation, bypassing both the target skeleton and the transitional target.

In addition, this study discovered the existence of confounding micro-concepts, which are seen to have an impact on the different ability groupings with higher ability level students being seemingly immune from such. Thus, in addition to determining that students of different ability levels perform mathematical translations in differing ways, this study discovered numerous dimensions which are constituent in, and helped to delineate the differences among, the different ability level groupings. While it is not clear if the observed disparity in processes employed by the different ability groupings were due to task complexity, many of the dimensions (i.e., confounding micro-concepts; fact mapping versus concept mapping; and the uses and distinctions of the target skeleton, the transitional target, and the final target), particularly as differentiated according to student ability level, simultaneously interconnect and transcend the findings of many previous studies which have considered student success in translating between mathematical representations.

Of particular note is that this study used its framework to investigate a single translation from a particular graph to a polynomial in factored form. Countless other mathematical contexts and additional translations exist (i.e., from equation to graph/table/verbal, from table to equation/graph/verbal, from graph to table/verbal, and from verbal to equation/table/graph) which were not investigated. It is important that future research consider the framework used in this study to verify findings in respect to other translations. Further research needs to connect the findings of this study directly to instructional practices and recommendations.

While it is valuable to have mapped the translation process and be able to better determine where in the translation process individual students may struggle, this investigation did not consider techniques for remediation to accompany blocks at particular locations in the process. However, one specific activity did seem to have some potential to establish student
understanding of the important ideas in a given representation prior to translation into another representations and then correlate student understanding of the constructs to student success translating between representations. Findings from this study made the researchers believe that students struggling with translations should be offered various representations and asked to respond to open ended prompts such as: state everything that you know about the representation; if a particular characteristic is changed (e.g., shifting the location of the real roots on a graph or changing the exponent on a factor of a polynomial in factored form), how does this affect the meaning of the representation; if a particular character combination or configuration is changed in one representation (e.g., graph), how would that affect the depiction of an equivalent representation (e.g., function notation); and given two different types of representations (e.g., table/function or function/graph), what ideas are common to both representations? Analysis of these questions will reveal that these prompts have the purpose of getting students to unpack ideas from representations, perform a preliminary coordination between representations by recognizing shared ideas, and begin determining the equivalence of different representational forms. The effectiveness of this type of activity in developing student ability to perform translations should be fodder for future research. Additionally, more research involving different mathematical relationships (other than polynomial functions) and representations (other than graphs and symbolic representations) needs to be undertaken to examine the veracity of the mapping.

REFERENCES


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