Using lattice models to determine Greatest Common Factor and Least Common Multiple

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Abstract
This article describes an alternative representation of whole numbers, one that can be constructed as a manipulative model. The material is particularly useful in providing a visual representation of the Greatest Common Divisor and the Least Common Multiple of numbers, concepts which have been shown in the literature to be calculated by students in rule-based manners for which students lack intuitive explanations.

I. Introduction
Knowledge about the multiplicative structure of natural numbers is applicable at all levels of mathematics. However, recent research indicates that elementary school students, and even prospective elementary teachers, have difficulties recognizing multiplicative relations between whole numbers, particularly when they are expressed in prime-factored form (Brown, Thomas, & Tolias, 2002; Zazkis & Campbell, 1996).

In this article I will describe an alternative representation of whole numbers, one that can be constructed as a manipulative model. My objective is to show how these models can be useful in visualizing and suggesting new forms of connecting processes and objects related to the multiplicative structure of whole numbers – in particular to the processes of finding the Greatest Common Divisor and the Least Common Multiple of numbers.

II. Research on prospective elementary teachers’ understanding of the multiplicative structure of the set of natural numbers.
Freudenthal (1983) has characterized the multiplicative structure of N as the whole of the relations \( a \times b = c \) as well as knowledge of properties of
commutativity, associativity, distributivity and equivalence of \( a \times b = c \) and \( c \div a = b \).

Knowledge of multiplicative structures are applied to different areas such as geometry, physics and probability. Therefore, it is important that students conceptualize these structures properly (Vergnaud, 1994). There is plenty of evidence that this is not the point at which some students are, though.

In particular, concerning the concept of Least Common Multiple (LCM), Brown, Thomas and Tolias (2002) interviewed prospective elementary school teachers with questions on number theory, and found that three approaches were used to find the least common multiple between two numbers:

- Set intersection: creating an ordered list of consecutive multiples for each number and finding the first one that appears in both lists;
- Creating a multiple and divide: creating an ordered list of consecutive multiples of one number and simultaneously checking each new entry for divisibility by the second number;
- Prime-factorization: comparing the prime factorization of the two numbers and finding the minimal product of prime powers that contains both of their factorizations (often referred to as “the higher exponent” rule).

Brown (2002) remarks that the first two methods can be wieldy because it may require a large number of steps, but that, on the other hand, justifying these two first methods is much more intuitively done than explaining why the “higher exponent” rule works.

Brown, Thomas and Tolias (2002) conclude that to be able to use multiplicative structure in reasoning one needs to deal flexibly with prime factorizations and to be able to identify multiples and factors through the application of meaningful processes. “A major goal of experiences of this type would be not only to teach students procedures (actions) needed for dealing with numbers in prime-factored form, but to get students to reflect on these actions sufficiently and in productive enough ways that they are able to move beyond successful actions to processes and objects.” (p. 78)

The reason why those authors point this out is because they found that students often change numbers from prime-factored representation to base-ten representation in order to perform mental operations on them. That is, they were not able to make connections between actions, processes, and objects as a *schema* – a process termed “thematization”. (Brown, Thomas & Tolias, 2002; Ferrari, 2002)
It is now that the use of different representations comes in. As Ferrari points out, there is a close link between this desired thematization and the use of different representations for mathematical objects. “Representations not only are necessary, for it is difficult to connect processes or objects without representing them, but also may themselves suggest some connections.” (p. 107) The advocacy for the use of multiple representations has been strong in mathematics education literature, specially since the work of Duval (1995).

The representation of whole numbers I will describe in the following section suggests connections about properties and relations between those numbers, and can be used as a valuable source of experiences for students.

III. Lattice models of whole numbers

This kind of manipulative model, which I have been calling lattice model, was proposed by Grossi (2000), and was based on Hasse diagrams.

To build the models you will need Styrofoam spheres and wooden sticks painted in different colors. The first thing to do to construct the models is to choose a different color to represent each prime number that is going to be represented. For example, we can stipulate that the color red will represent the prime 2, yellow will represent 3, blue will represent 5, green will represent 7, and so on. The wooden sticks should be painted in these colors, and they will represent the prime factors in the constitution of a number.

The Styrofoam spheres will represent the factors of a number, and will be labeled with a pen or marker.

Let us look at a concrete example: If the number we want to model is 2000=2^4×5^3, we start by getting 4 sticks of the color that represents the prime 2 and placing them in sequence. At the extremeties of each stick, we attach a foam sphere labeled with the sequence 2^0=1, 2^1=2, 2^2=4, 2^3=8, and 2^4=16. We then prepare another linear sequence of spheres and sticks for the sequence 5^0=1, 5^1=5, 5^2=25, 5^3=125 (we omit the factor 1 because we will use the sphere already labeled 1 in the previous sequence). We then attach it to the previous sequence, by pinning it perpendicularly at the sphere labeled “1”. We continue by filling up the array to represent the products of the powers of 2 by the powers of 5. The resulting model is shown in Figure 1.
So, for example, in the configuration in figure 1, the red stick represents the operation “times 2” if “read” from left to right, and “divided by two” if “read” from right to left. In other words, the red stick represents the prime two (because we chose red to represent two) as an operator in a multiplication or in a division.

Similarly, blue sticks would represent “times five” or “divided by five”, depending if you are moving from top to bottom or from bottom to top, respectively, in the diagram.

In general terms, if the number to be modeled is $m = a^\alpha b^\beta c^\chi d^\delta \ldots$, where the Latin letters $a$, $b$, $c$, $d$, etc. represent distinct primes, and the Greek letters $\alpha$, $\beta$, $\chi$, $\delta$, etc. represent their corresponding exponents, we start by getting $\alpha$ sticks of the color that represents $a$ and placing them in sequence. At the extremities of each stick, we attach a foam sphere labeled with the sequence 1, $a$, $a^2$, $a^3$ \ldots, $a^\alpha$. We then prepare another linear sequence of spheres and sticks for the sequence $b$, $b^2$, $b^3 \ldots$, $b^\beta$. We then attach it to the previous sequence, by pinning it perpendicularly at the sphere labeled “1”.

Figure 1
We continue by filling up the array to represent the products of the powers of $a$ by the powers of $b$. We continue the construction of the model for a general any number $m$ with general prime-factored form $m = a^{\alpha}b^{\beta}c^{\gamma}d^{\delta}...$, by choosing different directions in which to attach the sequences of powers of the primes $c$, $d$, ..., etc., and by completing the arrays with the combinations of products between those and the powers of the other primes already represented.

Figures 2 and 3 show models of numbers that have three and four distinct primes as their factors, respectively. The models reflect all the relations and properties of positive integers, including the uniqueness of their prime factorization: No two numbers have the same prime factorization – so no two numbers will have equal lattice models.

IV. The Greatest Common Divisor and The Least Common Multiple

When we look for the Greatest Common Divisor (GCD) or Greatest Common Factor of numbers, we have to compare the set of factors of those numbers, find which factors they have in common, and then, among those, which one is the greatest.
Our grid model gives us the factors of numbers. So we can easily compare two
grids and see the greatest common factor in those grids.

For example, looking at the models for the numbers 168 and 588, shown in Figure
4, we can see that the intersection between the two sets of factors (the part the two
models have in common) is the one whose foam balls were shaded blue. Since
those are the common factors, the greatest common factor is the greatest among
them, or 84. Note also that 84 is in the opposite corner from the number one in
the shaded common part of the models.

How can we use the lattice models to visualize the Least Common Multiple of
two or more numbers?

A multiple of a number \( n \) is any number whose lattice model includes the lattice
of \( n \). For example, since the lattice for 12 includes the lattice for 6, we know 12 is
a multiple of 6. Since the lattice for 18 also includes the lattice for 6, the number
18 is another multiple of 6 (Figure 5).
We see that, while we can visualize the factors of a number $n$ in its lattice model, to visualize a multiple of $n$ we would have to construct another model – one of which the model for $n$ is a part.

So when we look for the least common multiple between two or more numbers, we are looking for a number that we may have to construct, in whose grid we will find all the original numbers (that is, the numbers whose LCM we are looking for).

And among all the possible common multiples of these numbers, or all the models that can be built containing these original models as parts, which one is the least? In other words, what is the smallest model we can build which contains those the numbers whose LCM we are trying to determine?

Let us examine, for an example, the process of finding the LCM of 18 and 60. Their models are shown in figure 6 a) and 6 b).

The model for the LCM will have to include both these models among its parts, because both 18 and 60 are factors of the LCM. If we compare the two models, we see that the model for 60 has one more power of 5, and the model for 18 has one more power of 3. Figure 6 c) shows the two models combined, but in incomplete form– that is, the set form is not a model for any number.
We then complete the grid to find the smallest model that will contain that combined set, finding the model for 180, shown in Figure 6 d).

Figure 6: (a) The model for 60; (b) The model for 18; (c) An unfinished model combining the models for 60 and 18; (d) The finished model for the LCM.

I have been using the lattice models in my undergraduate courses for prospective elementary teachers, and have found that the experience gives them much more familiarity with the concepts of GCD and LCM. Visualization is a powerful tool,
and the importance of using multiple representations of mathematical concepts has been generally accepted. More than numerals and exponents, the lattice models allows students to construct a mental image of the relations between the natural numbers.

References


