Abstract

This study investigated the development of the concept of variables in middle grades mathematics textbooks during four eras of mathematics education in the United States. Findings revealed that each of the middle grades mathematics curricula examined used variables, but in varied proportions and levels of complexity. Formal definitions for variables were found in 11 of the 12 textbooks examined. The characteristics of the definitions for variables found in the different curricula were, however, different from one another. The uses of variables as a specific unknown quantity and as a label dominated the uses of variables in the mathematics curricula. The least used category of variables was as an abstract symbol. Overall, the data did not reveal any systematic or drastic change in the treatment of variable ideas during the 50 year period within which the study was situated. There was however, a steady increase in the use of variables as varying quantities across grade levels, and across the four eras of mathematics education in the United States. There were also some noticeable changes in the treatment of variable ideas found in the curriculum selected for the present NCTM era when compared with the treatment in the other three curricula.

Keywords: variables, curriculum, content analysis
Meaningful and long-lasting improvements in students’ learning require changes in many areas of our education system. At the center of this system is the curriculum, which is defined largely by the textbooks students and teachers use. A careful examination of the content, instructional strategies, and the treatment of fundamental concepts in these instructional materials is essential in determining their potentials for providing students the opportunity to learn important mathematics. This study contributes to that effort by examining the treatment of variables in middle-grades mathematics curricula selected from four eras of mathematics education in the United States.

The concept of variable is central to the teaching and learning of school mathematics and other related school subjects such as physics, chemistry, and economics. The evidence of the importance of this concept is seen in the many uses of variables (e.g., as labels, a specific unknown quantity, varying quantities, and abstract symbols) in school mathematics today. School algebra, for example, relies heavily on the use of variables for its presentation (Booth, 1988). The superiority of the use of variables over plain language in achieving the goal of school mathematics is evident when one tries to describe rules and undertake simple procedures in mathematics. For instance, in number theory, one can describe the product of two fractions in plain English as ‘the fraction whose numerator is the product of the two numerators and whose denominator is the product of the two denominators’. This same statement can be described more succinctly in algebraic language using variables as \( \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \), where \( b \) and \( d \) are non-zero real numbers.

In spite of these superior uses of variables to express mathematical ideas easily and concisely, the extant literature reveals that students, as well as many teachers, find it difficult to
work efficiently with variables (Clement, 1982; Macgregor & Stacey, 1997; Ursini & Trigueros, 1997). Booth (1988) for example, reported that students mistakenly perceived $5y$ to mean 5 yachts, or 5 yogurts, when it should have been conceived as 5 times the number represented by the variable $y$. Wagner (1981) documented that high school students erroneously identified different letters as always representing different numbers in equations. Specifically, they treated $n$ and $w$ in the equations $7n + 22 = 109$ and $7w + 22 = 109$ as representing two different numbers, and thought that there was a linear ordering correspondence between the alphabets and the number system. Thus, in their view, an $n$ would represent a smaller number than $w$. Usiskin (1988) observed that many students think that all variables are letters that stand for numbers. Yet, the roles of a variable are not always to represent numbers. Ortega and Ursini (2010) found that both males and females ninth grade students interpret variables incorrectly in many instances when they are used in functional relation and as general numbers, but that female students tend to have more difficulty working with the variables when they need to shift between different uses of variables as facets of the same mathematical object.

The preceding reports and many others found in the literature continue to incite researchers to investigate students’, as well as teachers’ understandings and misconceptions of variables (Boz, 2007; Clement et al., 1981; White & Mitchelmore, 1996) in order to address these difficulties. In addressing these issues however, very few studies examined the treatment of variables in the mathematics curriculum and the extent to which students have opportunity to learn it with relational understanding as opposed to instrumental, understanding (Skemp, 1976).

The purpose of this study, therefore, was to examine the presentation of the concept of variables in popular middle-grades mathematics textbooks. In addition, we sought to determine the extent to which the development and the presentation of this concept have changed, if they
have, in the school mathematics curriculum during the past 50 years (i.e., from 1957 through 2009). This time period (i.e., 1957 through 2009) was classified into four eras of mathematics education reforms in the United States: *New Math, Back to the Basics, Problem Solving,* and *NCTM Standards* era (Fey & Graeber, 2003; Payne, 2003) for the purpose of discerning trends.

The Origin and Historical Development of Variable

The research literature identified Diophantus as the first person to use symbols to solve mathematics problems (Boyer, 1991; Kieran, 1992). Specifically, Diophantus used symbols to represent an unknown quantity in mathematics problems in the middle of the third century (245-280). Prior to this period, mathematics problems were expressed and solved rhetorically – that is, exclusively using words without variables or symbols. Viete (1540-1603) was credited in the late sixteenth century for using consonants to indicate known magnitudes and vowels to indicate true unknowns. Thus, for the first time in the history of mathematics, letters were used as symbols to represent sets of numbers. Descartes in the early seventeenth century (1596-1650) established the tradition of using letters near the beginning of the alphabet to represent parameters and those near the end of the alphabet to represent varying quantities.

Sfard (1992) and Sierpinska (1992) noted that the general acceptance among the mathematical community of the concept of variable as a varying quantity and as a generalizer of a set of numbers progressed slowly in the 200 years following Viete’s momentous contribution. Reports from the extant literature indicated that by the 16th century, many more new uses of variables emerged within the mathematics community. As this concept developed historically, however, new uses of variable did not replace its uses during prior times, but rather expanded it into a complex repertoire. Some of these uses are often barely distinguishable from one another, similar in some ways, opposing in other ways, overlapping, and often difficult to articulate.
clearly. In school mathematics today, variables are used as *labels, unknowns or placeholders, varying quantities, generalized numbers*, and *abstract symbols*.

Frameworks on the Conceptions of Variable

The multifaceted nature of the concept of variables just described in the preceding page makes it difficult for students and teachers to understand and use variables efficiently. Consequently, in the late 1970s through early 1990s, several research reports (e.g., Küchemann, 1978; Philipp, 1992; Schoenfeld & Arcavi, 1988; Usiskin, 1988; Wagner & Parker, 1993) presented frameworks on the uses of variables in school mathematics for the purpose of constructing this concept into a manageable topic for students to learn and use more efficiently.

Küchemann (1978, 1981) identified six ways in which British middle school students interpret and use variables. He described these uses as: variable *evaluated; variable ignored; variable as objects; variable as specific unknowns; variable as generalized numbers; and variable as varying quantities*. Philipp (1992) categorized United States middle school students’ uses of variables as *labels, constants, parameters, unknowns, generalized numbers, varying quantities, and abstract symbols*. He also identified various misconceptions that middle school students have with this concept, and designed instruction to remedy these misconceptions. Philipp (1992) clarified that his categorization of variables includes only the uses of letters as variables, and acknowledged that other symbols in mathematics may be used to denote variable ideas in school mathematics. In a slightly different line of investigation, Usiskin (1988) identified four major conceptions of variables as they relate to their use in school algebra. He described these uses as: *generalized arithmetic; a study of procedures for solving certain kinds of problems; the study of relationships among quantities; and the study of structures*.

Using the preceding frameworks on variables, we developed a comprehensive analytical
framework on variables that is applicable to curriculum analysis, and used it to examine the presentation of variables in popular middle-grades mathematics textbooks. Table 1 presents the framework on the uses of variables employed in this study.

Table 1: *Categorization of Variables by their Use in this Study*

<table>
<thead>
<tr>
<th>Roles of Variables</th>
<th>Definitions</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Label</td>
<td>Shorthand for the name of object</td>
<td><em>f</em>, <em>y</em> in <em>3f</em> = <em>1y</em> (3 feet in 1 yard)</td>
</tr>
<tr>
<td>Constant</td>
<td>Stands for a quantity with a fixed value in a specified context</td>
<td><em>π</em>, <em>e</em>, <em>c</em></td>
</tr>
<tr>
<td>Specific Unknown</td>
<td>Stands for unique unknown number that can be found by solving an equation</td>
<td><em>x</em> + 5 = 8</td>
</tr>
<tr>
<td>Continuous Unknown</td>
<td>Stands for unknown <em>values</em> to be found in equations, expressions or inequalities</td>
<td><em>x</em>² − 3<em>x</em> = 28, 2<em>x</em> − 4 &lt; 7</td>
</tr>
<tr>
<td>Generalized Number</td>
<td>Expresses patterns or sequences of set of numbers that give true statement</td>
<td><em>a</em> + <em>b</em> = <em>b</em> + <em>a</em></td>
</tr>
<tr>
<td>Varying Quantity</td>
<td>Expresses functional relationships between quantities or the changing nature of a phenomenon</td>
<td><em>x</em> and <em>y</em> in <em>y</em> = −2<em>x</em> + 6</td>
</tr>
<tr>
<td>Abstract Symbol</td>
<td>Stands for literal symbols without number referent</td>
<td><em>e</em>, <em>x</em> in <em>e</em> <em>x</em> = <em>x</em></td>
</tr>
</tbody>
</table>

*Labels*: variables are used as labels when they stand as shorthand for objects rather than for a characteristic of them. An example is the use of *f* and *y* in *3f* = *1y* to denote 3 feet in 1 yard. Other uses of variables as labels are *m* to denote meter, *l* to denote liter, *t* to denote ton, etc.

*Constants*: variables are used as constants when they represent quantities with a fixed value, either in a specified context or in general form. For instance, the symbol *π* that denotes the ratio of the circumference of a circle to its diameter, and having a numeric approximation of 3.1416 is a constant. Some other examples of *constants* include physical constants, such as density of materials (*p*), and gravity (*g*) which is defined as *g* = 9.8 *m/sec*² for an object close
to the Earth’s surface, and $g = 2.3 \text{ m/sec}^2$ for an object on the moon.

Specific Unknowns: variables are used as specific unknowns when they represent unique unknown numbers that can be found in equations. For example, in the equation $x + 5 = 8$, the value of $x$ stands for a single number (in this case, 3) that can be found by solving the equation.

Continuous Unknowns: variables are used as continuous unknowns when they are employed in equations whose solutions yield more than one numeric value for the variable used. An example is the solution to the quadratic equation $x^2 - 3x = 28$ which yields two values for the variable $x$. Another example of this use of variables is when variables are employed to represent a signifier of a continuous and related set of numbers such as their role in inequalities. Thus, solutions to inequalities usually result in a set of values for the variable as opposed to a single value typically found in equations. This use of variable provides opportunities for students to recognize that a particular variable can assume more than one (or an unlimited) number of values. An example is the use of the variable $x$ in $2x - 4 < 7$. Küchemann (1978) provided yet another situation where the variable is employed as a continuous unknown. Specifically, when students are asked to complete the following equation: if $e + f = 8$, and $e + f + g = ?$, then $? = 8 + g$, one observes that the values of $e, f$, and $g$ remain unknown even after students solved the problem.

Generalized Numbers: variables are employed as generalized numbers when they are used in mathematical expressions to represent patterns or sequences. An example of such use of variables is $n$ in $2n + 1$ to generalize an expression for odd numbers. Another example is the use of variables to describe the commutative property of addition denoted by $a + b = b + a$.

Varying Quantities: when variables are used as varying quantities, they offer students the opportunity to see the changing nature of a phenomenon. In this way, students have the chance to
explore functional relationship between two or more variables as their values systematically change. An example is the relationship between $x$ and $y$ in $y = mx + b$. Another example of the use of variables as *varying quantities* is their role in formulas such as $A = lw$ and $A = \frac{1}{2}ab$.

*Abstract Symbols:* variables are used as abstract symbols when they do not have number or numerical referent. For example, when students are given that $a \cdot b = 2a + b$, and then asked to prove that $a \cdot (b + c) = (a \cdot b) \cdot c$, the variables are used as abstract symbols. In this situation, the variables are little more than arbitrary marks on paper that allow for algebraic manipulations or justifications of mathematical structures.

We employed the preceding seven uses of variables to examine variable ideas in the selected middle-grades mathematics textbooks. We wish to clarify here that we considered *variable ideas* to be those uses of variables represented by *letters* only. Similar to Philipp (1992), we were aware of the fact that not all variable ideas in school mathematics are represented by letters (e.g., a symbol in an open sentence such as $3 + \square = 5$), and that not every use of letters in mathematics problems represent variable ideas (e.g., $cm, kg, km$, used in metric systems).

Four Eras of Mathematics Education Reforms

For the purpose of examining changes in trend of the development of variables in school mathematics, we divided the time period within which this study is situated (i.e., 1957 through 2009) into four eras of mathematics education reforms in the United States: *New Math, Back to the Basics, Problem Solving*, and *NCTM Standards* era (Fey & Graeber, 2003; Payne, 2003). These eras and the major events that marked their emergence are described below.

*New Math*

The *New Math* education reform came as a result of United States response to the Soviet Union’s technological advancement shown by the launching of the Sputnik satellite in 1957
(DeVault & Weaver, 1970; Osbourne & Crosswhite, 1970). This event led to the need to focus school mathematics programs on developing “high quality mathematics for college-capable students, particularly those heading for technical or scientific careers” (National Advisory Committee on Mathematics Education [NACOME], 1975). The curricula developed as a result were dominated by attention to formal structure, properties, deductive proof, building of numeric systems relying heavily on sets, and on relations and functions with the major aim of linking school mathematics with university mathematics.

A few years after its implementation, however, there were complaints from parents and teachers about the abstract nature of the New Math curricula. This complaint was also fueled by students’ poor performance, and their lack of practical understanding of the mathematics that was being taught (Payne, 2003). These complaints and others led to the emergence of a different program - the Back to the Basics program to rectify the weaknesses of the New Math program.

**Back to the Basics**

By the beginning of the mid to the late 1970s, a new movement known as “Back to the Basics” movement emerged to address some of the problems identified in the New Math program. The growth of this program was fueled, among others, by the complaints from the business community that the existing curriculum was producing graduates that lacked simple mathematics skills needed in business (Confrey, 2007) and also that students were unable to compute accurately. Porter et al. (1991) described the Back to the Basics movement as “Guaranteeing basic skills became the agenda; easy content for all students” (p. 12). This Back to the Basics movement, however, was short-lived as it failed to equip many of its graduates with basic problem solving skills to be able to solve problems that arise in real-world situations.
**Problem Solving**

As a result of the failure of the *Back to the Basics*, two national reports (*An Agenda for Action*, and *A Nation at Risk*) were released in the 1980s advocating a new look at the school mathematics programs in order to “prevent the eroding educational foundations and threats to the future of the American society” (Gardiner, 1983, p. 5). Specifically, after a decade of focused attention on procedures and algorithms (*Back to the Basics*), the NCTM (1980) and other organizations (College Board, 1983; National Commission on Excellence in Education, 1983) issued reports calling for focus on *Problem Solving* in mathematics classrooms during the 1980s. Usiskin (1985) summarized these recommendations as follows: “Taken as a body, reports from inside and outside mathematics education agree almost unanimously that … emphasis should be shifted from rote manipulation to problem solving” (p. 15).

**NCTM Standards**

In 1989, the NCTM published *Curriculum and Evaluation Standards for School Mathematics*, and subsequent documents: *Professional Standards for Teaching Mathematics* (NCTM, 1991), *Assessment Standards for School Mathematics* (NCTM, 1995), and *Principles and Standards for School Mathematics* (NCTM, 2000) calling again for reform of mathematics education on a wide scale. Scholars concerned with the results of the United States students in international comparisons, for example, raised concerns that United States practices were insufficiently competitive in mathematics and science. The need for revising the mathematics curriculum found further support from students’ poor performance on internal assessments such as the NAEP, decreases in performance on the SAT, and decreases in the numbers of students entering scientific and mathematical fields.

NCTM provided recommendations, among others, for mathematical content that ought to
receive increased or decreased attention in the classroom, and outlined important mathematics processes such as problem solving and communication that should be encouraged and fostered as students do mathematics. NCTM has revised their curricula further by identifying *Curriculum Focal Points for Prekindergarten through Grade 8* (NCTM, 2006) and *Common Core Standards* for Mathematics to address the nation’s mathematics education needs.

**Methods**

*Research Design and Sample of the Study*

We employed content analysis (Krippendorff, 2004; Magid et al., 2000) to examine the presentation of variables in popular middle-grades mathematics textbooks selected from four eras of mathematics education in the United States. The middle-grade curriculum was chosen, in part, because it is regarded as a crucial stage for students’ transition from arithmetic to algebra (Herscovics & Linchevski, 1994; Kieran, 2004). Second, the extant literature reveals a lack of detailed knowledge on how the concept of variable is treated in middle-grades mathematics curriculum and the degree to which the treatment has changed over time.

*Popular* middle-grades mathematics textbooks for the “average-students” were chosen because we want to investigate the materials on the treatment of *variables* that were available to the majority of the students during each of the four eras of mathematics education in the United States in order to gain a broader perspective on the intended opportunities students have to engage with this vital concept in school mathematics.

*Specific Textbook Selection Criteria*

To be included in this study, the middle-grades mathematics curriculum must be classified as *popular* or have a large market share in United States middle schools during one of the four eras of mathematics education reforms described earlier. We drew on the work of Jones...
and Tarr (2007) that used textbook market share data from Weiss (1978, 1987) and Weiss et al. (2001), and on consensus from mathematics education experts (Glenda Lappen, Betty Phillips, Sharon Senk, Zalman Usiskin, Denisse Thompson, Douglas Grouws, Barbara Reys, and Robert Reys) who are familiar with the middle-grades mathematics curriculum to identify textbook series from each of the four eras of mathematics education reforms in the United States.

Using the results from Jones and Tar (2007), Table 2 identifies the mathematics textbook series we selected for examination with regards to their treatment of variable ideas.

Table 2: Eras in Mathematics Education, Timelines and Textbooks Selected

<table>
<thead>
<tr>
<th>Name of Era</th>
<th>Year</th>
<th>Textbook</th>
<th>Publisher</th>
</tr>
</thead>
<tbody>
<tr>
<td>New Math</td>
<td>1957-1972</td>
<td>Modern School Mathematics</td>
<td>Houghton Mifflin</td>
</tr>
</tbody>
</table>

Each of the selected textbook series was designed for grades 6, 7, and 8. The *Modern School Mathematics* textbook series were labeled 6, 7 and 8 respectively. The *Holt School Mathematics* textbook series were labeled Levels 64 – 76 (grade 6), Levels 77 – 91 (grade 7) and Levels 92 – 106 (grade 8). The *Mathematics Today* textbook series were labeled Brown (grade 6), Silver (grade 7), and Gold (grade 8). And the *Math Connects* textbook series were labeled as Course 1 (grade 6), Course 2 (grade 7), and Course 3 (grade 8). Thus, a total of 12 textbooks (three from each of the four textbooks series) were selected and examined in this study.

Unit of Analysis and Data Collection

The unit of analysis in this study was variable ideas. In particular, any task that contains the use of letter variables and fits any of the seven uses of variables described in the analytical
framework was examined. To identify these tasks, we examined each instructional page of the selected textbooks, beginning from the first page to the last page of each textbook. Specifically, we examined the uses of variables within the narratives, worked examples, assignments, exercises, activities, projects, chapter reviews, chapter summaries and chapter test blocks in the selected mathematics textbooks.

Through this process, we collected data on: a) the number of instructional pages in each textbook, b) the number of instructional pages containing variable ideas in each textbook, c) the presence, and the characteristics of the definition of variable provided in the textbooks, d) the frequency counts of the uses of variables of each type: as labels, constants, specific unknowns, continuous unknowns, varying quantities, generalized numbers, and abstract symbols, e) the content areas within which a particular use of variable was found, and f) the context within which variable ideas were explored (i.e., in solving real-world phenomena, in symbolic manipulations, in tables and graphs, whether technology was used to explore variable concepts, among others, in each of the 12 textbooks).

In doing the above counting, a set of variable tasks that build on one another was considered a single task. For example, questions 8-11 in Figure 1 was counted as single task. The same was true for questions 12-15 because these questions used the same values for the variables to evaluate similar expressions.

![Figure 1](image_url)

However, we counted the variable tasks that were very similar with one another with respect to what they include (see questions 40 and 41 in Figure 2) as different tasks since they may include multiple occasions for learners to experience this kind of opportunity.

40. A general rule for finding a man’s shoe size in the United States is to multiply the length of his foot in inches by 3 and then subtract 22. A formula describing this rule is $S = 3\ell - 22$, where $S$ is U.S. men’s shoe size and $\ell$ is the length of a man’s foot in inches. Nate’s foot is 11 inches long. Find his shoe size.

41. The formula $C = 0.6n + 4$ estimates the temperature in degrees Celsius when $n$ is the number of cricket chirps in 15 seconds. If a cricket chirps 25 times in 15 seconds, what is an estimate for the temperature?

Figure 2. Variables used as varying quantity, Transition Mathematics p. 127 Q. 40-41. Taken from The UCSMP: Transition Mathematics, 3rd Edition (2008), by Viktora, Cheung, Capuzzi, Usiskin, et al., published by Wright Group/McGraw Hill.

If a task contained more than one conception of variable, that task was multiple coded as containing as many uses of variables that was found in it. An example of such a task is shown in Figure 3. This task incorporates two conceptions of variables: one as a label (i.e., $r$, $s$ and $m$ to denote lines and angles in the task) and the other as a specific unknown (i.e., $x$) to be determined.

Figure 3. Variables used as a label and specific unknown, Transition Mathematics p. 396. Taken from The UCSMP: Transition Mathematics, 3rd Edition (2008), by Viktora, Cheung, Capuzzi, Usiskin, et al., published by Wright Group/McGraw Hill.
Using the data generated from the process described above, we calculated the proportion of the instructional pages on which variables were found in the textbooks, the proportion of the uses of variables in each of the seven categories of variables (as labels, unknowns, varying quantities, generalized numbers, and abstract symbols), and so on.

Reliability

A key ingredient in content analysis is the reliability of the coding procedure. To ensure the reliability of our coding, two coders performed check-coding (Miles & Huberman, 1994) on 25% of the pages in 6 of the 12 textbooks examined in this study. The reliability results for the number of variable pages and the proportion of variable categories are summarized in Table 3.

Table 3: Inter-Coder Reliability Estimates for Variable Pages and Variable Categories

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Agreement with Coder A</th>
<th>Agreement with Coder B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Initial</td>
<td>Final</td>
</tr>
<tr>
<td>Variable Pages</td>
<td>99%</td>
<td>100%</td>
</tr>
<tr>
<td>Variable Category</td>
<td>86%</td>
<td>100%</td>
</tr>
</tbody>
</table>

Based on Cohen’s guidelines for interpreting reliability estimates: $K < 0$ – no agreement; $0.0 \leq K \leq 0.2$ – slight agreement; $0.21 \leq K \leq 0.40$ – fair agreement; $0.41 \leq K \leq 0.60$ – moderate agreement; $0.61 \leq K \leq 0.80$ – substantial agreement; and $0.81 \leq K \leq 1.00$ – almost perfect agreement, the data in Table 3 indicates an almost perfect agreement between the two coders on the two criteria examined.

Results of the Study

Proportions of Variable Pages

Table 4 summarizes information on the number of instructional pages, the number of instructional pages that contain variable ideas, the proportion of pages that contain variable ideas (reported in percent), the first, and the last page of variable ideas in each of the 12 middle-grades mathematics textbooks examined. A cursory examination of the table indicates that variable ideas
were found in each of the 12 mathematics textbooks examined. The variables were used mainly in these curricula to solve equations and inequalities, to develop formulas, to label lines and angles, and to represent functions.

Table 4: *Instructional Pages, Variable Pages, and Proportion of Pages of Variable Ideas*

<table>
<thead>
<tr>
<th>Textbooks</th>
<th>Grade</th>
<th>No. of Inst. Pages</th>
<th>No. of Var. Pages</th>
<th>Percent of Var. Pages</th>
<th>First Var. Page</th>
<th>Last Var. Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modern School Mathematics</td>
<td>6</td>
<td>341</td>
<td>234</td>
<td>69</td>
<td>2</td>
<td>341</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>523</td>
<td>288</td>
<td>55</td>
<td>8</td>
<td>526</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>492</td>
<td>323</td>
<td>66</td>
<td>7</td>
<td>492</td>
</tr>
<tr>
<td>Holt School Mathematics</td>
<td>6</td>
<td>354</td>
<td>111</td>
<td>31</td>
<td>26</td>
<td>353</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>386</td>
<td>173</td>
<td>45</td>
<td>2</td>
<td>383</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>386</td>
<td>206</td>
<td>53</td>
<td>18</td>
<td>385</td>
</tr>
<tr>
<td>Mathematics Today</td>
<td>6</td>
<td>392</td>
<td>86</td>
<td>22</td>
<td>40</td>
<td>388</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>440</td>
<td>264</td>
<td>60</td>
<td>12</td>
<td>439</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>438</td>
<td>248</td>
<td>57</td>
<td>5</td>
<td>437</td>
</tr>
<tr>
<td>Math Connects</td>
<td>6</td>
<td>669</td>
<td>304</td>
<td>45</td>
<td>22</td>
<td>669</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>665</td>
<td>308</td>
<td>46</td>
<td>35</td>
<td>665</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>665</td>
<td>411</td>
<td>62</td>
<td>29</td>
<td>665</td>
</tr>
</tbody>
</table>

Information from the table also reveals that different middle-grades mathematics curricula employed variable ideas in different proportions. The proportion of pages containing variable ideas ranged from 22% to 69% with the mean of 51% and standard deviation 10.6 for the entire sample of textbooks examined. Overall, the *Modern School Mathematics* curriculum selected from the *New Math* era (1957-1972) recorded the highest proportions of pages containing variables, while the *Holt School Mathematics* curriculum selected from the *Back to the Basic* era (1973-1983) has the least proportions of pages containing variables.

The first page of variable ideas occurred within the first quarter of the instructional pages in all the textbooks examined. In some of the textbooks (e.g., *Modern School Mathematics* grade
6, and *Holt School Mathematics* grade 7), variable concepts appeared as early as on the 2nd instructional page. Similarly, the last page of variable ideas occurred within the fourth quarter of the instructional pages in all the textbooks examined, and as late as on the last instructional page of the majority of the textbooks. There was less use of variable ideas in the 6th-grade textbooks than in the 7th- and 8th-grade textbooks. There were no major observed differences between the frequencies of use of variables in the 7th- and 8th-grade textbooks.

*Introduction and Definitions of Variable by Curricula*

With the exception of the *Math Connects* curriculum, there was no formal introduction of the concept of variables in the other three mathematics curricula examined. Specifically, variables were simply introduced in these curricula by using them. Formal definitions for variables were found in 11 of the 12 textbooks examined. The exception was the 6th grade textbook of the *Mathematics Today* series that did not provide a formal definition for variables.

The length of the definitions (and the accompanying explanations that followed), ranged from few words: e.g., “in the equation \(x + 7 = 12\), \(x\) is a variable, the solution is 5, because \(5 + 7 = 12\) is true” (*Holt School Mathematics Grade 7*, p. 53) to paragraphs - “A variable is a symbol, usually a letter, used to represent a number. The expression \(2 + n\) represents the sum of two and some number. Any letter can be used as a variable. The letter \(x\) is often used as a variable. It is also common to use the first letter of the value you are representing.” (*Math Connects*, Grade 6, p. 42).

The definitions of variables found in different mathematics curricula were different from one another. In particular, the four mathematics curricula examined defined variables in three different ways: the *Modern School Mathematics* from the *New Math* era defined variable in terms of its use as generalized numbers. In contrast, the *Holt School Mathematics* from the *Back
to the Basics and the Math Connects from NCTM Standards eras defined variables as specific unknowns. The definition found in the Mathematics Today curriculum selected from the Problem Solving era identified variable with its use as continuous unknowns.

Except for the 8th-grade textbook of the Math Connects curriculum that defined variables prior to using them, all the other textbooks examined used variables prior to providing a formal definition for them. In addition to the definitions offered, a few of the textbooks explained how variables are used. These explanations were given in the contexts of equation solving and/or writing mathematical expressions. It is also worth stating that none of the textbooks provided any historical information about the origin of variables. Table 5 reports the presence of definitions, and explanation on the uses variable ideas in the textbooks.

Table 5: Presence of Definitions and Explanation of Variable ideas in Middle Grade Mathematics Textbooks from the Various Eras

<table>
<thead>
<tr>
<th>Math Ed Era</th>
<th>Definition</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6th</td>
<td>7th</td>
</tr>
<tr>
<td>New Math</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Back to Basic</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Problem Solving</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>NCTM Standards</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

*Proportions of Variables Categories by Mathematics Curricula*

Across all curricula, the use of variables as a specific unknown and label dominated the uses of variables in the middle grades mathematics textbooks examined. As can be seen in Figure 4, the use of variables as specific unknown constituted 37%, and the use as label constituted 31% in the Modern School Mathematics curriculum. These percentages of usage were similar in the other three curricula examined: 43% and 28% in the Holt School Mathematics curriculum, 37% and 33% in the Mathematics Today curriculum, and 33% and 19% in the Math Connects
curriculum respectively for *specific unknown* and *label* categories. Thus, the uses of variables in these two categories constituted a combined percentage of approximately 70% of all the uses of variables in the middle-grades mathematics curricula examined. We also found that the constitutions of the *label* and *specific unknown* categories were higher in the 6th- and 7th-grade textbooks than they were in the 8th-grade textbooks.

Another category of use of variable that stood out among the five remaining uses of variables was that of *varying quantities*, constituting 12%, 13%, 17% and 24% respectively, as shown in Figure 4. The least proportion of use of variable in the curricula was that of *abstract symbol* category, appearing only in two (Modern School Mathematics and Math Connects) of the four mathematics curricula examined.

When examined by content areas, the use of variables as a *specific unknown* dominated number and operations, as well as algebra contents. In geometry, measurement, and data analysis and probability content areas, the use of variables as a *label* was dominant. In addition, there was a high usage of variables as *varying quantities* in geometry than in the other content areas.

For the most part, the order of occurrence of variable ideas in the selected textbooks was - *labels, specific unknowns, continuous unknowns, generalized numbers, and varying quantity*. More specifically, the use of variables as continuous unknowns, and varying quantities occurred
mostly towards the final quarter of the instructional pages in majority of the textbooks examined.

*Trends in the Treatment of Variables*

*Changes in the proportions of pages containing variables:* As can be seen in Table 4, the overall proportions of pages containing variable ideas in the textbooks from the respective curricula did not follow any systematic pattern; they were the highest in the curriculum selected from the *New Math* era, became relatively low in the curricula selected from the *Back to the Basics* and *Problem Solving* eras, and then got relatively high again in the curriculum selected from the *NCTM Standards* era.

*Changes in the definition of variable:* In terms of the definition of variable, the data revealed that, even though the characteristics of the definitions of variable differ by mathematics curricula, there was no specific pattern to the differences observed: the definition of variables changed from its conceptualization as *generalized numbers* in the textbooks selected from the *New Math* era, to their conceptualization as *specific unknowns* in the textbooks selected from *Back to the Basic* era, to the characterization of variables as *continuous unknowns* in the textbooks selected from *Problem Solving* era, and back to variables being viewed as *specific unknowns* in the textbooks selected from *NCTM Standards* era.

*Changes in the proportions of variable categories:* When examined as a whole, the data did not reveal any drastic change in the proportions of use of variable categories over the 50 year period in the middle-grades mathematics textbooks. To be specific, the middle-grades mathematics curriculum consistently employed variable ideas predominantly as *specific unknowns* and *labels*. We, however, observed a steady increase in the proportion of use of variables as *varying quantities* across the four curricula over the 50 years period (see Figure 4).

*Trends in the use of variable by content area:* Overall, the data revealed that algebra and
geometry contents employed variable ideas the most in the presentation of their contents, whereas measurement and data analysis and probability contents employed variable ideas the least. In addition, we observed a sharp decrease in the use of variables within number and operations content area across the four eras of mathematics education. On the contrary, there was a sharp increase in the use of variables in algebra content area during the same period. There were, however, no consistent changes observed in the uses of variables in geometry, measurement, and data analysis and probability content areas during the same period.

![Figure 5: Proportions of Uses of Variables in Different Content Areas by Eras](image)

**Influences of Reform Recommendations:** Apparently, reform recommendations in the respective eras seemed to have a direct influence on the materials covered in the respective curriculum, and in turns, influenced the treatment of variable ideas in these curricula. For example, the presentation of variable ideas in the Modern School Mathematics curriculum (*New Math* era) was at a relatively higher level of abstraction than was the case in the other three curricula examined. Similarly, the Math Connects curriculum (*NCTM Standards* era) records a relative increase in the use of variables to model “real-world problems”, employed technology to explore variable ideas, and used concrete materials to represent variables ideas.

**Notable Changes in the Treatment of Variables in the Math Connects Curriculum:** There were some noticeable differences in the treatment of variable ideas in the Math Connects
curriculum (NCTM Standards era) that are worth commenting on. First, the context characteristics of the majority of tasks employing variable ideas in this curriculum relate to the real-world situations of the learners. In the 6th-grade textbook, for example, students were asked to create a real-world problem in which they would solve the equation \(a + 12 = 30\) (Math Connects, Grade 6, p. 60, question 38), or to “write about a real-world situation that can be represented by equations” such as \(y = 5x\) and the “explain what the variable represents in the situation” (Math Connects, Grade 6, p. 353, question 24). This was generally not the case in the treatment of variable ideas in the other three curricula. Second, the Math Connects curriculum used manipulative (cups and counters, see Figure 6) to represent variables in developing meaning for inverse operations in equation solving situations.

Figure 6. Solving equations using models (Math Connects, Grade 7, p. 134).

In these activities, the cups were used to stand for the unknown while the counters represent the numbers. In addition, it was only the Math Connects curriculum in the entire sample of textbooks examined that employed technology to explore variable ideas, distinguished
between *independent* and *dependent* variable and showed evidences of discussing research issues regarding students’ misconceptions about variables.

**Discussions**

This study investigated the treatment of variables in popular middle-grades mathematics textbooks, and the extent to which the presentation has changed during a 50 year period. The results of this study indicated that each of the middle grades mathematics curricula examined employed variable ideas, but in different proportions. Overall, the *Modern School Mathematics* curriculum records the highest proportion of variable pages, while the *Holt School Mathematics* series accounts for the least proportion of variable pages in the sample. The nature of variable ideas found in the *Modern School Mathematics* curriculum also seemed to be relatively more complex than those found in the other three curricula. This might be due to the influences of the reform recommendations of the *New Math* era, which among others, called for the creation of curricula dominated by attention to formal structure, properties, deductive proof, and building numeric systems relying heavily on the ideas of set, relation and function with the major aim of linking school mathematics with university or higher mathematics as soon as possible.

Formal definitions for variable were found in 11 of the 12 textbooks examined. The definitions were, however, different. To be more specific, the four mathematics curricula employed three different definitions of variables. This finding is consistent with Schoenfield and Arcavi’s (1988) report that different textbooks, different researchers, and even different experts describe the concept of variables in different ways. Accordingly, many mathematics education researchers have not accepted a common definition for this fundamental concept. Given this lack of consensus, curriculum designers find themselves in the position to choose among the available definitions of variable to use in their materials. The type of definition they choose influences the
treatment of variable ideas in their curriculum, and hence determines the learning opportunities
available to students using these curricula.

Although the set of middle grades mathematics curricula examined employed all the
seven uses of variables in school mathematics, the results indicated that these curricula provided
far more opportunities for students to engage with variables as *labels* and *specific unknowns*,
than they provided for students to learn about the other uses of variables. It was also observed
that the use of variables as *specific unknowns* and *labels* occurred at the beginning of the
textbooks whilst the use of variables as *varying quantities* and *continuous unknowns* generally
occurred towards the third quarter of the textbooks examined. An important implication of these
findings is that students who are taught mathematics using these curricula will more likely
develop the conceptions of variables as *labels* and *specific unknowns* than they will with the
other uses of variables in school mathematics. The findings may shed light on possible links
between students’ understanding of variable and its presentation in the curriculum. Specifically,
the relatively large emphasis on the use of variables as *specific unknowns* in the majority of the
curricula examined could explain the finding reported repeatedly in the research literature that, in
spite of the many uses of variables in school mathematics, the majority of students think of
variables as representing a specific unknown – a *placeholder* (Kieran, 1992; Mohr, 2008).

Opportunities for students to engage with variables as *abstract symbols* were even more
limited in the textbooks examined. Specifically, only two curricula (*Modern School Mathematics*
and *Math Connects*) provided opportunities for students to engage with the use of variables as an
*abstract symbol*. This finding supports Usiskin’s (1988) claim that the least used category of
variable in school mathematics is that of an abstract symbol. In fact, the abstract symbol category
was employed sparingly, even in those curricula in which it was used. This might be due to the
fact that many of the concepts that employ variables as an abstract symbol are not of major focus in the middle-grades mathematics curriculum.

Over 30 years ago, Küchemann (1978, 1981) articulated four hierarchies of students’ understanding and use of variables. In this hierarchy, Küchemann placed the use of variables as labels and placeholders in level 1 and level 2 (the lowest levels of his classification). A variable that varies (varying quantity) is considered to be of a higher level of formality than the variable as generalized number or continuous unknown, which are again more formal than a variable as a placeholder. At the top end is the use of variable as an arbitrary symbol (Herscovics & Linchevski, 1994; Filloy & Rojano, 1989). The results of this study, however, indicated that the uses of variables that are prevalent in the middle-grades mathematics curricula examined are those at the lowest level of Küchemanns’ hierarchy (labels and specific unknowns/placeholders).

It can therefore be argued that, if these curricula are good representation of the materials that the majority of middle grades students learn from, then the conception of variables that these students will likely develop are those of low levels of use of variables, which according to Gray, Loud and Sokolowski (2005) will not be very useful to them when learning advanced mathematics, unless the curricula they use in the future grade levels provide more opportunities for them to engage with the advanced uses of variables. The data, however, showed a steady increase in the use of variables as a varying quantity across grade levels and across mathematics education eras, while the use of variables as labels decreased on the same measures.

Kieran (1981) observed that despite the importance of the concept of variable, many mathematics curricula continue to discuss the concept of variables as if it is a simple term. This observation was true in the treatment of variable ideas found in the textbooks examined. For example, even though almost all the textbooks defined variables, very limited explanation were
provided to help students understand variables were provided in the curriculum. Moreover, none of the curricula presented the concepts of variables in ways that students can clearly understand the richness of this concept without being overwhelmed by its complex nature. We agree with Wagner (1981) that, if we want students to gain an appreciation of the power of variables and yet not be overwhelmed by its complex nature, then textbook developers need to explicitly and carefully introduce students to the various uses of variables in school mathematics.

The results of this study also suggest that reform recommendations do influence the content of textbooks to some extent. For example, the Back to the Basics era recommended a shift in emphasis to the learning of the basics. Consequently, the Holt School Mathematics selected for that era treated variables at the most basic level, when compared to the treatment of variables in the other three curricula examined. Similarly, in conjunction with the NCTM (2000) reform recommendations that advocate the use of manipulative materials to build mathematical understanding, and called for increased emphasis on studying math within real-world context, the Math Connects curriculum employed variables to model real-world situations, and used manipulative materials to develop variable ideas, among others. These observations seemed to establish possible links between reform recommendations and the treatment of topics in the mathematics curricula from the respective eras, and in particular, the treatment of variable ideas.

Clearly, the research we report in this study is not exhaustive. However, we believe that our analysis of popular middle-grades mathematics textbooks selected to span a 50 year period of mathematics education in the United States provides some important insight into the state of the treatment of variable ideas in middle-grades mathematics curriculum. Future research can expand on this work to help the mathematics education community understand better the issues related to using variables in school mathematics.
List of References


