TURKISH PRIMARY SCHOOL STUDENTS’ STRATEGIES IN SOLVING A NON-Routine MATHEMATICAL PROBLEM AND SOME IMPLICATIONS FOR THE CURRICULUM DESIGN AND IMPLEMENTATION

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ABSTRACT. Turkish primary mathematics curriculum emphasizes the role of problem solving for teaching mathematics and pays particular attention to problem solving strategies. Patterns as a subject and the use of patterns as a non-routine problem solving strategy are also emphasized in the curriculum. The primary purpose of this study was to determine how primary school students who learn mathematics in this context approached non-routine problems. The secondary purpose of this study was to discuss how effective this context could be for non-routine problem solving and to develop some perspectives for curriculum design and implementation. With these purposes, a rich problem in terms of problem solving strategies was employed and eight 6th grade students (12-13 years old) participated in the study. The problem solution attempts made by the students for 5 weeks indicated that their non-routine problem solving strategies were quite poor and inflexible. The results of the study also showed that the students used the pattern-seeking strategy as a simple search for regularity. These results were discussed in terms of both internal and external consistency problems to be taken into account when designing and implementing a curriculum.

KEY WORDS: problem solving, problem solving strategies, non-routine problem, patterns, 6th grade.

INTRODUCTION

Problem solving has been considered as a main activity in teaching mathematics and many studies on problem solving have been carried out (e.g., Schoenfeld, 1985; Silver, 1985; Charles & Silver, 1989; Schoenfeld, 1992). However, the questions of how effectively different problem solving strategies are employed by students and what sort of contribution or harm can the emphasis placed on problem solving bring in to mathematics education are still relevant. In this study non-routine problem solving strategies of students, and particularly, how they use the pattern seeking strategy are analyzed. Students’ approaches are analyzed in the light of the curriculum’s approach to problem solving and pattern use.

In this section, firstly, the concepts of problem, problem solving and non-routine problem solving will be addressed. Then, how the pattern concept is used as a problem solving strategy will be examined.

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Problem, problem solving and non-routine problem solving

While various definitions are given in the relevant literature, a problem is generally expressed as a situation which an individual has to solve but does not know the way to solve it (Charles & Lester, 1982; Chi & Glaser, 1985; Hiebert et al., 1997; Zeits, 2007). Problem solving is a series of cognitive activities to get out of this situation and achieve the determined objective (Cooper, 1986; Schoenfeld; 1989; PISA, 2003). Problem solving activity requires the use of some types of knowledge. These can be classified into various categories as strategic knowledge, procedural knowledge, and semantic knowledge (Mayer, 1992; Mayer & Hegarty, 1996). The individual may already have this knowledge required for solving the problem, or may acquire it during the problem solving stage. Polya (1945) describes the problem solving activity as a four-step process. These steps are understanding the problem, devising a plan, carrying out the plan and looking back. Knowing and applying these steps can facilitate the solution of the problem but does not ensure it. Thus, the problem solving strategies known or used by the individual become crucial. The oldest strategy used in problem solving is trial-error. Trial-error strategy is the repeated and different attempts to solve a problem. Trial-error is not a systematic approach adopted when faced with a problem, but applied in situations where nothing is known about the solution of the problem or nothing can be hypothesized (Van Der Linden et al., 2001). Systematic exploration can be considered as the opposite of the trial-error strategy. Systematic exploration is an approach that involves steps like hypothesis testing, planning, and assessing the results of an action (Van Der Linden et al., 2001). The fundamental characteristic of human problem solving approach is using heuristic strategies (Lucas, 1974). Heuristic strategy is a complex process that is applied to solve a problem instead of applying a known algorithm or trying to come up with a solution through trial and error (Romanycia & Pelletier, 1985). It is Polya (1945) that developed heuristics as a method. Polya describes some heuristic strategies that can be used in each of the problem solving stages mentioned above. Some of these strategies are: looking for a pattern, drawing a picture or diagram, making a model, using a formula, making a table, using logical reasoning, working a simpler problem first, and making a list.

The most important factor to determine which strategies will be used is through which strategies the problem can be solved, in other words, what the nature of the problem is. Various types of problems defined in the literature are open-ended and context-based (Arsac et al., 1991), complex (Crahay, 2005), non-routine (Nancarrow, 2004) and real-life problem (Maasz & O’Donoghue, 2011). However, the most frequent distinction made between
problems is classifying them as routine and non-routine. Non-routine problems are those where the method of solution is not obvious and more thinking is necessary compared to routine problems (Polya, 1945). It is pointed out in the literature that solving non-routine problems is a cognitively complex task requiring the use of different strategies and the use of learned procedures in unfamiliar ways (Nancarrow, 2004; Kolovou, 2011). In contrast with routine problems, which usually have a straightforward solution involving routine calculations, non-routine problems also require creative thinking and the use of some heuristic strategies (Elia, van den Heuvel-Panhuizen & Kolovou, 2009). In respect to this, Elia et al. (2009) stress that flexibility in using heuristic strategies in non-routine problem solving is important and this flexibility is related to performance. They define strategy flexibility as the behavior of switching strategies and distinguish two kinds of strategy flexibility: intra-task strategy flexibility, which occurs during the solution of a problem and inter-task strategy flexibility, which occurs between problems.

Studies show that students, at all grades, do not sufficiently master skills required to solve mathematical application problems (Schoenfeld, 1992; De Corte, Greer & Verschaffel, 1996) and have difficulties in using effective strategies when confronted with non-routine problems. Many students do not spontaneously, or not at all, apply heuristic strategies (Lester, Garafolo & Kroll, 1989; Schoenfeld, 1992; De Bock, Verschaffel & Janssens, 1998), and students’ non-routine problem solving strategies are inflexible (Elia, van den Heuvel-Panhuizen & Kolovou, 2009).

Consequently, in recent years, the question of how students’ problem solving skills and strategies can be improved has received significant attention. For instance, The National Council of Teachers of Mathematics (NCTM) standards place problem solving in the center of mathematics teaching and define mathematical activity as problem solving, reasoning and proving, communicating, connecting and representing (NCTM, 2000). Many other countries seem to have adopted a similar approach for their own curricula (e.g., Ministry of National Education (MONE) Ontario, 2005; MONE Singapore, 2006; MONE Turkey, 2009a). It appears that different types of problems (open-ended, context-based, complex, non-routine, real-life problem) have been integrated into the objectives of these curricula.

**Pattern as a problem solving strategy**

The issue of “pattern” has been drawing a great deal of interest in the field of mathematics education. It is a popular view among mathematics educators that the understanding and the use of patterns are important in the study of mathematics and patterns play a key role in
teaching many concepts (Reys et al., 1984; Schoenfeld, 1992; Orton, 1999; Devlin, 2003; Vogel 2005). Nevertheless, there is no clear, agreed upon definition of “pattern”. Mulligan (2010) defines a mathematical pattern as “any predictable regularity, usually involving numerical, spatial or logical relationship” (p.48). Resnik (1981) provides another definition for pattern:

[Pattern] is a complex entity consisting of one or more objects, which I call positions, standing in various relationships (and having various characteristics, distinguished positions and operations). A position is like a geometrical point in that it has no distinguishing features other than those it has in virtue of being that position in the pattern to which it belongs (p. 532).

Instead of giving a definition, Orton (1999) prefers to explain the pattern concept, but admits that, since the word ‘pattern’ has many meanings, it is not easy to do so:

It is not easy to define what we mean by pattern even in mathematics. One of the difficulties is that the word has a variety of different meanings. On the one hand, ‘pattern’ can be used simply in relation to a particular disposition or arrangement of shapes, colours or sounds with no obvious regularity. Indeed, sometimes the arrangement might form a recognizable representation or picture. On the other hand, it might be required that the arrangement possesses some kind of clear regularity, perhaps through symmetry or repetition. In mathematics we more often than not use the word pattern in relation to a search for order, so regularity is more likely than not. (p. vii)

The quote above from Orton shows that whether regularity is necessary for pattern is not clear, and indeed the meaning of the word ‘regularity’ itself is vague. While the definitions vary, the researchers agree that teaching patterns can help students make the transition from arithmetic to algebra, support the development of algebraic thinking and functional relationship, enable students to make mathematical connections and develop problem solving strategies (Tall, 1992; Herbert & Brown, 1997; Hargreaves, Shorrocks-Taylor & Threlfall, 1999; Zazkis & Liljedahl, 2002; Van De Walle, 2004; Papic & Mulligan, 2005; Warren & Cooper, 2006; Dekker & Dolk, 2011).

“Looking for a pattern” is described as a heuristic problem solving strategy in Polya’s study (1945). Looking for a pattern can be described as seeking for a relationship or rule among the data or events in the problem. In this sense, looking for a pattern (shortly pattern seeking), is a part of inductive reasoning. Inductive reasoning is a process of reaching a general conclusion through repeated observations of specific events (Chalmers, 1976). In mathematics, inductive reasoning can be defined as a type of reasoning that is based on the
principle of analyzing different situations, discovering patterns and coming up with a conclusion (Bello, Britton & Kaul, 2009). Finding and using pattern is considered as an important strategy for mathematical problem solving (Stacey, 1989). Therefore, Stacey states that many curriculum materials include problems where specific situations are analyzed, the results are systematically organised, patterns are discovered and these patterns are used to find the solution. These problems usually have numerical or geometrical patterns. While the relevant studies generally group the pattern seeking strategies into two categories as recursive strategies where the relationship among the different terms of pattern is sought, and the explicit strategies where the relationship among the variables are sought (Stacey, 1989; Orton & Orton, 1999), many different strategies are also described as well.

The strategies that students use in pattern seeking and generalizing the patterns that they have found have been the focus of many studies (e.g., Lee & Wheeler, 1987; Stacey, 1989; Threlfall, 1999; Orton & Orton, 1999; Zazkis & Liljedahl, 2002). These studies, which often explicitly ask students to find and generalize patterns, found that they have a lot of difficulties in doing this. For example, Lee and Wheeler (1987) state that most students are not flexible when solving a problem, they cannot go beyond the first pattern they see, and they do not check whether the pattern rule they have found works or not. Orton and Orton (1999) found that students are able to see multiple patterns in a given situation, but cannot decide which one to pick for generalization. Stacey (1989) found that when students generalize the patterns of linear relationships, they make over-generalization errors, and there are inconsistencies in the methods that they use for similar tasks.

Therefore, patterns have an important place in mathematics education and pattern seeking is a valuable problem solving strategy. However, it is obvious in most studies that it is necessary to use pattern seeking strategy to solve the problems; otherwise, students are explicitly asked to solve the problems by using this strategy. Nevertheless, many strategies that can be used to solve non-routine problems are either previously unknown, or these problems are unsolvable with a single strategy. This raises the question of how students will approach non-routine problems where pattern seeking and pattern generalizing are not easily seen or not helpful when used alone.

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Curricular perspectives on problem solving

In mathematics education literature, problem solving is often examined from a cognitive point of view and curricular perspectives on problem solving have rarely been taken into account. However, some studies emphasize that problem solving should also be considered from a curricular perspective. Problem solving is dealt with differently in different cultures and the approach towards problem solving in the curriculum of a country may change over time (Artigue & Houdement, 2007; Doorman et al., 2007; Törner, Schoenfeld & Reiss, 2007). This may explain the observation of differences in both teacher practices and student achievements. Furthermore, it is a frequently observed phenomenon that ambitious curricular changes introduce internal consistency and external consistency problems. For a curriculum, having internal consistency means that the curriculum has logical contingencies among its elements. Having external consistency means that the vision and approaches of the curriculum are understood similarly by all the stakeholders (curriculum developers, practitioners, supervisors, etc.) of the education system (Kessels & Plomp, 1999). Ambitious curricular changes often create a gap between curricular expectations and what can be effectively understood by textbook authors and teachers (Artigue & Houdement; 2007, Erdogan, 2014). When teachers cannot fully realize the expectations of a curriculum, they only follow recommended practices with prescribed sequences and instructional format (Ben-Peretz, 1990; Choppin, 2011). Consequently, analyzing curricular questions like how problem solving is included in a curriculum, how consistent the curriculum approaches are towards problem solving, and how these approaches affect mathematics education is thought to be important.

Purpose of the study

In this study, the idea that pattern seeking is a part of inductive reasoning, and that non-routine problem solving requires conscious and systematic use of various strategies is adopted. Based on the emphasis placed upon problem solving and pattern seeking as a strategy of problem solving in curricula and literature, this study focuses on students’ non-routine problem solving strategies and aims to answer to these questions:

- What strategies do students apply when trying to solve a non-routine problem?
- If pattern seeking comes up as a non-routine problem solving strategy, how do students use this strategy?
The Turkish primary mathematics curriculum was significantly modified in 2005\(^2\). Central concepts within this curriculum are non-routine problem solving and patterns. After the introduction of the curriculum, there have been many studies regarding both of these concepts. Most of these studies have investigated students’ approaches and strategies, and a few studies have questioned this curriculum from a curricular perspective and analyzed its content and coherence (e.g., Babadogan & Olkun, 2006; Ubuz et al., 2010; Incikabi, 2011). Based on students’ approaches, another aim of this study was to discuss how effective the context offered by the curriculum for non-routine problem solving could be and to develop some perspectives for curriculum design and implementation.

**BACKGROUND INFORMATION**

*Problem solving, problem solving strategies and patterns in Turkish primary mathematics curriculum*

Turkish primary curricula (1-8 grades), including subjects such as mathematics, Turkish, science and technology, and social sciences, underwent significant changes in 2005. The reformed curricula, widely influenced by the constructivist approach, aimed to ensure student-centered learning and meaningful teaching with new pedagogies and methods. Improving students’ problem solving skills is mentioned as one of the main common objectives in these curricula. Problem solving skill is defined as the ability to solve problems encountered in daily life covering sub-skills, such as understanding the problem, identifying the steps, making a plan, and using strategies.

With regard to Turkish primary mathematics curriculum, problem solving constitutes one of the main objectives of the curriculum and almost five pages are devoted to problem solving in the introductory document of the mathematics curriculum. The role of the problem solving activities for mathematics teaching, teacher’s and students’ respective positions and responsibilities in implementing and carrying out problem solving activities, and the use of problem solving strategies are explained in detail and exemplified in the introductory document. In this document, mathematical problems are defined in the meaning of unfamiliar mathematical situations requiring the skilful use of different skills and knowledge. It is stressed that “problem solving is not a subject but a process. In this process, the mathematics curriculum aims at acquisition of problem solving skills and this acquisition have a great importance” (MONE Turkey, 2009a, p.13). It is therefore expected that the teacher create an

\(^2\) Fundamentally changed in 2005, the curriculum was revised again in 2009 and 2013. Since it was not in effect at the time of the data collection for this study, the 2013 curriculum is not taken into account.
environment in which students can solve a given problem with different ways, share freely their ideas with classmates and their teacher. In line with the common core skills (problem solving, reasoning and proof, communication, connecting and representing), the mathematics curriculum also encourages the teachers to discuss and explain how a problem is solved and represented and how the representation chosen facilitated the solution.

As for problem solving strategies, it is underlined that students should use different strategies and represent them differently, explain these strategies and additionally examine what happens if some of the given conditions are changed. Furthermore, 16 problem solving strategies, largely influenced by Polya (1945), were listed (trial and error, looking for a pattern, simplification, anticipation, drawing a diagram, etc.) in the mathematics curriculum.

This emphasis on problem solving is accompanied by a vision of mathematics, which claims “mathematics is the science of patterns and regularities” (MONE Turkey, 2009a, p.7 and 44). The word “pattern” is used more than 50 times in the introductory document of the mathematics curriculum. In accordance with the literature, patterns are introduced as a subject of teaching and, “pattern-seeking” is mentioned as an effective problem solving strategy.

Discovering the relations in patterns and generalising them help students better develop their skills for a better understanding of the world around them. Furthermore, different representations of patterns, especially symbolic representation, will largely contribute to the acquisition of fundamental concepts of algebra (MONE Turkey, 2009a, p.98).

When relations between numbers are studied, activities such as constructing a number pattern, finding the rule of a number pattern and explaining this rule should be organized. Activities such as anticipating the following items in a number pattern and explaining how this anticipation is made contribute to the development of reasoning and communicating (MONE Turkey, 2009b, p.23).

Thus, from the 1st grade (7 years old) up to 8th grade (14-15 years old), patterns are progressively introduced in geometry sections under the subsection “patterns and tessellations”: finding the relation in a pattern and determining the missing item in a pattern (1st grade), constructing a given pattern with different materials (2nd grade), paving the plane with triangles and rectangles, (3rd grade) and so on. Number patterns are introduced from the 2nd grade up to 5th grade and, algebraic representations of patterns are introduced in algebra section under the subsection “patterns and relations” from the 6th grade up to 8th grade.
As a conclusion, we observe that Turkish primary mathematics curriculum places problem solving at the centre of mathematics teaching and gives a privileged place to patterns as a subject of teaching. It seems that by combining problem solving activities with the teaching of patterns (see, the excerpts analyzed in Appendix), Turkish primary mathematics curriculum hopes to offer favourable conditions to improve students’ problem solving skills and strategies.

**METHODOLOGY**

The study has its origin in our observations described above, related to the place of problem solving activities and patterns in Turkish primary mathematics curriculum. Thus, a non-routine problem, which is called “tiling a kitchen”, was chosen for its potential with regard to the problem solving strategies and the use of patterns. The problem was designed for 6th grade students (12-13 years old) and eight voluntary students from different sociocultural and educational backgrounds were involved. Out of school time, the students worked on this problem for 2 hours in a week, for five weeks. At the beginning of the activity, a notebook with squared sheets was distributed to all students. The students were asked to keep writing any kind of solutions, comments and ideas chronologically and not to erase their work at all. The students’ work was observed without any intervention in terms of knowledge and strategies. Students’ sheets and the notes taken during the observations of the sessions were analyzed and students’ solutions and strategies were identified according to two steps of the problem described below.

*Tiling a kitchen*

The problem was introduced as follows:

Let us suppose that we have a rectangular kitchen and we want to tile our kitchen. We would like to have very special tiles, they are very expensive but we can afford it. We go to a tiles store, which sells very special tiles but only in the shape of dominoes (two small adjacent squares, Figure 1-a).

So, we wonder which kinds of the kitchens we can tile with these tiles. Of course, we should not break tiles since they are expensive.

Once you think that you reached a solution, what can you say about tiling a rectangular kitchen with tiles in shape of L (Figure 1-b).

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(a) ![Tile Shape A](image1.png)  
(b) ![Tile Shape B](image2.png)

*Figure 1. The types of tiles*
This is a specific kind of non-routine problems named as Research Situation for the Classroom (RSC). The following criteria are given to characterize a RSC (Grenier & Payan, 2003; Grenier & Godot, 2008):

1) A RSC belongs to the professional research domain. It must be close to unsolved problems,
2) The initial problem should be easy to understand,
3) There should be basic strategies without specific prerequisites,
4) Several strategies in the research and several ways to advance should exist; this is the case for the activity (construction, proof, calculation) and the mathematical concepts as well,
5) A solved question should lead to new questions.

The RSC “Tiling a kitchen”, which has been experimented many times (Grenier & Payan, 1998; Condat et al., 2006; Deloustal-Jorrand, 2008; Grenier & Godot, 2008), is now a classical RSC. Below we explain how this RSC meets such criteria and why this is an appropriate problem for our study:

1) This is a problem related to the paving of the plane. As the paving is the covering of a surface without overlapping or overflow, the general mathematical question is that: which kind of surfaces can be tiled with identical tiles? It is an open-ended question and remains unsolved because it is possible to change a given surface and the shape of tiles at will. In this problem, the students are asked only to work on rectangular surfaces and explore whether these surfaces can be tiled with only two types of tiles. Thus, the problem is reduced to a level that can also be explored by primary school students.

2) The initial problem is easy to understand, as the formulation of the task does not contain any specific mathematical words. In addition, the task is quite familiar for Turkish students because they encounter similar formulations when they study patterns. In geometry sections under “tessellations” title, they also have to solve numerous problems related to the paving of the plane.

3) Basic strategies exist. By trial and error, tiling small kitchens with both dominoes and L-shaped tiles constitutes a basic strategy. With this strategy, students can get an idea about which kitchens can be tiled and put some hypotheses forward. Especially, tiling kitchens like 1x2, 1x3, 2x2, and 2x3 with dominoes they can discover that the multiplication of a kitchen dimensions must be even to be tiled (Figure 2).
For the tiling with L tiles, it is possible through trial-error to have an idea about whether kitchens with dimensions like 2x2, 2x3, 3x3 can be tiled or not (Figure 3).

However, since there is not always a systematic relationship among the changing dimensions, deciding whether bigger dimensioned kitchens can be tiled or not with L tiles requires using more sophisticated strategies.

4) Several ways of tiling and strategies are possible to employ: After simple cases are observed and concluded, students can use more advanced strategies. For example, as can be seen in Figure 4, after they see that a 2x3 kitchen is tiled with L tiles (Figure 4a), they can make an inference that a kitchen with 2x6, 4x3 and 4x6 dimensions can also be tiled (Figure 4b).
This strategy can be named as composition-decomposition. If the student, for instance, saw that a 2x3 kitchen is tileable he/she can infer that kitchens with 2x6 or 4x3 dimensions can also be tiled by composing two 2x3 kitchens. On the other hand, if the student, for instance, is exploring which of the 2xn kitchens can be tiled, when he/she reaches the kitchen with 2x6 dimensions, he/she can decompose it into two 2x3 kitchens. Because he/she previously knows that each of the 2x3 kitchens can be tiled, he/she can infer that 2x6 can also be tiled.

Similarly, the students can also make use of the symmetry concept. For example, because a 2x3 kitchen and a 3x2 kitchen are the same in terms of tiling, students can reduce the number of cases to be analyzed.

The strategies presented above are specific heuristic strategies that students can use for the case of this problem based on symmetry, area and dimension concepts. In addition to these, students can also use more general heuristic strategies. For instance, by illustrating the results of various tilings with a table, they can more easily observe whether there is a relationship among changing cases.

As we can see in the brief analysis below, this problem offers many opportunities in terms of problem solving strategies listed in the curriculum: trial and error for checking the possibility of tiling, using a table to represent the solutions, using previous steps by composing or decomposing a kitchen, using similarity by considering symmetrical kitchen (mxn and nmx), reasoning by the areas, total number of tiles and the shape of tiles and so on. Since this problem requires inductive reasoning, seeking for a pattern between the possibility of tiling and the dimension of the kitchen is an inherent strategy to this problem. However, the problem “Tiling a kitchen with L tiles” cannot be totally solved by identifying and generalising a unique pattern. Some solution can be easily reached by pattern seeking strategy but an effective approach to the problem require that this strategy should be combined with other strategies.

**FINDINGS**

In what follows, we first present strategies used by the students to solve the first part of the problem called “tiling a kitchen with dominoes” and then present the strategies related to the second part called “tiling a kitchen with L tiles”.

**Tiling a kitchen with dominoes**

The students tried to tile the kitchens with the dimensions of 1xn, 2xn, 3xn… by changing the value of n. It is observed that the relation between the possibility of tiling and the dimensions
of the kitchen progressively appeared from the second session onward. At the end of this session, the majority of the students found the solution. However, the students explained their solutions differently and most of them could not give a general formulation (see, Table 1).

<table>
<thead>
<tr>
<th>Types of kitchen</th>
<th>Solution number</th>
<th>Students’ solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>2xn, 4xn…</td>
<td>1</td>
<td>The tiling is possible</td>
</tr>
<tr>
<td>3xn, 5xn…</td>
<td>2</td>
<td>A pattern: (the tiling is) possible-impossible-possible-impossible…</td>
</tr>
<tr>
<td>Unspecified</td>
<td>3</td>
<td>The tiling is possible for some kitchens and impossible for some others</td>
</tr>
<tr>
<td>Any type</td>
<td>4</td>
<td>If the number of boxes is even, the tiling is possible. If the number of boxes is odd, the tiling is impossible and one tile should be broken.</td>
</tr>
</tbody>
</table>

*Table 1. Students’ solutions related to the possibility of tiling with dominoes*

Some students affirmed that for the kitchens with the dimensions of 2xn, 4xn… they did not have to break tiles (solution 1) but for the kitchens with the dimensions of 3xn, 5xn… they explained that there is a kind of pattern (solution 2); the tiling is possible for some cases (when n is even) and the tiling is impossible for some other cases (when n is odd). Some other students were satisfied with observing that in some cases the tiling is possible and in some other cases it is impossible (solution 3). Only two students could give a general formulation (solution 4). To reach these solutions, the students who used the trial and error strategy only observed the possibility of tiling by varying the dimension of the kitchens.

*Tiling a kitchen with L tiles*

The students continued tiling with the previous way by writing down the result of their tiling for the kitchens of all dimensions (2xn, 3xn…9xn). In this way, each student tiled more than 100 kitchens. Table 2 shows, without changing the students’ formulations, the solutions found for each kind of kitchen.

<table>
<thead>
<tr>
<th>Types of Kitchen</th>
<th>Solution Number</th>
<th>Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>2xn</td>
<td>1</td>
<td>For (n=) 3 and its multiples, I did not break tiles</td>
</tr>
<tr>
<td>3xn</td>
<td>2</td>
<td>For the multiples of 6 (the dimension of the kitchen) we did not break tiles.</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>(For the consecutives values of n: 1, 2, 3...) 2TB, noTB, 2TB... There is a pattern.</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>When the number of boxes is odd we have 1 TB and 1 EB, when the number of boxes is even we can tile without breaking tiles. 6 and its multiples: (For the consecutives values of n) 1EB, noTB, 1EB, noTB, 1EB, noTB, 3EB, no TB, 1EB, noTB: we could not obtain a pattern.</td>
</tr>
<tr>
<td>4xn</td>
<td>5</td>
<td>When the number of boxes is even we break tiles, except 12 and its multiples (the dimension of the kitchen).</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>(For the consecutives values of n): 2TB, 1TB, noTB, 1EB, 2EB, noTB, 1EB, 1TB, noTB: There is a pattern such as it is not possible for two (consecutives values of n)</td>
</tr>
</tbody>
</table>
and it is possible for 1 (the third consecutive value of n).

| 5xn | 7 | (For the consecutives values of n): 3TB, 1TB, 2TB, 3TB, 1TB, 2TB, 3TB, 1TB, 2TB... That continues as 3-1-2 |
| 8 | There is no rule. |
| 6xn | 9 | A pattern: 3 TB – the tiling is possible- the tiling is possible-3 TB- 3 TB- the tiling is possible- the tiling is possible... |
| 10 | When it is even (the value of n), we did not break tiles, when it is odd we broke tiles. |
| 7xn | 11 | No relation. |
| 8xn | 12 | (For the consecutive values of n): 4TB, 1TB, noTB, 4TB, 1TB, noTB, 4TB, 1TB, noTB, 4TB... That continues as 4-1-0. |

Table 2. Students’ solutions related to the possibility of tiling with L tiles

As can be seen from Table 2, the students studied the kitchens with the dimensions of 2xn, 3xn... step by step by varying the value of n. In this way, the students could obtain some partial solutions as seen in solutions 1, 2, 4 and 5. For example, by varying the value of n, the students observed that for the kitchens 2xn (solution 1), the tiling is possible when n is a multiple of 3. By using the same approach, they found that for the kitchen 3xn (solution 2), the tiling is possible when n is a multiple of 2. These solutions are related to a partial solution of the problem: 2mx3n kitchens can be tiled for different values of m and n. This solution involves using the composition-decomposition strategy and expressing the observed pattern with a single formula. However, the table 5 shows that the students focused on the possibility of finding a pattern in a sequence of tiling (for example, 3x2, 3x3, 3x4...) rather than on the link between the previous results of the tilings for different kitchens. The students did not use composition-decomposition strategy, which allows such deductions and significantly facilitates the task of tiling. For example, the student who found that the tiling is possible for kitchens with the dimensions of 4xn (solution 5) when the dimension of the kitchen is a multiple of 12 did not obtain this solution on the basis of the solutions previously found for the kitchens of 2x3m or 3x2m dimensions.

On the other hand, the students did not realize that some kitchens were symmetrical. For example, a student found that the kitchen 4x6 could be tiled but for the kitchen 6x4 he affirmed that 3 tiles should be broken (Figure 5).

Table 2. Students’ solutions related to the possibility of tiling with L tiles

As can be seen from Table 2, the students studied the kitchens with the dimensions of 2xn, 3xn... step by step by varying the value of n. In this way, the students could obtain some partial solutions as seen in solutions 1, 2, 4 and 5. For example, by varying the value of n, the students observed that for the kitchens 2xn (solution 1), the tiling is possible when n is a multiple of 3. By using the same approach, they found that for the kitchen 3xn (solution 2), the tiling is possible when n is a multiple of 2. These solutions are related to a partial solution of the problem: 2mx3n kitchens can be tiled for different values of m and n. This solution involves using the composition-decomposition strategy and expressing the observed pattern with a single formula. However, the table 5 shows that the students focused on the possibility of finding a pattern in a sequence of tiling (for example, 3x2, 3x3, 3x4...) rather than on the link between the previous results of the tilings for different kitchens. The students did not use composition-decomposition strategy, which allows such deductions and significantly facilitates the task of tiling. For example, the student who found that the tiling is possible for kitchens with the dimensions of 4xn (solution 5) when the dimension of the kitchen is a multiple of 12 did not obtain this solution on the basis of the solutions previously found for the kitchens of 2x3m or 3x2m dimensions.

On the other hand, the students did not realize that some kitchens were symmetrical. For example, a student found that the kitchen 4x6 could be tiled but for the kitchen 6x4 he affirmed that 3 tiles should be broken (Figure 5).
Furthermore, the students did not notice that a kitchen could be tiled in different ways although different results and the similarity between some kitchens should have led them to think of this possibility. For instance, we can consider the case of the student shown in Figure 5. As an L tile is based on three boxes, the existence of four empty boxes should create a doubt. However, the student did not realize that his sequence could be due to the way he tiled, i.e., he placed the tiles. In relation with this tiling approach, some students did not try to tile a (large) kitchen while restarting but they continued tiling the new kitchen on the basis of the preceding one. For example, after having tiled the kitchen with the dimensions of 6x2, a student systematically added a column to the left of his preceding kitchen to obtain a new kitchen and tried to tile this new kitchen without changing his preceding tiling (Figure 6).

*Figure 6. A student’s worksheet about adding column*
DISCUSSION AND CONCLUSION

In this study our purpose was to determine how 6th graders approached a non-routine problem. Grounded on the emphasis that mathematics curriculum places on problem solving and patterns, this study focused on the questions of “What strategies do students apply when trying to solve a non-routine problem?” and “If pattern seeking arises as a non-routine problem solving strategy, how do students use this strategy?” Another purpose of this study was to discuss how effective this context could be for non-routine problem solving and to develop some perspectives for curriculum design and implementation. Thus, the students were asked an inductive reasoning based non-routine problem that allowed many strategies including pattern seeking. The students worked on this problem for two hours a week, for a total of 5 weeks. The problem was composed of two parts. For the first part, which was called “tiling a kitchen with dominoes”, they were asked to tile kitchens with 2xn and 3xn dimensions while finding out the regularity between the dimensions of a kitchen (odd or even). Consequently, reasoning by the area could allow students to find the general solution. In order to observe this regularity and to establish its relationship to the dimensions of the kitchen, making a table could be an effective strategy.

<table>
<thead>
<tr>
<th></th>
<th>n=1</th>
<th>n=2</th>
<th>n=3</th>
<th>n=4</th>
<th>n=5</th>
<th>n=6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1xn</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>2xn</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>3xn</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>4xn</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

Table 3. Tiling with dominoes

From the table 3, by observing that 2xn, 4xn, etc. kitchens are tiled, and kitchens like 1xn, 3xn, are tiled only for n’s even values, the students could have reached the general conclusion that, for a rectangular kitchen to be tiled with dominoes, its area must be even.

However, most of the students could not explain their solution with a general formulation, and none of them used a table to represent the solutions found for different kitchens. Some students mentioned the word “pattern” in the meaning of regularity (possible-impossible-possible-impossible) between the dimensions of a kitchen and the possibility of tiling without looking for a general formulation.

For the second part of the problem, which was called “tiling a kitchen with L” tiles, there was no general rule or regularity for all cases. For a rectangular kitchen to be tiled with L tiles, the
area of the kitchen (that is, the number of boxes) must be the multiples of 3. However, as can be seen in Figure 7, not every kitchen with a number of boxes which is a multiple of 3 can be tiled with L tiles.

![Figure 7. All possible placements to tile a 3x3 kitchen with L tiles](image)

Nonetheless, in solving the problem, by excluding the kitchens with number of boxes not a multiple of 3, the number of cases to be explored could have been reduced. Similarly, by observing that a 2x3 kitchen is tiled, and by using the composition-decomposition strategy and symmetry, the inference that 2x6, 2x12, 4x3, 4x6, 4x12 kitchens can be tiled could have been made. Finally, with the help of a table presented below (table 4), the partial conclusion that 2mx3n kitchens can be tiled could have been drawn.

<table>
<thead>
<tr>
<th></th>
<th>n=1</th>
<th>n=2</th>
<th>n=3</th>
<th>n=4</th>
<th>n=5</th>
<th>n=6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1xn</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2xn</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3xn</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4xn</td>
<td>✓</td>
<td></td>
<td></td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5xn</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>✓</td>
<td></td>
</tr>
</tbody>
</table>

*Table 4. Tiling with L tiles*

In other words, contrary to the domino problem, in the L tile problem, it was necessary to use different strategies to obtain some solutions and to reach a partial generalization. However, the results showed that the students strictly approached the problem with the way they approached the domino problem. The students studied step by step on the kitchens with dimensions of 1xn, 2xn, 3xn… by varying the value of n without using any specific strategy. They focused on the sequences of empty boxes and on the number of tiles to be broken which they found by changing the value of n from 1 to 9. While seeking regularities between consecutive steps of a sequence of tiling (for example 2xn, according to the value of n), they explained the results of their research in terms of pattern between the possibility of tiling, empty boxes and tiles to be broken in this sequence. The students’ approach can be explained as: (1) applying the rule to the simplest case and writing down the result, (2) changing the
case step by step up to testing a large number of cases, (3) observing regularities between the results of various cases. This approach seems to be a kind of naive induction that can be summarized as “I observe, I multiply the observed cases and I come to a result.”

The students’ approach to the problem can be interpreted in different ways. Firstly, it is evident that the strategies that students use to solve problems were limited. For both the domino problem and the L tile problem, the students used no other strategy than the trial-error strategy. Trial-error strategy is a non-systematic strategy used when one has no idea whatsoever about how to solve the problem (Van Der Linden et al., 2001). Trial-error strategy was an entry-level strategy for the problem in this study, but it was expected that this strategy would improve over time and the observed situations would encourage the use of other strategies. However, the students did not change this strategy for 5 weeks and did not try to develop problem-specific strategies by using heuristic problem solving strategies such as table-drawing or the concepts of symmetry or area in the curriculum. The fact that the strategies employed by students in non-routine problem solving are poor has been emphasized in some previous studies (Schoenfeld, 1992; De Bock et al., 1998; Verschaffel et al., 1999; Elia, van den Heuvel-Panhuizen & Kolovou, 2009). Our result is in line with the findings of these studies.

On the other hand, the students approached the two parts of the problem in the same way although they were quite different in many aspects (complexity degree, existence of a general solution, strategies and mathematical knowledge required, etc.). For the L tile problem, the students did not change their approach to the problem although their exploration did not yield any effective results. In other words, the students changed their strategies in neither moving from the first part to the second part nor when working on only one part. This indicates that they do not have flexible strategies, and as reported by Elia, van den Heuvel-Panhuizen and Kolovou (2009) that they do not have inter-task strategy flexibility (switching strategies between problems) and intra-task strategy flexibility (switching strategies during the solution of a problem).

Secondly, it is clear that students’ pattern seeking and pattern generalizing strategies remain very narrow in scope and follow a certain style. The students continuously sought a pattern from the beginning to the end of the problem. However, to identify a pattern, they did not transform the kitchens that they were working on and they just tried to observe a pattern in the same type kitchens (e.g., 2xn) by using only the trial and error strategy. When they found such a pattern, they did not try to generalize the pattern by considering it along with the cases
that they had previously concluded. Their search for generalization was limited to the regularity that they sought only for the same type kitchens. Therefore, the majority of the solutions were explained on the basis of pattern concept without looking for a general formulation: “there is pattern”, “there is no pattern”, “there is no rule”, “no relation”, “that continues as…” This shows that, without trying to generalize, the students tried to identify a repeating pattern like ABCABC… (Zazkis & Liljedahl, 2002) or tried to obtain a linear pattern like 4, 10, 16, 22…(Stacey, 1989). In other words, it can be said that, instead of finding a full or partial solution by identifying a pattern and generalizing it, the students only focused on finding a pattern. Many previous studies established that while seeking patterns, students are stuck only with the most distinct patterns and have difficulty in generalizing (Lee & Wheeler, 1987; Stacey, 1989; Threlfall, 1999; Orton & Orton, 1999, Zazkis & Liljedahl, 2002). The findings of this study support these studies as well.

It is thought that students’ non-routine problem solving approaches and styles of pattern seeking strategy use identified in this study are related to the approach of the curriculum to this matter. In the curriculum we observed an over-insistence on problem solving strategies, pattern concept and the use of patterns as a non-routine problem solving strategy. However, it is evident that the examples provided in the curriculum for pattern seeking and pattern generalizing, and also the examples presented to demonstrate how patterns can be used as a non-routine problem solving strategy ignore some important difficulties that students may encounter. As can be clearly seen in the teaching scenario in Table 6 in the appendix, the curriculum gives the impression that patterns are immediately visible and generalizable relationships when particular cases are explored and a regularity is observed.

While Turkish primary school mathematics curriculum adopts such an approach, the question of how this will be perceived and applied by textbook authors and teachers is not adequately addressed. It is thought that there is a “gap” between the practitioners’ and curriculum developers’ expectations, as stated by Artigue and Houdement (2007). We observed that this approach of the curriculum seem to have affected various teaching resources. Some textbooks have directly used examples given in the curriculum to illustrate the acquisitions concerning patterns while some others, especially those are prepared for national examinations, seem to have misinterpreted the curricular expectations by proposing a huge number of routine tasks of finding and generalising patterns. When this problem, which can be named as the problem of external consistency (Kessels & Plomp, 1999), is added to the other institutional factors (limited class hours, multiple-choice national tests, the performance anxiety of the institutions
that prepare for the national tests and the textbooks prepared for these tests, etc.), the gap between the expectations of the curriculum and the practitioners’ approach becomes wider.

Another consistency problem in the curriculum is the inconsistency of its overall objectives in view of the approach and content adopted in its learning sub-domains. This problem, which can be named as “internal consistency problem” (Kessels & Plomp, 1999), can be more clearly understood when the 117-page introductory document of the curriculum is compared with the learning sub-domains. The curriculum emphasizes non-routine problem solving and use of problem solving strategies in the introductory document, but, as found by Babadogan and Olkun (2006), it seems to adopt a subject centred, not a problem centred approach in its learning domains.

As a conclusion, it is crucial for the curriculum to go beyond having a non-routine problem solving vision. It is necessary to express what kind of non-routine problems can be used in various learning domains. As such, instead of stressing the use of patterns, it would be better for the curriculum to deal with them along with other concepts and problem solving strategies.

REFERENCES


mathématiques 2002, V. Durand-Guerrier &C. Tisseron (Eds.) Paris (pp.189-205), IREM de Paris 7, ARDM.


APPENDIX

I-Analysis of an example given in the 6th grade curriculum to illustrate how patterns can be introduced (MONE, 2009a, p.206).

Have students to choose a number pattern: 2 4 6 8 ...

Three models of this pattern that can be made from different materials, the number of materials used for each number and the table showing the relation are given below:

**Model 1:** For each number “1”, the pattern can be modelled by one cube.

<table>
<thead>
<tr>
<th>The range of the number in pattern</th>
<th>The number of cube used for the number</th>
<th>Numerical relations between the number and the total number of cube used</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>First choice: 1+1=2, Second choice: 2⋅1=2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>2+2=4, 2⋅2=4</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>3+3=6, 2⋅3=6</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>4+4=8, 2⋅4=8</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>n</td>
<td>n + n</td>
<td>2n</td>
</tr>
</tbody>
</table>

The teacher should ask the students to express verbally the numerical relation between choices 1 and 2 on the table

**Table 5.** A pattern task in 6th grade mathematics curriculum

This excerpt is related to the first acquisition of the “patterns and relations” sub-section of 6th grade algebra section. The acquisition is formulated as “He/she will be able to model number patterns and explain the relation in patterns with letters”. Two pattern seeking strategies are underlined. Both of them are explicit strategies, given with different formulations (n+n, 2n). Recursive strategy is not mentioned but may constitute a choice. Thus, these strategies constitute a subject of teaching for 6th grade curriculum.

II-Analysis of an example given in the curriculum to illustrate how a teacher can benefit from different problem solving strategies in class

Besides introducing patterns as a subject of teaching, the curriculum considers “pattern-seeking” as a non-routine problem solving strategy as well. The expectations and recommendations of the curriculum about this strategy are especially found in the introductory document addressing to all grades. The scenario below (MONE Turkey, 2009a)
may give an idea about the status of this strategy for the curriculum. We present this scenario with our analysis related to the processes of problem solving involved in this scenario (Table 6).

<table>
<thead>
<tr>
<th>Scenario given in the curriculum</th>
<th>Processes of problem solving</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teacher: How can we find the sum of the interior angles of any polygon? I mean convex polygons. Could you explain the problem with your own phrases?</td>
<td>Strategy 1: Formulating the problem differently</td>
</tr>
<tr>
<td>Aysegul: We should calculate the sum of the interior angles of polygons.</td>
<td></td>
</tr>
<tr>
<td>Teacher: What can we do to solve this problem?</td>
<td></td>
</tr>
<tr>
<td>Niyazi: We can use pattern-seeking strategy. To do this, one can draw a table on which the name of the polygon, the number of sides, and the sum of the interior angles can be written.</td>
<td>Strategy 2: Seeking a number pattern</td>
</tr>
<tr>
<td>Table: Pattern-seeking about the sum of the interior angles of the polygons.</td>
<td>Strategy 3: Using a table</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Name of the polygon</th>
<th>Number of sides</th>
<th>Sum of the interior angles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle</td>
<td>3</td>
<td>180°</td>
</tr>
<tr>
<td>Quadrilateral</td>
<td>4</td>
<td>360°</td>
</tr>
<tr>
<td>Pentagon</td>
<td>5</td>
<td>540°</td>
</tr>
<tr>
<td>Hexagon</td>
<td>6</td>
<td>720°</td>
</tr>
<tr>
<td>…</td>
<td>…</td>
<td>…</td>
</tr>
<tr>
<td>…</td>
<td>…</td>
<td>…</td>
</tr>
<tr>
<td>n-gon</td>
<td>n</td>
<td></td>
</tr>
</tbody>
</table>

Aslihan: The sum of the interior angles of the polygons can be written as a multiple of 180°. For example, for a quadrilateral, an angle of 360° can be expressed as 2 180°, and for a pentagon, an angle of 540° can be expressed as 3 180°.

Teacher: Can we establish a relation with 2 and 3 here and another information on the table?

Yasemin: These factors can be written as 2 less of the number of sides of the polygon in question.

Teacher: How can we generalize this situation?

Utkun: We can multiply 180° and 2 less of the number of sides of the polygon.

Demet: We can write the mathematical phrase of the sum of the interior angle of an n-side polygon as (n-2) · 180°

Teacher: Through multiplying 180° and 2 less of the number of sides of the polygon we can calculate the sum of the interior angle of a polygon. Let’s try it for nonagon.

Oktay: (9-2) · 180° = 1260°
**Hanife:** We can seek pattern by drawing polygons. There is a relation between the number of triangles in a polygon and the measures of its angles. That’s to say I should multiply $180^\circ$ by the number of triangles. As the number of triangles is 2 less of the sum of the interior angles of a polygon, we can find the sum of the interior angles of a polygon with the operation $(n-2) \cdot 180^\circ$.

**Kerem:** By drawing triangles in a nonagon, I calculated the sum of the interior angles of a nonagon and I found 1260.

**Teacher:** We can find the sum of the interior angles of an n-side polygon by multiplying 180 by 2 less of the number of sides. In other words, with the help of $(n-2) \cdot 180$, we can calculate the sum of the interior angles of convex polygons.

| **Table 6. Scenario given in the curriculum to illustrate how to use problem solving strategies and its analysis** |
|------------------------|--------------------------------------------------|
| **Strategy 4: seeking a figural pattern** | **Institutionalization** (Brousseau, 1997) |

This scenario aims to demonstrate how a non-routine problem situation can be handled in the classroom in the light of the spirit of the curriculum and hence help the teachers to see how the students can discover the knowledge aimed in the situation at the end of their own activity. In this scenario, once the question is asked and reformulated, Niyazi suggests that they can use pattern-seeking strategy as if he knew already which strategy is appropriate in this situation and what the teacher wants to hear. The teacher and students develop Niyazi’s suggestion and Hanife suggests, in her turn, another pattern-seeking strategy, with a geometrical approach. Finally, the teacher institutionalizes, in terms of Brousseau (1997), the mathematical knowledge behind the activity. Although this scenario is given to illustrate different problem solving strategies, we can notice that it is particularly centered on pattern-seeking strategy.