Errors in the Teaching/Learning of the Basic Concepts of Geometry

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1. Activities in Teacher Education

The work that we are presenting was carried out with prospective primary teachers (PPTs) studying in the Education Faculty of the University of Extremadura (Spain). The content of the work formed part of the obligatory course “Didactics of Geometry” designed to be taken in the third year of the official Plan of Studies. The basic objective of the course is that the student should “acquire the pedagogical content knowledge (Blanco, 1994; Mellado, Blanco y Ruiz, 1998) related to the teaching/learning of Geometry in Primary Education”.

Our intention is that the activities which we develop might generate simultaneously mathematical knowledge and knowledge of the teaching/learning of Geometry. Also we take it that the curricular proposals imply an epistemological change with respect to school-level mathematical content and to the classroom activity which may result in the generation of this knowledge.

Preceding investigations have indicated to us that our PPTs have basic errors concerning mathematical content, and in particular about geometrical concepts. They also have deeply-rooted conceptions about the teaching/learning of mathematics deriving from their own experience as primary and secondary pupils, and which present contradictions with the new school-level mathematical culture. Our aim therefore is not only to broaden or correct their mathematical knowledge relative to the specific content of school-level mathematics, but also to put forward activities designed to encourage reflection on how mathematical knowledge is generated and how it is developed, taking into account the process of working towards a new mathematical culture suggested in the current curricular proposals and in recent contributions about the teaching/learning of Geometry.

These activities should lead them to reconsider their prior conceptions on mathematics and its teaching/learning. And consequently, this learning environment must enable them to generate the metacognitive skills that will allow them to analyse and reflect on their own learning process as it is taking place at that moment. An important variable in the process of learning to teach is the capacity to be able to think about one’s own learning process and the way in which it has developed.

The proposed tasks will enable epistemological change with respect to mathematical knowledge: how this mathematical knowledge is generated and developed; and how this knowledge is learnt.

Mathematical knowledge Reflections on the learning process

Working in groups, Conjecturing, Generalizing, Communicating, ...

Figure 1. Proposed task objectives
Our teaching experience and the conclusions of various studies suggest to us the advisability of posing chosen situations from school-level mathematics which the prospective teachers might have difficulties in resolving. This will make it possible to analyse and evaluate, and consequently, to correct and develop the PPTs’ mathematical knowledge.

2. Errors concerning geometry concepts

We are going to look at various activities which showed up major conceptual and procedural errors when the students teacher resolved them. I consider that the cause has to be sought in the teaching process that they themselves went through in primary school.

Activities about the altitude of a triangle

It has been found that PPTs have problems in performing activities related to the concept of altitude of a triangle (Gutiérrez y Jaime, 1999; Azcárate, 1997). This suggests situations that we may present as educational tasks to allow us to analyse the difficulties and errors presented by the teaching/learning of geometry in primary education.

As I mentioned at the beginning of this article, my teaching activity is with prospective teachers of primary education, and this is the context in which the resolution of the following activities is developed.

Activity 1. Draw the orthocentre of an obtuse triangle.

The activity described is set by way of the following mathematical task:

- Define altitude of a triangle
- Define the orthocentre of a triangle
- Draw the orthocentre of the following triangle

Figure 2. Activity 1. Draw the orthocentre of the triangle

This mathematical situation is an activity which brings out major errors of concept and procedure of the PPTs with respect to the specific concept of the altitude of a triangle, but also with respect to the process of the teaching/learning of geometrical concepts.

The analysis of the students’ responses to this set activity presents an interesting contradictory situation. Thus most of the students write down correctly the definition of altitude of a triangle and of orthocentre. They draw the altitudes incorrectly, however, and consequently also the orthocentre of the triangle of the figure. They usually place the orthocentre inside the triangle as the following figure shows.
Figure 3. Photocopy of the response of a student to activity 1 (It is the point of intersection of the three altitudes of a triangle. The altitude of a triangle is the perpendicular line which goes from the vertex of the triangle to the opposite side or its prolongation).

It is interesting that the students are unaware of the contradiction their response presents until we initiate with them an analysis of the process which they followed in resolving the activity.

The interaction that we provoke with the students leads us to reject the hypothesis of confusion with some other concept such as that of median, or bisector of a vertex, or perpendicular bisector, or with the representation of any of them. And that is why this situation allows us to go deeper into the process of acquisition of geometrical concepts on the basis of the students’ own process of learning the concepts we are concerned with.

A similar situation to the above occurs when we ask the students to draw the circumcentre of an obtuse triangle.

Activity 2. Draw the altitude of different triangles.

The errors in representing the altitudes of triangles are equally manifest when we set the following activity.

Figure 4. Activity 2.

The students manifest major difficulties in drawing the altitude of some of the triangles in the figure. Indeed, the errors in representation and answers left blank formed a high percentage.
Recognition of specific prisms. Activity 3.

In our classes, we use a basic dictionary of geometrical concepts as a resource for the students. From the definitions, we carry out different activities to establish relationships of similarity and difference between concepts. This will help us to go deeper into these concepts, into their characteristics, and to recognize different criteria of classification and inclusion.

Well, these activities lead us into paradoxical situations which have elements in common with that described before from the perspective of triangle geometry.

Thus, for instance, at one point in the course, we focus on the definitions of polyhedra, and specifically on the concept of prism. Now, at the beginning of the work on the concept of prism, once the definition has been established and memorized by the prospective teachers, we ask them to identify different specific prisms amongst the polyhedra of the dictionary.

Well, I have to say that, in spite of knowing the definition of prism and using the dictionary of geometrical concepts, they find it hard to recognize further examples of prisms other than the right or oblique prisms or the triangular or pentagonal prisms which are specifically given in the dictionary. In most cases, they do not recognize the cube or rectangular prism (called orthohedra in Spanish use) as particular cases of prisms.

In other words, they have difficulties in setting up relationships of similarity between different geometrical definitions, and therefore in being able to understand and set up different classification criteria.

3. Analysis of these situations. Definition and representation of a geometrical concept

The analysis of these situations shows us that the students’ errors have common elements that are interesting to highlight.

To understand the situation we are faced with, we have to look at the analysis of the concepts involved and the different subconcepts that make them up, and assume that the solving procedure followed by the students is closely related to their own stage as primary school pupils. In other words, the errors that the students manifest are mainly based on the teaching/learning process that they went through in primary school.

Let us go back to activity 1, and analyse the procedure followed as a function of the recognition and use of the properties of the concepts involved (Figure 5).

![Figure 5. Variables of the altitude concept](image-url)
When we look further into the variables of the altitude concept, especially into perpendicularity onto the opposite side or onto its prolongation, the students begin to recognize their error in the representation of the orthocentre.

The recognition of the error is an effective, interesting, and motivating starting point from which to continue the activity and propose specific new activities both designed to fill these gaps in their mathematical knowledge, and on the teaching of Geometry.

Bearing in mind the map represented in the previous Figure, we shall proceed by recognizing each of the concepts and subconcepts indicated, giving emphasis to their appropriate representation. This line is similar to that put forward by Gutiérrez & Jaime (1996)\(^4\). In this regard, those authors note the following subconcepts of the concept of altitude and the associated activities:

1) The subconcept of perpendicularity:

![Figure 6. Activity 4.](image)

2) The subconcept of perpendicularity from a point:

![Figure 7. Activity 5.](image)
3) The subconcept of opposite vertex: Identify the vertex of a triangle that is opposite a certain side.

Indicate with a B the vertex opposite the side b of each triangle. Then draw a line segment that goes perpendicularly from the vertex opposite b to this side b or to its prolongation.

Figure 8. Activity 6.

4) The concept of altitude of a triangle: Draw the altitude of a triangle onto a certain side.

It is at this point that we can return to proposing activity 2 (Figure 4), which will be resolved by the students taking into account these variables that we have been discussing.

Let us return now to the students’ responses. It is interesting to look again at the contradiction that occurs when the students write down the correct definitions of altitude and orthocentre, but draw a graphical representation that does not correspond to what they had written (Figure 3). Even more important, however, is the failure to recognize their error until we initiate the analysis of the variables of the concept altitude.

This situation allows us to speak of the difference between definition and representation of a concept, and consequently we can look more deeply into the mental image that the students have of the concepts involved. In this case, it is the specific mental image that they have associated with the concept of altitude of a triangle.

Thus, on recalling their time as primary education pupils, the students recognize an image associated with the height of an acute triangle standing on a horizontal base and arranged so that the representation of the altitude lies within the triangle.

Figure 9. Traditional representation of the altitude of a triangle in textbooks
The abuse of this representation also helps to create the image of the altitude of the triangle as a perpendicular line segment which is unique for each triangle. (The expression h is “the” altitude of the triangle makes this explicit). This idea has its field of validity in the everyday use of the word “altura” (= height).

This image explains why some students resolve activity 7 in the sense indicated in the figure itself.

Activity 7. Draw the altitude of the following triangle

![Figure 10. Activity 7. Draw the altitude of the triangle.](image)

And, in general, it partly explains the difficulties the students have in resolving activity 2.

Likewise, the image of the orthocentre appears linked to the representation of the orthocentre of this acute triangle, so that the orthocentre is located inside it. In other words, they capture the image of the orthocentre in a particular example.

The orthocentre of acute triangles is always in the interior of the triangle

![Figure 11. Orthocentre of a acute triangles](image)

As a consequence of the situation that the students are experiencing, we can make three important observations:

First, the mental image of the orthocentre in the interior of the triangle predominated over the recognition and use of the variables of the concept, as expressed in the definition.

Second, the mental image is lasting, despite the contradiction that arises between the definition and the representation.

And third, the students begin to become aware of their contradiction when we initiate the analysis of the variables of the concept, and not before.
The predominance of the metal image associated with a concept is the reason why, when we propose activity 8, they identify it just with a rhombus, excluding from our responses the concept of square.

![Activity 8](image)

Figure 12. Confusión between square and rhombus concepts

It is clear that in this example, the student has been carried along by the mental image associated with the rhombus, instead of by the analysis of the properties of the figure, which in this case has all four sides equal and the four right angles.

Similarly, the difficulties in analysing the variables of a concept, and the image that they have associated with particular cases of that concept, constitute the cause of the students’ difficulties in finding similarities and differences or relationships of inclusion between mathematical concepts in general, and geometrical concepts in particular.

And this leads us on to the third activity in which, to resolve it, we asked our students to make a square, in which there appear the names of prisms, rectangular prism (called orthohedra in Spanish use), and cube, and underneath, the subconcepts and relationships which make up the definition of each of these names.

At this point, it is convenient to recall that to acquire a concept means building a conceptual scheme of that concept. “Therefore, memorizing the definition of a concept is no guarantee of understanding its meaning. In reality, understanding means having a conceptual scheme so that certain meanings are associated to the word that designates the concept: mental images, properties, procedures, experiences” (Azcárate, 1997, 29).
4. Teaching geometrical concepts

The recognition by the prospective teachers of the contradiction and their expectations as future teachers of Mathematics motivate them to analyse the causes of the situation. And this allows us to initiate the analysis of the methodological process with respect to the teaching of Geometry lived through by the prospective teachers during their stage as primary level pupils. We could represent this schematically in the following way:

i. Definition - recognition of the specific figure - memorization drills
   ii. Examples of figures - description of their mathematical characteristics - definition - drills of recognition and recognition of the specific figure

Figure 13. Scheme shown in Gutiérrez & Jaime (1996, 145).

Respect to the concepts involved, the prospective teachers recognize that their teaching was based on:
   i. Standard examples that displayed the various concepts. Thus, the triangles were usually presented in the same position; the altitudes of the triangles were drawn normally on equilateral and acute triangles standing on a horizontal base.
   ii. The greatest emphasis was given to the definition, obviating analysis of the properties, and with no importance given to the fact that the visualization produces a more lasting and influential effect than spoken or written text.
   iii. The activities reflected a static and repetitive process on template examples chosen almost exclusively from the textbook.
   iv. Lack of specific experimentation with other possible situations aimed at going deeper into the comprehension of the concepts and aid in transferring the knowledge to other problems.
   v. A paucity of materials and resources. In most cases the textbook was practically the only resource.
In my opinion, this model of teaching as recalled by the prospective teachers lies at the origin of the difficulties they have in working with different geometrical concepts. Fortunately it is worth recalling that this model does not correspond with the current proposals on teaching geometry. These proposals point to the need for the students to share in the construction of the mathematical concepts actively and creatively, with the idea that this will encourage communication, the elaboration of conjectures, problem solving, etc.

![Diagram](image-url)

Figure 14. Prospective teachers during their stage as primary level pupils.
Note:
1. In Spanish, an altitude of a triangle is simply called the “altura”, an everyday word which also means “height”.

References