The Effect of Alternative Solutions on Problem Solving Performance

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Abstract

The purpose of this study was to investigate the effect of instruction in alternative solutions on Taiwanese eighth-grade students’ mathematical problem solving performance. This study was exploratory rather than experimental. Alternative-Solution Worksheet (ASW) was developed to encourage students’ engagement with alternative solutions to mathematical problems during instruction. Data for this study consisted of students’ problem solving scores on the pretest, posttest and ASWs. The results of this study indicated that students improved their problem solving performance after instruction in alternative solutions utilizing ASW techniques and students who performed better on ASWs tended to improve more on their problem solving performance.

Keywords: Mathematics; Problem solving; Alternative solutions; Multiple solutions
Introduction

The National Council of Teachers of Mathematics (NCTM, 2000) indicated that students should develop their “flexibility in exploring mathematical ideas and trying alternative solution paths” (p. 21). Hiebert et al. (1997) indicated that students should be provided with problem-solving tasks that for which they “have no memorized rules, nor for which they perceive there is one right solution method” (p. 8). It is important to study teaching and learning alternative solutions in mathematical problem solving because “you can learn more from solving one problem in many different ways than you can from solving many different problems, each in only one way” (an aphorism of unknown origin, cited in Silver, Ghousseini, Gosen, Charalambous, & Strathun, 2005).

Alternative solutions are an important feature of effective problem-based mathematics instruction (Cai, 2003). Teaching mathematics through problem solving provides a learning environment for students to explore problems on their own and to invent ways to solve the problems. Such activities allow them to facilitate connections of related ideas, to consolidate their mathematical knowledge, and to think creatively (Polya, 1973; Kalman, 2004; Krulik & Rudnick, 1994). As suggested by Schroeder & Lester (1989), teaching through problem solving offers the promise of fostering student learning.

Although problem solving with alternative solutions may foster students’ mathematics learning, there were, however, limited empirical studies that directly addressed how mathematical problem solving with alternative solutions could influence students’ problem solving performance (Große & Renkl, 2006; Silver et al., 2005). Mixed results have been found on the relationship between alternative solutions and problem solving performance. While some studies indicated multiple solutions or representations enhanced students’ problem solving performance (Fouche, 1993; Brenner et al., 1997), other studies didn’t show improved problem
solving performance (Brenner & Moseley, 1994). This investigation was an attempt to understand better how teaching alternative solutions could influence students’ problem solving performance.

Teachers play an essential role in teaching through problem solving. NCTM (2000) indicated that teachers must “decide what aspects of a task to highlight, how to organize and orchestrate the work of the students, what questions to ask to challenge those with varied levels of expertise, and how to support students without taking over the process of thinking for them and thus eliminating the challenge” (p. 19). Hiebert and Wearne (1993) found that teachers in problem-solving classrooms used fewer problems, spent more time on each problem, and asked more conceptual questions than teachers in more traditional classrooms. Despite the importance of the teacher’s role in problem-solving instruction, little research suggests how teachers learn to teach through problem solving (Cai, 2003). For example, instruction in alternative solutions is rarely observed in mathematics classrooms, and little is known about how to support more frequent use of alternative solutions by mathematics teachers in their classroom instruction (Silver et al., 2005). In the present study, Alternative-Solution Worksheets (ASWs1) were developed to support students’ engagement with alternative solutions to mathematical problems during instruction.

This study was exploratory rather than experimental. The purpose of this study was to investigate the effect of the use of ASW on Taiwanese eighth-grade students’ problem solving performance. The specific questions to be answered by this study included:

1. What instructional procedures support the use of ASWs?
2. What relationship is there between problem solving scores on pre and post tests and students’

1 We use the abbreviation ASW to represent Alternative-Solution Worksheet developed as part of this study to encourage students to generate alternative solutions while solving problems.
performance on ASW activities?

**Methods and data sources**

**Subjects**

This exploratory study involved one teacher and twenty-eight students. The teacher is the investigator and the subjects were from an eighth-grade class of a small-sized junior high school in Taiwan.

**Materials and instruments**

*Alternative-solution worksheets.* Alternative-Solution Worksheets (ASWs) consisted of two sections, Initial Solution and Alternative Solutions. The section of “Initial Solution” was for the solution students obtained first and the section, Alternative Solutions, was for other solutions they might find later. One ASW problem was adapted from *Algebra in a technological world* (Heid, 1995). The other ASW problems were adapted from the schools system’s instructional materials. Students were encouraged to provide as many different solutions on the worksheet as they could. Figure 1 shows an Alternative-Solution Worksheet with an ASW problem and a student’s solutions.

<table>
<thead>
<tr>
<th>Problem: Factor $(x-2)^2 + a(x-2) + 3(x-2) + 3a$ completely.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Initial Solution:</strong></td>
</tr>
<tr>
<td>$(x-2)^2 + (a+3)(x-2) + 3a = (x+1)(x-2+a)$</td>
</tr>
<tr>
<td>$\frac{(x-2)}{(x-2)} + 3$</td>
</tr>
<tr>
<td>$\frac{+a}{(x-2)}$</td>
</tr>
<tr>
<td><strong>Alternative Solutions:</strong></td>
</tr>
<tr>
<td>$(x-2)^2 + a(x-2) + 3(x-2) + 3a$</td>
</tr>
<tr>
<td>$= (x-2)(x-2+a) + 3(x-2+a)$</td>
</tr>
<tr>
<td>$= (x-2+a)(x+1)$</td>
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</tbody>
</table>

*Figure 1.* An Alternative-Solution Worksheet with an ASW problem and a student’s solutions.
The pretest and the posttest instruments. Each student was given the pretest before and the posttest after the ASW instruction. Four open response problems that required students to provide different solutions for each problem made up both pretest and posttest. The problems on these two examinations were adapted from the schools system’s instructional materials and required comparable problem solving knowledge and skills. Students were instructed to show all the work and not to erase or black out anything that they had written for each problem.

To determine the reliability of both of the tests, Cronbach’s alpha was calculated and the values obtained were .89 for the pretest and .90 for the posttest. Figures 2 and 3 provide a pretest and a posttest problem with solutions given by the same student, respectively.

**Problem:** Below there are four different sizes of rectangles A, B, C and D, each of which has sides of x, x and b, x and 1, b and 1, respectively. Suppose there is a big rectangle which consists of 3 As, 1 B, 9 Cs and 3 Ds. Find the length and width of this big rectangle in terms of x and b.

<table>
<thead>
<tr>
<th>A</th>
<th>x</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td></td>
<td>b</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>D</td>
</tr>
</tbody>
</table>

**Initial Solution:**

\[3x^2 + xb + 9x + 3b\]
\[= 3(x + 3) + b(x + 3)\]
\[= (x + 3)(3x + b)\]

**Alternative Solutions:**

\[(3x^2 + xb) + (9x + 3b)\]
\[= x(3x + b) + 3(3x + b)\]
\[= (3x + b)(x + 3)\]

*Figure 2. A pretest problem with a student’s solutions.*
Problem: There are 6 mosaic rectangle pieces. As shown below, they are 2 As, 1 B, 2 Cs, and 1 D, where A, B, C, and D are of sides of x, b and x, 1 and x, 1 and b, respectively. Now a new mosaic rectangle piece is made up of these six pieces. Find the length and width of this new mosaic rectangle piece in terms of x and b.

![Diagram of mosaic pieces]

Initial Solution:

\[
2x^2 + xb + 2x + b \\
= 2x(x + 1) + b(x + 1) \\
= (x + 1)(2x + b)
\]

Alternative Solutions:

\[
2x^2 + xb + 2x + b \\
= x(2x + b) + (2x + b) \\
= (2x + b)(x + 1) \\
x^2 + x^2 + xb + x + x + b \\
= (1 + 1)x^2 + (b + 2)x + b \\
= 2x^2 + (b + 2)x + b \\
= \frac{2x + b}{x + 1} \\
= (2 + b)x + x \\
= 2x^2 + bx + 2x + b \\
= (2x + b)(x + 1)
\]

Figure 3. A posttest problem with the solutions given by the student solving the problem in Figure 2.
Scoring. As shown in Table 1, ASW scoring rubric (adapted from Charles, Lester, & O’Daffer, 1987 and Schoenfeld, 1985) was used to rate each attempted solution approach to the problems on the Alternative-Solution Worksheets, pretest, and posttest. Lester (1983) claimed that both quantitative and qualitative criteria should be employed to assess problem solving performance; therefore, ASW scoring rubric was based not only on how many solutions were generated by the student, but also on how well the solutions were done. For each attempted solution approach, this grading procedure considered not only correctness but also completeness to identify both the breadth and depth of the students’ problem solving knowledge. No credits were awarded to a solution approach that repeated a used solution idea. A total score was then computed by adding up points for each ASW, pretest and posttest. For example, a total score of 11 was awarded to the solutions in Figure 3, with 4 points given to both the initial solution and the first alternative solution, and 3 points given to the last alternative solution. The last alternative solution was given 3 points because the student demonstrated a right problem solving approach at the beginning but didn’t complete the final step of the approach. The problem was then completed with a used solution idea.

Table 1

<table>
<thead>
<tr>
<th>ASW Scoring Rubric</th>
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<tbody>
<tr>
<td>4 points</td>
</tr>
<tr>
<td>Student demonstrates a right problem solving approach and correctly solves the problem.</td>
</tr>
<tr>
<td>3 points</td>
</tr>
<tr>
<td>Student demonstrates a right problem solving approach but solves the problem with few errors or little incompleteness. For example, the following solution to the equation $4(2x-3)^2-36=0$ would be given 3 points. $4(2x-3)^2-36=0$, $4(2x-3)^2-6^2=0$, $[2(2x-3)+6][2(2x-3)-6]=0$, $(4x)(4x-12)=0$, $x=-4$ or $x=3$.</td>
</tr>
</tbody>
</table>

(continued)
2 points
Student demonstrates a right problem solving approach but solves the problem with some errors or some incompleteness. For example, 2 points would be given to the following solution to the equation $4(2x-3)^2-36=0$.  
$4(2x-3)^2-36=0$, $4(2x-3)^2-6^2=0$, 

1 point
Student shows minimal understanding of the problem. For example, student is aware of a right problem solving approach but doesn’t pursue it at all. For example, the following solution to the equation $4(2x-3)^2-36=0$ would be given 1 point.  
$4(2x-3)^2-36=0$, $2^2(2x-3)^2-6^2=0$.

0 point
Student demonstrates wrong problem solving approaches or shows incorrect understanding of the problem.

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**Instructional procedures**

Instructional procedures supporting the use of ASWs were developed based on the “open-ended approach” (Shimada & Becker, 1997) promoting instruction in alternative solutions in Japan. Allowing sufficient time for students to explore the problem fully, discussing the solutions with the entire class, comparing these solutions, and summarizing what has been learned were important elements of the instructional procedures utilizing the ASW techniques. This ASW instruction did not, however, include any specific training on heuristics in problem solving. Pretest was administered by the investigator before the study began.

The subjects participated in the ASW instruction for fifty minutes, two days a week for a period of four weeks. Each week during one class students were required to complete independently two Alternative-Solution Worksheets, one of which consisted of a mathematical
problem relevant to the pre and post tests and the other, for the sake of variety and stimulation, consisted of an irrelevant one. More specifically, the relevant ASW problems were related to the pre and post tests in that they required the same problem solving knowledge but they were presented in different forms. Students were asked to provide alternative solutions in addition to their initial solution to the problems on the ASWs, pretest, and posttest.

To encourage students’ engagement in generating different solutions to the ASW problems, the teacher explained how their responses on the worksheets would be evaluated, emphasizing that each problem solving approach demonstrated on the worksheets would be given credit based on its accuracy and completeness; hence, students also were instructed to write down all work and not to erase or black out anything they had written. After students started to work on the worksheets, the teacher moved around the classroom to help them, but her role was to facilitate rather than demonstrate problem solving ideas. In case a hint was needed, it was given to the entire class instead of to individuals.

To evaluate students’ performance on ASWs, an ASW scoring rubric, explained in the “scoring” section, was used to rate each attempted solution on ASWs. In addition to this assessment, positive comments were written for the students who pursued an interesting solution approach or tried more than one solution method to the problems. In those worksheets that were less than successful, encouraging comments also were written to urge students to improve their work.

After graded, the worksheets were given back to the students in the following class. During that class, the teacher had some students present their solutions and discuss them with the whole class. The solutions were presented and listed on the chalkboard based on the viewpoints used (Sawada, 1997). The teacher’s task focused on guiding the students to analyze and compare these different solutions listed on the chalkboard by the presenters. Discussions usually started
with “what similarities or differences do you think there are between these two solutions?” Often a response given by some student resulted in the class looking more closely at the relationship between the solutions and the problem structure. This provided the opportunity for the students to enhance their understanding about the problems and develop their problem solving abilities. The class was then concluded by the teacher summarizing what has been learned. After the ASW instructional procedures, the posttest was administered to the entire class by the teacher.

Analysis of data

Data for this exploratory study were comprised of students’ problem solving scores on the pretest, posttest and ASWs. Data generated from the pre and post tests were analyzed statistically to determine if the posttest problem solving scores differed significantly from the pretest scores. Difference in the number of responses from pretest to posttest was also noted. Furthermore, correlation analysis was used to examine the relationship between the pre and post test problem solving scores and students’ performance on ASW activities.

Results

There was an increase between the means of the problem solving scores and the number of responses on both tests. The mean for pretest was 5.86 compared to the mean score of 6.93 for posttest. While there was not much increase between the means of the problem solving scores on both tests, 50% of the students increased the number of responses from pretest to posttest, regardless of correctness or completeness. Moreover, the mean of the number of responses increased .60 from 2.54 to 3.14, which indicated that each student on average gave .60 more responses on the posttest than on the pretest.

A t-test was conducted on the problem solving scores to detect statistically significant differences between pre and post tests. Based on this analysis, it was concluded that students’ problem solving performance was not significantly different between pre and post tests.
Despite the lack of statistically significant results on problem solving performance between pre and post tests, a t-test of the number of responses indicated that students made more responses from pretest to posttest by an amount that approached statistical significance (p=.05, p=.098).

A correlation analysis was performed to examine the correlation between the average ASW scores and the difference of the pre and post test problem solving scores. The difference of the pre and post test problem solving scores was used in this correlation analysis because of the significant initial difference each subject could have in their problem solving abilities. There was a .544, statistically significant, correlation between the average ASW performance and the difference of pre and post test problem solving performance. It suggested that students who performed better on ASWs tended to improve more from pretest to posttest.

The results of this exploratory study indicated that students improved their problem solving performance after the ASW instruction and students who performed better on ASWs tended to improve more on their problem solving performance.

**Discussion of findings**

An unexpected result of this study was the only slightly improved problem solving performance from pretest to posttest. There were several possible reasons for this result:

First, individual ASWs, pretests and posttests indicated that most students who did not improve their problem solving performance on the posttest were not well equipped with either mathematical content knowledge or procedural facility or both. The ASW instructional technique was intended to provide a way for each student to reflect on and learn from their problem solving process; however, if the students did not possess the required problem solving knowledge to make appropriate progress, they could have too few resources to benefit from ASW and improve their problem solving ability. Knowledge base plays an important role in students’ problem
solving performance (Polya, 1973; Schoenfeld, 1985). As indicated by Polya (1973), “it is hard to have a good idea if we have little knowledge of the subject, and impossible to have it if we have no knowledge. Good ideas are based on past experience and formerly acquired knowledge” (p. 9).

Second, it takes time to improve problem solving performance within a four-week instructional episode implementing a new approach to the learning of mathematics. Perhaps time was insufficient for some students to make significant problem solving improvement. According to Lesh (1985), the understanding and ability to use problem solving processes develop slowly over time.

Suggestions for future research

In the present study, instruction in alternative solutions was implemented over a period of four weeks with twenty-eight eighth-grade students. To determine the effectiveness of instruction in alternative solutions in improving students’ problem solving performance, the methods of this study should be implemented by other classroom teachers, over a longer period, or with a larger group of students. It also would be beneficial to conduct experimental studies with both a treatment group receiving instruction in alternative solutions and a control group without receiving such instruction in order to understand better the effect of instruction in alternative solutions on students’ problem solving performance.

As suggested by Große and Renkl (2006), the effects of alternative solutions may depend on the type of learning outcomes such as procedural skills or conceptual knowledge. It would be of interest to know if the effect of instruction in alternative solutions would vary with different types of learning outcomes considered. Studies should also be undertaken in a variety of mathematical content areas to determine if instruction in alternative solutions would be effective across areas of content.
Epilogue

This exploratory study provided an instructional approach to implement alternative solutions in mathematics classrooms. More research is needed to refine and validate this instructional approach or to develop other instructional approaches in alternative solutions to understand better the relationship between alternative solutions and problem solving performance.
References


