Empowering Learning with Rich Mathematical Experience: Reflections on a Primary Lesson on Area and Perimeter

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Abstract
In this paper, a Hong Kong primary school lesson on area and perimeter is analysed with a perspective to discuss the meaning for students to have rich mathematical experiences, and how pre-designed pedagogical tools could enrich mathematics classroom learning environment which promote re-shaping, shaping and even creation of mathematical knowledge.

Mathematical Experience
What does it mean to say that a student has experienced something mathematical? Does it mean that the student has memorized a formula, executed a computation algorithm, written down a string of symbols, proved a proposition, or recognized a pattern? All these no doubt are commonly regarded as mathematical, but what are the critical features that characterize the student’s experience as mathematical? Instead of pursuing a more traditional path of just asking what mathematics is to address this query, I will focus on what empowers a mathematical experience for students. If I take an experience as a “bridging” between a human subject and an observable phenomenon and in particular in an educational context as how a student connects him/herself to what is being taught or learnt, then a learning experience is about interactions connecting a learner (the human subject) and the matter being learnt (the observable phenomenon). In a pedagogical setting, these interactions usually consist of processes of dynamic feedback between the learner and the object being learnt through tools (for example, a ruler, a graph paper, or an ICT learning environment) that are situated for specific pedagogical intentions in specific contexts. Mathematics is often referred to as the science of patterns (Resnick, 1997). Mathematical patterns can be interpreted as invariant structures concerning numbers and/or shapes that emerge when the situation in consideration undergoes variation. In this vein, undergoing a mathematical experience means that a mathematical pattern is discerned and ways to re-produce or re-present that pattern are conceived.

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Mathematical Experience: The discernment of invariant pattern concerning numbers and/or shapes and the re-production or re-presentation of that pattern.

What generates a mathematical experience for students? As mentioned, the kernel of a learning experience is the interaction between the learner and the matter to be learnt. For a learning experience to be a mathematical experience, this interaction must enable the induction of critical features that are mathematical; that is, those elements in the interaction that are most potent to bringing about discernment of invariant structures that concern numbers and/or shapes, and ways to re-produce or re-present the structures. Variation is a key mechanism that produces mathematical patterns. The richer the variation, the richer the mathematical experience. In this article, I will use these ideas to analyse (re-think) a Primary Four mathematics lesson; and in doing so, further explore the key components of a rich mathematical experience.

The Lesson

Background
The lesson chosen for this discussion was one of many research lessons taken from a Hong Kong government-funded project on improving primary classroom teaching and learning based on the Japanese Lesson Study Model and Marton’s Theory of Variation (Lo, Pong and Chik, 2005). The author was the researcher in charge of the chosen lesson. It was a Primary Four lesson on the relationship between the perimeter and the area of a rectangle in a local Hong Kong Primary School.

The Intended Lesson
The design of the lesson was based on a teaching strategy that followed a sequence of learning activities:
Two 70-minute lessons were planned with each lesson beginning with the teacher posing a question to students. Students would then engage in the above learning sequence by following carefully designed activities supplemented by tailor-made resources prepared by the teacher. The two questions were:

**Question One (Lesson One)**
Chun Ming’s and Mei Mei's homes are both rectangular in shape. The perimeter of Chun Ming’s home is 16 meters and the perimeter of Mei Mei’s home is 20 meters. Is Mei Mei's home bigger than Chun Ming's home?

**Question Two (Lesson Two)**
Gardener Uncle Wah has a 36 meters long fence and he plans to enclose a rectangular flowerbed with it. Can you help Uncle Wah to think of a way to enclose the largest flowerbed?

Question One was open-ended while Question Two had an intended answer. In both cases, there were two objects of learning that the teacher wanted students to focus on. The primary object of learning was to explore the relationship between the concepts of ‘area’ and ‘perimeter’ by varying certain aspects (the length and width of the two houses in Question One, and the length and width of the flowerbed in Question Two) while keeping an aspect fixed (the perimeter of each house in Question One and the length of the fence in Question Two). By allowing a dimension of variation (in this case the perimeter) being kept constant while varying other associated dimensions of variation (length and width), discernment of mathematical patterns (invariant structure) concerning area and perimeter could be induced. A secondary object of learning was to introduce students to the concept of solving a problem in a real life situation by comparing quantities in a systematic fashion using tabulation. In doing so, it was hoped that students' individual differences on understanding perimeter and area would be observed. This variation in students’ understanding of what are learnt might provide a space of learning where a collective understanding of the mathematical concepts could be developed. Furthermore, students may develop comparison techniques that could re-produce or re-present the discerned mathematical patterns. The variations set out in the lesson design aimed to bring about at least four concepts on the relationship between perimeter and area:
1. If only the perimeter of a rectangle is given, its area cannot be determined. Therefore, the ‘size’ of Chun Ming’s and Mei Mei’s homes cannot be compared.
2. Increasing the perimeter of a rectangle does not necessarily increase the area of the rectangle.
3. If either the length (or width) of a rectangle is fixed, then increasing its perimeter will increase its area.
4. The square has the largest area among rectangles with the same perimeter.

The learning sequence that students engaged in was as follows:

(1) Conjecturing
Students were divided into groups and each student was asked to try to guess the answer (conjecturing) to the question posed by the teacher and to write their answers on a worksheet and to share it with their group-mates. In particular, each student drew their rectangles on a worksheet and compared them with their group-mates’ rectangles. The teacher conducted a discussion with students on their answers without giving any hint on the correctness of students’ suggestions.

(2) Problem Solving
Different resources (situated tools) were then made available for students to start a group investigation of the posed problem: worksheets, grid papers, colour pens, strings with pre-cut lengths (16 cm and 20 cm), a pin-board made of Styrofoam, hint cards for students who need help, and Overhead transparency to record findings. Students were asked to draw and tabulate the length and width of those rectangles with the same perimeter. Here is an example of a table in the worksheet:

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<tr>
<th>Width (cm)</th>
<th>Length (cm)</th>
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<th>Area (cm²)</th>
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(3) Exploration

Students were encouraged to discover patterns in their drawings and tables to propose possible answers to the posed problem. Each group was asked to present and explain its findings to the whole class using an overhead transparency.

The Implemented Lesson

The teacher performed an excellent job to follow the intended lesson plan by acting as a facilitator to initiate open discussions, instruct students on how to use the tools, guide students through different stages of the lesson and to moderate the group presentations. The teacher never discouraged students who offered “wrong” answers, and did not give students the “correct” answers. Rather students’ collective findings were used to construct the experienced objects of learning for the lesson.

Students seemed to enjoy the learning activities and were quite eager to participate in the investigation. One student commented on the worksheet that “Mathematics is wonderful” while another wrote, “Mathematics is a little strange”. The technique of using systematic tabulation to tackle a mathematical problem was new to them. Some students were able to grasp the idea very quickly and began to make unexpectedly interesting conjectures from their observations, while others were slower in adjusting to the approach but eventually rediscovered the formulas for perimeter and area. On the one hand, the teacher was amazed by two students who were able to draw from the tabulation strategy raw ideas on approximation, infinity and sequence of numbers using their own language. These concepts were regarded to be far more advanced than those at Primary Four level. On the other hand, there were students who could barely make sense of the whole learning experience. Below is a collection of what students claimed to have found in the lessons. Some are common to most students (those marked with an asterisk) while others are findings from individuals or small groups.

Summary of students' findings:

1. The perimeter of a rectangle equals to the sum of all of its sides. Perimeter of a rectangle = (length + width) × 2. (*)
2. The area of a rectangle is the amount of surface that it covers. Area of a rectangle = length × width. (*)
3. A rectangle with a fixed perimeter can have many possible combinations of length and width. For example, a 16 cm long string can outline 4 different rectangles provided that only integral values for length and width are use. (*)

4. Students observed from the combinations of length and width in the tables that whenever the length is increased by 1, the corresponding width will decrease by 1. (*)

5. Students observed from the tables that (length + width) = perimeter ÷ 2.

6. The longer the perimeter, the more combinations of length and width there are.

7. Generally if the given perimeter is divisible by 4, then there are (perimeter ÷ 4) combinations of length and width. Hence there are (perimeter ÷ 4) rectangles with this given perimeter (see actual student work in Figure 1a and a translated version in Figure 1b).

8. If numbers other than integers can be used, then there are ‘infinitely many’ (the student who wrote this comment actually used the ∞ sign) combinations of length and width for a rectangle with a fixed perimeter (Figure 1b).

9. If only the perimeter of a rectangle is given, its area cannot be determined. It is necessary to know either the length or the width. (*)

10. Increasing the perimeter of a rectangle does not necessarily increase its area.

11. When two rectangles have either the same length or the same width, then the rectangle with the longer perimeter would have the larger area.

12. A rectangle with a fixed perimeter would have the largest area when its length and width are equal (a square). (*)

13. The situation in point no.12 is true not only for integral values for the length and width, but also for decimal values.
Figure 1a. Examples of a student’s work
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<tr>
<th>Width (cm)</th>
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<td>17 cm</td>
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<td>2 cm</td>
<td>16 cm</td>
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<td>9 cm</td>
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<td>36</td>
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</table>

Discovery
All the strings that the teacher gave us happened to have lengths divisible by 4, if we divide them by 4, we can then get: “the number of different rectangles that can be outlined”.

Conclusion
Conclude that as long as the length of a string is a multiple of 4, divide the lengthy by 4, then can quickly calculate the number of different rectangles that can be outlined.

Record again in ascending order

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<tr>
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<td>2 cm</td>
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<td>9 cm</td>
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Discovery
Discovered that for a rectangle, the bigger the width, the shorter the length, the bigger the area, because as the shape of a rectangle is close and like a square, the area get bigger. Opposite, the bigger the length, the smaller the width, the smaller the area.

Conclusion
Think
3. The school has 80 meters of fencing, how to use it to outline a largest rectangle?
   \[80 \div 4 = 20\] \[\text{Area} = 400 \text{ m}^2\]
   ”If allow decimal numbers has \( \infty \) types, or else only 20 types.
4. What is there is 120 meters of fencing?
   \[120 \div 4 = 30\] \[\text{Area} = 900 \text{ m}^2\] Same as above.
14. There were some interesting discoveries made by one student while he was observing Table 1 (see actual student work and a translated version in Figure 2):

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<th>Width (cm)</th>
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<tr>
<td>9</td>
<td>9</td>
<td>36</td>
<td>81</td>
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</table>

Table 1

a. The area increases as the width increases while the length decreases. The more the rectangle approximates a square, the larger its area becomes (Figure 1b).

b. There is a sequence of differences between two consecutive areas (the right hand column of numbers in the table): 15, 13, 11, 9, 7, 5, 3, 1

The student further made two interesting conjectures about this sequence:

(a) The first term of the sequence can be obtained by the formula

\[(\text{Perimeter} \div 2) - 3\]

(b) The number of terms in the sequence can be obtained by the formula

\[(\text{Perimeter} \div 4) - 1\]
In the Discover and Conclusion shown in Figure 2, what the student tried to express was \((36 ÷ 2 - 3) = 15\) is the first term of the sequence; \((36 ÷ 4 - 1) = 8\) is the number of terms in the sequence. The last area minus one gives the second last area, minus it by 3 gives the third last area, so keep subtracting by the consecutive terms of the sequence (starting with 81 and 1).
will give the consecutive areas. Notice in the Conclusion, the student seemed to have miswritten 11 and 13 with 1 and 3 respectively.

Analysis

Learning can be seen as experiencing variations and different perspectives in seeing objects in the world. In this lesson, an object of learning is the relationship between area and perimeter of a rectangle. The realization of this object of learning in multi-representation via different tool usages (e.g. hands-on experience in constructing rectangles, visual perception of the change in size and shape of different rectangles, sequences or patterns of number in a tabulation exercise) empowers students’ mathematical capability and insight, resulting in a rich understanding of the knowledge concerned. These different ways of understanding via different tools is crucial in collectively acquiring the essential knowledge embedded in the objects of learning. Without a multi-perspective interpretation, mathematics loses its power and meaning.

Tools

Different tools usages in the lesson enabled students to foster different levels of understanding on the relation between the area and perimeter of a rectangle. In particular, the use of a table to organize numerical records into discernable patterns seemed to be a powerful means for students to engage in rich mathematical experience. These tools instrumented patterns of variations that brought about discernment of invariant structures and eventually generation of formulas. From the summary of findings, I will try to categorize the 14 student findings into three (hierarchical) categories of understanding and discuss how tool usages might have produced the kinds of variation that could have contributed to their formation. Three tools were available to students simultaneously during the problem solving and exploration phases.

1. Rectangular grid paper

Students were asked to draw rectangles having the same perimeter on grid paper. The grid paper was at the same time a measuring tool and a template for making rectangles. Students used this tool to visualize different rectangles (in size and shape) having the same total perimeter but with different lengths and widths. It thus created a
contrasting experience and generated an intuitive relation between the area and perimeter of a rectangular for students.

2. Pin-board and pieces of string
Pin-boards custom made with Styrofoam (with grids on them), pins, pieces of string with lengths 16 cm and 20 cm were provided for students. Students used them to construct different rectangles. This tool was designed for students to experience connections between the physical world and the mathematical world: between a piece of string and a fixed perimeter; between a physically enclosed region and an area. It situated students in a hands-on context with real objects and thus allowing students to have a simultaneous focus of attention on real world representation and paper-pencil representation (an interaction between different domains of experience). Simultaneity is a critical variation condition for discernment of invariant concepts.

3. Table for tabulating numerical values
The use of a table to record and interpret numerical findings was a new problem solving strategy for students. The Table tool was considered to be the most critical component in the design of the lesson and was regarded as a secondary object of learning for the lesson. The focus of attention of this tool was on the variation of numerical patterns which provided students a chance to transit from geometry to algebra thus creating a fusion of experiences that could be conducive to higher order thinking.

Categories of Understanding
There appears to be three categories of understanding arising from the students’ findings. I depict them in Figure 3 and further explain their features:
(I) First category of understanding: Re-shaping prior mathematical knowledge

Students re-discovered the concepts of perimeter and area of a rectangle in the contexts provided by the tools. In particular, they wrote down the familiar formulas:

- The perimeter of a rectangle equals the sum of all of its sides.
- Perimeter of a rectangle = (length + width) \times 2.
- The area of a rectangle is the amount of surface that it covers. Area of a rectangle = length \times width.

It has been commonly observed that primary students have difficulty distinguishing between the perimeter formula and the area formula and relating these formulas to geometrical situations. During the lesson, students had concrete visual experiences (for example, via grid paper drawing) enabling them to re-state these two formulas situated in mathematically relevant context (for example, via a numerical table seeing a number pattern). In particular, students were open to opportunities to create for themselves a comparison between their prior knowledge and new experiences. This was conducive to the understanding of the concepts behind the two formulas and thus re-shaped students’ understanding. Furthermore, by observing a simple numerical pattern from the table, some students were able to re-present the perimeter formula:
• Students observed from the tables that \((\text{length} + \text{width}) = \text{perimeter} \div 2\).

This new “formula”, as articulated during a group presentation, was not due to an algebraic manipulation of the known perimeter formula; rather it was a numerical pattern observed by some students from the numerical tables that they had constructed. This alternative formula indicated a re-shaping of a stable mathematical concept.

(II) Second category of understanding: Shaping new mathematical knowledge

By keeping the perimeter fixed (concretized by a pre-cut string), students were able to generate variations in length and width and hence constructed rectangles of different shapes. This enabled them to observe some obvious invariant patterns:

• In these combinations of length and width, whenever the length is increased by 1, the corresponding width will decrease by 1.
• The longer the perimeter, the more combinations of length and width there are.

This kind of separation (keeping an aspect fixed while varying other aspects) activity was a new mathematical experience for students. A student was able to delve deeper and made the following statement about combination of numbers:

• For a rectangle with a fixed perimeter, there are many possible combinations of length and width. For example, a 16 cm long string can outline 4 different rectangles provided that we only use integral values for length and width.

Notice this statement is a primitive mathematical proposition situated in a geometrical context concerning the nature of number. This student eventually advanced to the third category of understanding which I will discuss in the next section. After experiencing the same patterns using different given tools, most students were able to make the following generalizations (see examples of two students’ work in Figure 4):

• Increasing the perimeter of a rectangle does not necessarily increase its area.
• If we are given only the perimeter of a rectangle, we cannot determine its area. We need to know also either the length or the width.
• When two rectangles have either the same length or the width, then the rectangle with the longer perimeter would have the larger area.
• A rectangle with a fixed perimeter would have the largest area when its length and width are equal (a square).

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Discovery
There are many combinations for their houses. If compare Chun Ming’s largest area combination with Mei Mei’s smallest area combination, then Chun Ming’s can be bigger, but conversely, if compare Mei Mei’s largest area combination with Chung Ming’s smallest area combination, then Mei Mei’s can said to be bigger.

Conclusion
Therefore, can only look at which shape.

Figure 4. Examples of students’ work with translations
These conclusions reflected that most students achieved the objectives set out by the teacher (i.e. the four concepts on the relationship between perimeter and area). In this stage of understanding, students shaped new ideas using their own language.

(III) Third category of understanding: Creating/discovering mathematical knowledge beyond the present context

Two students made exceptional mathematical discoveries that were beyond the expectations of the teacher. They used the numerical data in their tables to make mathematical conjectures about numbers and number patterns. These conjectures transcended the concrete geometrical context of the lesson to the domains of number system and number sequence. They therefore demonstrated a higher level of understanding above the second category of understanding. There were three types of conjecture: nature of numbers, concept of convergence and formulas representing number patterns.

1. Nature of numbers
   - If numbers other than integers can be used, then there are ‘infinitely many’ (the student who wrote this comment actually used the $\infty$ sign) combinations of length and width for a rectangle with a fixed perimeter.
   - A rectangle with a fixed perimeter has the largest area when its length and width are equal (a square). This is true for integral and decimals values for the length and the width.

2. Concept of convergence
   - The area increases as the width increases while the length decreases. The more the rectangle is approximately like a square, the larger its area will become.

3. Formulas representing number patterns
   - Generally if the given perimeter is divisible by 4, then there are $(\text{perimeter} \div 4)$ combinations of length and width. Hence there are $(\text{perimeter} \div 4)$ rectangles with this given perimeter.
   - There is a sequence of differences between two consecutive areas: 15, 13, 11, 9, 7, 5, 3, 1.
     (a) The first term of the sequence can be obtained by the formula
(Perimeter ÷ 2) – 3

(b) The number of term in the sequence can be obtained by the formula

(Perimeter ÷ 4) – 1

All of these findings were results of two students being able to “see” the tables of numerical data organized in a simple ascending and descending manner in different ways. The stable structure of the tabulation enabled their focus of attention to vary simultaneously among different domains: imagining infinite number of decimals between integers, getting closer and closer to a known shape, finding ways to encapsulate number patterns. This fusing together of different perspectives could have triggered insights that contributed to the two students’ creative findings. In particular, the two students showed a coherent understanding of mathematical ideas.

Discussion

In the beginning, I asked the question what generates a mathematical experience for students, and suggested that pedagogical tools should play a critical role to mediate between students and mathematical knowledge. The above lesson illustrated that simple tools like grid paper, pin-board and table to record numerical values, when used appropriately, can empower students to re-shape, shape and even create mathematical knowledge. In particular, these tools have the potential to help students create variation experiences. Variation is a driving force for learning and awareness (Marton and Booth, 1997) and for gaining mathematical knowledge (see for examples, Watson and Mason, 2005; Leung, 2008). Dienes (1963) attributed the abstraction and the generalization processes in mathematical thinking by what he called the perceptual variability principle and the mathematical variability principle:

“The perceptual variability principle stated that to abstract a mathematical structure effectively, one must meet it in a number of different situations to perceive its purely structural properties. The mathematical variability principle stated that as every mathematical concept involved essential variables, all these mathematical variables need to be varied if the full generality of the mathematical concept is to be achieved.”

(Dienes, 1963, p.158)
The tools used in the lesson provide opportunities for students to experience these two variability principles: comparing old and new knowledge, contrasting what is and what is not, separating out patterns with respect to fixed conditions, making generalizations after seeing the same phenomenon under different tool usages, and simultaneously fusing together different varying aspects to construct new knowledge. With these usages, the tools become pedagogical instruments. I therefore suggest that a *mathematically rich learning environment* is one that provides students with pre-designed tools (in the sense of situating tools in a pedagogical context) with suitable constraints that allow students to construct variation experiences that are able to induce (a) discernment of invariant structure/pattern and (b) formulation of representations for the discerned mathematical structure/pattern. Such an environment produces rich mathematical experiences for students. On the basis of the findings, I have identified three categories of understanding from the students’ learning outcome (Figure 3). Even though these categories appear to be in a hierarchical (or coherent) order, by no means is it a labelling of students’ abilities. Rather, it shows that the pedagogical tools were able to cater for individual differences, allowing students to learn multi perspectives to view the object of learning. The crucial finding is that most of the student outcomes matched the teaching objectives set by the teacher. The hierarchy is an observed pedagogical trajectory that can be meaningful in designing future lessons.

Ma (1999) studied elementary mathematics teachers’ subject matter knowledge and coined the abbreviation PUFM (Profound Understanding of Fundamental Mathematics) to mean “an understanding of the terrain of fundamental mathematics that is deep, broad, and thorough” (Ma, 1999, p.120). PUFM has four interrelated properties: connectedness, multi perspectives, basic ideas and longitudinal coherence. With respect to these four interrelated properties, deriving from students’ learning outcomes, the learning environment of this lesson seems to have nourished a collective students’ PUFM on the relationship between area and perimeter of a rectangle.

The purpose of this paper is not to advocate any conclusive framework for teaching and learning of elementary mathematics, rather it is a deep reflection on a successful mathematics lesson and an attempt to learn from it critical features of such a lesson.
References


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Dr Allen Leung obtained a PhD in mathematics at the University of Toronto. He is currently an Assistant Professor at the Department of Mathematics and Information Technology, The Hong Kong Institute of Education. Dr Leung’s research areas include the pedagogical and epistemological potentials of dynamic geometry environments, application of the Theory of Variation in teaching and learning, Lesson Study, and language and mathematics. In particular, he pioneered the use of the Theory of Variation as an interpretative tool for dynamic geometry environments. He was a member of the organizing team for Topic Study Group 22: Technology in the teaching and learning of mathematics for the 11th International Congress on Mathematics Education, 2008, Monterrey, Mexico.