Teaching a New Method of Partial Fraction Decomposition to Senior Secondary Students: Results and Analysis from a Pilot Study

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Abstract
In this paper, we introduce a new approach to compute the partial fraction decompositions of rational functions and describe the results of its trials at three secondary schools in Hong Kong. The data were collected via quizzes, questionnaire and interviews. In general, according to the responses from the teachers and students concerned, this new approach has potential to be introduced at the senior secondary level, as an alternative to the method of undetermined coefficients described in common secondary mathematics textbooks. Some remarks on the related pedagogical issues are included.

Keywords: Partial fraction decomposition; the Improved Heaviside approach; cover-up techniques.

1. Introduction
The problem of decomposing a rational function into partial fractions is often encountered in the study of calculus, differential equations, discrete mathematics and control theory, etc. According to Norman [4], there are two common methods for computing the unknown partial fraction coefficients. One method is to use the method of undetermined coefficients, where the unknowns can be found by solving a system of linear equations. However, the drawback is that the calculations involved could be quite tedious (see [2-3], [17]). Another method is to apply the Heaviside’s cover-up technique, which uses simple substitutions to determine the unknown coefficients of the partial fractions with single poles, and successive differentiations to handle those with multiple poles (see [4-5]). In recent years, an improved Heaviside approach for finding the partial fraction decompositions (PFD) of proper rational functions was introduced and developed by Man [5-7], which involves substitutions and polynomial divisions only to determine the unknown partial fraction coefficients. Compared with the other techniques described in [8-12, 17-18], the distinguishing feature of this new approach is that there is no need to use differentiation or to solve a system of linear equations. In order to study its potential application in secondary mathematics education, a study of its trials at three secondary schools in Hong Kong were conducted in 2010-11. The data were collected for analysis via quizzes, questionnaires and face-to-face interviews. In this paper, we will
report and discuss the results. The paper is organized as follows. In section 2, a brief introduction of the new approach to PFD is provided. In section 3, we will describe the backgrounds of the study, including the objective, the participants and the instruments involved. In section 4, the results will be described and discussed. Then, we will conclude with some final remarks in the last section.

2. The Improved Heaviside approach

In this section, we provide a brief introduction of the improved Heaviside approach to PFD with some examples below. For details of its theoretical background, please refer to [5].

The improved Heaviside approach:

Input: a rational function \( F(x) = \frac{a(x)}{b(x)} \), where \( a(x) \) and \( b(x) \) are polynomials such that \( \deg a(x) < \deg b(x) \), \( b(x) = (x - \alpha_1)^{n_1}(x - \alpha_2)^{n_2} \cdots (x - \alpha_s)^{n_s} \), \( \alpha_1, \alpha_2, \ldots, \alpha_s \) are constants and \( n_1, n_2, \ldots, n_s \) are positive integers.

Output: The PFD of \( F(x) \), namely

\[
F(x) = \sum_{i=1}^{s} \left( \frac{a_{i,1}}{(x - \alpha_i)} + \frac{a_{i,2}}{(x - \alpha_i)^2} + \cdots + \frac{a_{i,n_i}}{(x - \alpha_i)^{n_i}} \right),
\]

where \( a_{i,j} \) are constants.

Procedure: The unknown coefficients \( a_{i,j} \) can be found by the following steps.

S1: \( a_{i,n_i} = \frac{a(x)}{b(x)} \cdot (x - \alpha_i)^{n_i} \bigg|_{x=\alpha_i} \) \hspace{1cm} (Note: This is the so-called cover-up technique);

S2: \( a_{i,n_i-j} = \left[ \frac{a(x)}{b(x)} - \sum_{k=0}^{j-1} \frac{a_{i,n_i-k}}{(x - \alpha_i)^{n_i-k}} \right] (x - \alpha_i)^{n_i-j} \bigg|_{x=\alpha_i} \);

where \( 1 \leq i \leq s \) and \( 1 \leq j \leq n_i - 1 \).

Thus, the cover-up technique is applied to compute \( a_{i,n_i} \) first. Then, the known partial fractions are subtracted from \( F(x) \) and simplified to become a new function\(^1\). Next, we can apply the same technique to handle the new function(s) obtained again and again, until all the unknown coefficients \( a_{i,n_i-j} \) are found.

\(^1\) The simplifications involved can be done by means of polynomial division(s).
Example 1

Let us consider the function \( F(x) = \frac{4 + x - x^2}{(x + 3)(x - 1)^2} \).

The PFD concerned can be represented as:

\[
F(x) = \frac{a}{x - 1} + \frac{b}{(x - 1)^2} + \frac{c}{x + 3},
\]

where \( a, b \) and \( c \) are unknown constants to be determined.

Applying the cover-up technique, we have:

\[
b = \frac{4 + x - x^2}{x + 3} \\
\left. \frac{4 + x - x^2}{x + 3} \right|_{x = 1} = \frac{4 + 1 - 1}{4} = 1; \\
c = \frac{4 + x - x^2}{(x - 1)^2} \\
\left. \frac{4 + x - x^2}{(x - 1)^2} \right|_{x = 3} = \frac{4 - 3 - (-3)^2}{(-3 - 1)^2} = -\frac{1}{2}.
\]

Then, we can subtract the known partial fractions from \( F(x) \) to obtain the value of \( a \):

\[
\frac{4 + x - x^2}{(x + 3)(x - 1)^2} - \frac{b}{x - 1} - \frac{c}{x + 3} = \frac{4 + x - x^2 - (x + 3) + (x - 1)^2/2}{(x - 1)^2(x + 3)}
\]

\[
= \frac{3 - 2x - x^2}{2(x - 1)^2(x + 3)}
\]

\[
= -\frac{1}{2(x - 1)}
\]

\[\therefore a = -1/2.\]

Hence, \( F(x) = \frac{-1}{2(x - 1)} + \frac{1}{(x - 1)^2} - \frac{1}{2(x + 3)}. \)

Note: We can use factorization or polynomial division to simplify the expression \((3 - 2x - x^2) / 2(x - 1)^2(x + 3)\) to obtain the last partial fraction.

Example 2

Let us consider the function \( F(x) = \frac{3x + 2}{(x + 1)(x^2 + 2)}. \)

The PFD concerned can be represented as:

\[
F(x) = \frac{a}{x + 1} + \frac{bx + c}{x^2 + 2},
\]

where \( a, b \) and \( c \) are unknown constants to be determined.

Applying the cover-up technique, we have:

\[
a = \left. \frac{3x + 2}{x^2 + 2} \right|_{x = -1} = \frac{3(-1) + 2}{1 + 2} = -\frac{1}{3}.
\]

Then, we can subtract the known partial fraction from \( F(x) \) to obtain:

\[
\frac{3x + 2}{(x + 1)(x^2 + 2)} - \frac{a}{x + 1} = \frac{3x + 2}{(x + 1)(x^2 + 2)} + \frac{1}{3(x + 1)} = \frac{x^2 + 9x + 8}{3(x + 1)(x^2 + 2)} = \frac{x + 8}{3(x^2 + 2)}
\]
\[ b = \frac{1}{3} \quad \text{and} \quad c = \frac{8}{3}. \]

Hence, \[ F(x) = \frac{-1}{3(x + 1)} + \frac{x + 8}{3(x^2 + 2)}. \]

Note: We can use factorization or polynomial division to simplify the expression \((x^3 + 9x + 8)/3(x + 1)(x^2 + 2)\) to obtain the last partial fraction.

3. **Background of the study**

3.1 **Objective**

The main objective of this study was to explore the potential application of the new approach to PFD in secondary mathematics education, via trials at secondary schools in Hong Kong.

3.2 **Participants**

3.2.1 **Participating schools**

Letters were sent to five local secondary schools to invite their participation at the beginning of this study. Three of them replied that they could participate in the academic year 2010-11 and could meet the working schedule of the study, so they were chosen to be the participating schools. For convenience, they were named School A, School B and School C below. School A and School C were co-educational schools, while School B was a girls school. They were all non-private government subsidized schools.

3.2.2 **Participating teachers**

In each participating school, one mathematics teacher with at least 3 years of senior secondary mathematics teaching experience, was invited to conduct the trial lessons in his/her own senior secondary mathematics class. For convenience, they were named Teacher A, Teacher B and Teacher C below. Teacher B was a female teacher, while the other two were male teachers. During the trial lessons, each one taught the method of undetermined coefficients (Method I) and then the method of Improved Heaviside’s approach (Method II) to his/her mathematics class. All the trial lessons conducted by each participating teacher were observed by the investigators.

3.2.3 **Participating students**

A total of 51 students participated in the trials, with 18 students from School A, 18 students from School B and 15 students from School C. They were all secondary sixth students (also known as F. 6 or A-level\(^2\) students in Hong Kong) and their ages were between 17-18 years.

\(^2\) It refers to the pre-university studies in Hong Kong, which prepare the students to sit for the public A-level examination and then apply for admission to local universities if satisfactory results are obtained in the examination. The normal period of the A-level studies is two years.
3.3 Quizzes

There were a total of four quizzes. Each quiz had three problems, namely, one with non-repeated linear factors, one with repeated linear factors, and one with a linear factor and an irreducible quadratic factor, in the denominators of the given rational functions (see Appendix A). After learning Method I, the students were asked to complete the first two quizzes (the problems were similar) by Method I, and then the answer scripts were collected. Similarly, after learning Method II, the students were asked to complete another two quizzes (same as the first two quizzes) by Method II, and then the answer scripts were collected. The time allowed for each quiz was about 15~20 minutes. Marks were awarded according to the performances in the following domains: (a) accuracy in calculations (1-3 marks: from 1=low to 3=high); (b) number of correct partial fraction coefficients obtained (0-3 marks: from 0=none to 3=all) and (c) mastery of the method concerned (1-4 marks: from 1=unsatisfactory to 4=excellent). Also, there were two self-evaluation questions included at the end of each quiz (see appendix B), which were used to measure the students’ perception of difficulties of the problems and their self-confidence in answering them correctly. Then, the responses were recorded by a five point Likert scale (1-5 points: from 1=very unconfident /very easy to 5=very confident/very difficult) for subsequent analysis. For each participating school, the time spent on teaching the topics and the quizzes concerned were almost the same, as shown in Table 1 below.

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Topic*</th>
<th>Length of the lesson</th>
<th>Time spent on teaching</th>
<th>Time spent on quizzes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lesson one</td>
<td>Method I</td>
<td>135 min</td>
<td>95 ~ 105 min</td>
<td>30 ~ 40 min</td>
</tr>
<tr>
<td>Lesson two</td>
<td>Method II</td>
<td>135 min</td>
<td>95 ~ 105 min</td>
<td>30 ~ 40 min</td>
</tr>
</tbody>
</table>

Note*: Method I refers to the method of undetermined coefficients, while Method II refers to the Improved Heaviside approach.

3.4 Questionnaire

At the end of the trial lesson, a questionnaire was distributed to the students to collect their views on the following questions: (1) Which method (I/II) is easier to understand? (2) Which method (I/II) is easier to use for finding PFD? (3) Should the new approach be introduced to students or not? (4) Which level(s) is/are more suitable to introduce the new approach to the students if the answer to Q3 is affirmative? (5) Which method (I/II) is more interesting? (see appendix C) Additional comments were allowed and they were treated as supplements to the responses collected at the other parts of the questionnaire.
3.5 Interviews

After the trial lessons, each participating teacher and three students randomly chosen from those who had been present for both quizzes were interviewed by the investigators, in an individual face-to-face mode. The interviews were aimed at collecting more in-depth responses or feedback (if any) from the participants. Each interview lasted for 15~20 minutes. For convenience, the three student interviewees from School A were named by A₁, A₂, A₃. Similarly, those from School B and School C were named by B₁, B₂, B₃, and C₁, C₂, C₃, respectively.

3.6 Data analysis

To analyze the students’ performances in the quizzes, the paired $t$-test was used to study the mean scores obtained by using the two methods in each school. The null hypothesis was to test if the mean scores obtained by the methods were equal or not in several domains, namely “accuracy in calculation” (abbreviated as calculation), “number of correct partial fraction coefficients obtained” (abbreviated as coefficients), “mastery of the method concerned” (abbreviated as mastery), “perception of difficulties of the problems” (abbreviated as difficulties) and “the self-confidence in answering the problems correctly” (abbreviated as confidence). The $F$-test was used to test if the standard deviations of the scores obtained by the two methods were significantly different from each other or not. The level of significance adopted in the tests was $\alpha=0.05$.

4. Results and discussions

The results and the statistical analysis of the data collected are described in the sections below. The students chosen were from those who had been present for both quizzes. Since two students were absent in School B during the quizzes for Method I, so the total number of students shown in the tables below was 16 instead of 18.

4.1 Results of the quizzes in the first three domains

<table>
<thead>
<tr>
<th>School A</th>
<th>Method</th>
<th>$N$</th>
<th>Mean</th>
<th>SD</th>
<th>$t$</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Method I</td>
<td>18</td>
<td>2.41</td>
<td>0.43</td>
<td>-0.94</td>
<td>1.16</td>
</tr>
<tr>
<td></td>
<td>Method II</td>
<td>18</td>
<td>2.54</td>
<td>0.40</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>School B</th>
<th>Method</th>
<th>$N$</th>
<th>Mean</th>
<th>SD</th>
<th>$t$</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Method I</td>
<td>16</td>
<td>2.46</td>
<td>0.38</td>
<td>-2.67*</td>
<td>2.51*</td>
</tr>
<tr>
<td></td>
<td>Method II</td>
<td>16</td>
<td>2.76</td>
<td>0.24</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>School C</th>
<th>Method</th>
<th>$N$</th>
<th>Mean</th>
<th>SD</th>
<th>$t$</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Method I</td>
<td>15</td>
<td>2.54</td>
<td>0.59</td>
<td>-0.49</td>
<td>1.24</td>
</tr>
<tr>
<td></td>
<td>Method II</td>
<td>15</td>
<td>2.64</td>
<td>0.53</td>
<td></td>
<td></td>
</tr>
<tr>
<td>School</td>
<td>Method</td>
<td>N</td>
<td>Mean</td>
<td>SD</td>
<td>t</td>
<td>F</td>
</tr>
<tr>
<td>--------</td>
<td>--------</td>
<td>----</td>
<td>------</td>
<td>-----</td>
<td>-----</td>
<td>------</td>
</tr>
<tr>
<td>A</td>
<td>I</td>
<td>18</td>
<td>2.13</td>
<td>0.67</td>
<td>-1.25</td>
<td>1.66</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>18</td>
<td>2.38</td>
<td>0.52</td>
<td>-1.25</td>
<td>1.66</td>
</tr>
<tr>
<td>B</td>
<td>I</td>
<td>16</td>
<td>2.28</td>
<td>0.49</td>
<td>-3.23*</td>
<td>2.85*</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>16</td>
<td>2.74</td>
<td>0.29</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>I</td>
<td>15</td>
<td>2.56</td>
<td>0.56</td>
<td>-0.52</td>
<td>1.31</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>15</td>
<td>2.66</td>
<td>0.49</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4. Mean scores and SD in the *mastery* domain

<table>
<thead>
<tr>
<th>School</th>
<th>Method</th>
<th>N</th>
<th>Mean</th>
<th>SD</th>
<th>t</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>I</td>
<td>18</td>
<td>2.76</td>
<td>0.52</td>
<td>-1.86</td>
<td>1.04</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>18</td>
<td>3.08</td>
<td>0.51</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>I</td>
<td>16</td>
<td>3.16</td>
<td>0.52</td>
<td>-2.99*</td>
<td>2.48*</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>16</td>
<td>3.62</td>
<td>0.33</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>I</td>
<td>15</td>
<td>3.17</td>
<td>0.83</td>
<td>-0.70</td>
<td>1.02</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>15</td>
<td>3.38</td>
<td>0.82</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From Tables 2-4, we can see that the mean scores (and the standard deviations) of Method II were larger (and smaller) than that of Method I in each domain. In particular, the students’ performances of School B in using Method II were significantly better than that of Method I in each domain, at \( \alpha=0.05 \). It implies that the students’ performances in using Method II could be comparable to (or even better in the case of School B) that of Method I in each school. According to the class observations by the investigators, all the three participating teachers adopted a similar teacher-centered approach to introduce the two methods. However, Teacher B had spent a bit more time (around 10~15 minutes) than the other two teachers to explain the rationale and mathematical background of Method II, rather than just focusing on providing examples to illustrate how it work. By comparing the mean scores of School B with the other two schools in each domain, we can see that the priority of the students’ performances of School B changed from the middle to the top when Method II was used after Method I. On the contrary, the priority of the performances of School A remained unchanged (lowest among the three schools), and that of School C changed from the top to the middle when the method of calculation was switched from Method I to Method II. We believe that the time spent on explaining the rationale and mathematical background of Method II had positive effects on the understanding and mastery of such a method and the approach adopted by Teacher B could enhance the students’ relational understanding (know why) of the concepts behind Method II, as well as their instrumental understanding (know how) of the calculation procedure involved (see [15-16]).
4.2 Results of the quizzes in the self-evaluation domains

Regarding the self-evaluations of difficulties of the problems and the confidence in answering them by using Method I and Method II, the results are shown in Tables 5-6 below.

Table 5. Perception of difficulty of the problems

<table>
<thead>
<tr>
<th>School</th>
<th>Method</th>
<th>N</th>
<th>Mean</th>
<th>SD</th>
<th>t</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Method I</td>
<td>18</td>
<td>2.71</td>
<td>0.71</td>
<td>0.13</td>
<td>1.27</td>
</tr>
<tr>
<td></td>
<td>Method II</td>
<td>18</td>
<td>2.68</td>
<td>0.63</td>
<td>-0.30</td>
<td>1.13</td>
</tr>
<tr>
<td>B</td>
<td>Method I</td>
<td>16</td>
<td>2.89</td>
<td>0.98</td>
<td>0.76</td>
<td>1.22</td>
</tr>
<tr>
<td></td>
<td>Method II</td>
<td>16</td>
<td>2.99</td>
<td>0.92</td>
<td>-0.41</td>
<td>0.52</td>
</tr>
<tr>
<td>C</td>
<td>Method I</td>
<td>15</td>
<td>2.78</td>
<td>0.83</td>
<td>0.11</td>
<td>1.46</td>
</tr>
<tr>
<td></td>
<td>Method II</td>
<td>15</td>
<td>2.56</td>
<td>0.75</td>
<td>-0.41</td>
<td>0.52</td>
</tr>
</tbody>
</table>

Table 6. Confidence in answering the problems

<table>
<thead>
<tr>
<th>School</th>
<th>Method</th>
<th>N</th>
<th>Mean</th>
<th>SD</th>
<th>t</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Method I</td>
<td>18</td>
<td>3.53</td>
<td>0.58</td>
<td>0.11</td>
<td>1.46</td>
</tr>
<tr>
<td></td>
<td>Method II</td>
<td>18</td>
<td>3.51</td>
<td>0.48</td>
<td>-0.41</td>
<td>0.52</td>
</tr>
<tr>
<td>B</td>
<td>Method I</td>
<td>16</td>
<td>3.76</td>
<td>0.68</td>
<td>0.92</td>
<td>0.81</td>
</tr>
<tr>
<td></td>
<td>Method II</td>
<td>16</td>
<td>3.88</td>
<td>0.94</td>
<td>0.92</td>
<td>0.81</td>
</tr>
</tbody>
</table>

From Tables 5-6, we can see that there were no significant differences between the mean scores of Method I and Method II in the difficulty and the confidence domains in each school, at \( \alpha=0.05 \). However, School B had the highest scores in the confidence domain when compared with that of the other two schools. These results were consistent with the performances of School B in the other three domains, as described in section 4.1. On the other hand, the mean scores of School B in the difficulty domain were also higher than the other two schools. This may be due to the general perception of mathematics by the female students as abstract and difficult at the secondary level. According to our lesson observations, Teacher B had attempted to explain Method II more clearly to the students and a few minutes of time was spent on stimulating the students to think why Method II works, apart from giving examples to demonstrate how it works. Although the students might not totally master the concepts behind, there were opportunities for them to be motivated to think and discuss the new topic in class. Such an integration of subject knowledge and pedagogical content
knowledge ([1], [13-14]) is important and crucial for the successful delivery of a mathematics lesson by a teacher.

4.3 Responses and feedbacks from questionnaire and interview
The responses and feedbacks collected from the questionnaire and the interviews are summarized and discussed below.

4.3.1 Ease of understanding

<table>
<thead>
<tr>
<th></th>
<th>Method I</th>
<th>Method II</th>
<th>No comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>School A</td>
<td>13 (72.2%)</td>
<td>5 (27.8%)</td>
<td>Nil</td>
</tr>
<tr>
<td>School B</td>
<td>8 (50.0%)</td>
<td>8 (50.0%)</td>
<td>Nil</td>
</tr>
<tr>
<td>School C</td>
<td>11 (73.3%)</td>
<td>4 (26.7%)</td>
<td>Nil</td>
</tr>
</tbody>
</table>

From Table 7, it shows that more than 70% of students in School A and School C perceived that Method I was easier to understand than Method II. During the interview, Teacher A explained that the concept of comparing coefficients had been taught at junior secondary before, so Method I was not hard for the students. Similar comments were made by Teacher C and the student interviewees. However, the responses of the students in School B were more balanced towards Method I and Method II. So, we believe the effort and approach adopted by Teacher B in teaching Method II had positive effects on the students. Here are some extracts from the interviews:

Teacher A: Students had learnt how to compare coefficients of two equal polynomials in junior secondary before. So, they think Method I is easier to them. On the other hand, the improved Heaviside approach (Method II) is totally new to the students and even to the teachers, so they think it is harder to understand.

Students A3: I think Method I is easier to understand. We have learnt similar ideas in lower form before. So, this method is easier to me.
Interviewer: What do you mean by similar ideas?
Student A3: I mean the technique of comparing coefficients.

Student B1: Method one is easier to understand and the steps are very direct. For Method II, it is hard to imagine why it works at the beginning. After we have learnt it, I think that it is amazing.
4.3.2 Ease of use

Table 8. Responses on ease of use

<table>
<thead>
<tr>
<th></th>
<th>Method I</th>
<th>Method II</th>
<th>No comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>School A</td>
<td>6 (33.3%)</td>
<td>12 (66.7%)</td>
<td>Nil</td>
</tr>
<tr>
<td>School B</td>
<td>1 (6.2%)</td>
<td>15 (93.8%)</td>
<td>Nil</td>
</tr>
<tr>
<td>School C</td>
<td>8 (53.3%)</td>
<td>7 (46.7%)</td>
<td>Nil</td>
</tr>
</tbody>
</table>

From Table 8, we can see that over 90% of students in School B perceived that Method II was easier to use, as compared with those in the other two schools. These responses were consistent with the results described in sections 4.1–4.2. Here are some extracts from the interviews:

Teacher A: Method I is easier to understand. But, sometimes the calculations are tedious and easy to make mistakes. If there are more than two unknowns, the system of equations could be quite complicated. We need more time to solve them. For Method II, the algorithm is very clear. The first step is to do this and the second step is to do that, and so on. We can repeat the first two basic steps to lower the degree of the factor in the denominator at each time. It is easy to follow. However, the underlying concept is not as easy as Method I. I had tried to explain it to the students, but not many of them could get it.

Student A1: Method II works faster. It is convenient to use.

Student B1: When we use Method II, we do not make mistakes so easily.
Interviewer: Can you elaborate a bit?
Student B1: When we do the subtraction step, for instance \( f(x) - a \frac{x - c}{(x + 1)^2} \), we know the denominator of the result can be simplified to \( x+1 \). If it cannot be done, we are sure that there must be something wrong.

Student B3: Method II is easier to use. I think it is faster and more stable.
Interviewer: What do you mean by stable?
Student B3: I mean it is not so easy to make mistakes.

4.3.3 Suitability for introducing Method II to students

Table 9. Responses to suitability for introducing Method II to students

<table>
<thead>
<tr>
<th></th>
<th>Yes</th>
<th>No</th>
<th>No comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>School A</td>
<td>18 (100%)</td>
<td>0 (0%)</td>
<td>Nil</td>
</tr>
<tr>
<td>School</td>
<td>Method I</td>
<td>Method II</td>
<td>Notes</td>
</tr>
<tr>
<td>--------</td>
<td>----------</td>
<td>-----------</td>
<td>-------</td>
</tr>
<tr>
<td>B</td>
<td>15 (93.8%)</td>
<td>1 (6.2%)</td>
<td>Nil</td>
</tr>
<tr>
<td>C</td>
<td>13 (86.7%)</td>
<td>2 (13.3%)</td>
<td>Nil</td>
</tr>
</tbody>
</table>

From Table 9, we can see that most students agreed that Method II should be taught. Also, all the three participating teachers responded that Method II could be introduced to the students after they had learnt Method I. Though some students could not understand the theory behind Method II well nor did the teachers explain it more clearly to them, they did appreciate the computational efficiency of Method II when it was applied to compute the unknown partial fraction coefficients. In fact, some teacher (such as Teacher C) said that Method II could be a good alternative approach to Method I described in common secondary mathematics textbooks. Here are some extracts from the interviews:

*Teacher A:* I prefer to teach the method of undetermined coefficients (Method I) first. After the students have learnt this method, then I will teach Method II to allow them to learn one more method to find partial fraction decompositions.

*Interviewer:* Why not reverse the teaching sequence?

*Teacher A:* If we teach Method II first, the students may not feel the power of this method. Also, they may not understand the underlying concepts of Method II so easily, as compared with Method I.

*Teacher B:* Some students told me that Method II was more stable. It means there is less chance to make mistakes when they use Method II. Although finding the common denominator (in the subtraction step) may be tedious sometimes, they can improve the speed of calculations via more practices. I support to teach Method II.

*Teacher C:* It is good to learn more approaches to partial fraction decomposition. I agree to teach Method II.

*Student A1:* Method II should be taught. It is a good and interesting approach to me.

*Student A2:* I agree to teach Method II. But I think it is not good to teach this method first.

*Interviewer:* Why?

*Student A2:* After learning Method I, I will expect to learn a faster and a convenient approach, such as the cover-up approach (Method II). It is more natural to me.

*Student B3:* I like Method II. It is faster and safer.

*Interviewer:* What do you mean by “safer”?
Student B3: I mean we have less chance to make mistakes.

Student C1: I support to teach Method II. But if the cover-up method (Method II) is taught first, I guess many students may not have much interest to learn the method of undetermined coefficients (Method I).

4.3.4 Suitability for introducing Method II at secondary or undergraduate levels

Table 10. Responses to level(s) suitable for introducing Method II

<table>
<thead>
<tr>
<th></th>
<th>Junior secondary (F.1-F.3)</th>
<th>Senior secondary (F.4-F.5)</th>
<th>Senior secondary* (F.6-F.7)</th>
<th>Undergraduate (UG)</th>
<th>No response</th>
</tr>
</thead>
<tbody>
<tr>
<td>School A</td>
<td>2 (11.1%)</td>
<td>1 (5.6%)</td>
<td>10 (55.6%)</td>
<td>2 (11.1%)</td>
<td>3 (16.6%)</td>
</tr>
<tr>
<td>School B</td>
<td>0 (0.0%)</td>
<td>4 (25.0%)</td>
<td>10 (62.5%)</td>
<td>0 (0.0%)</td>
<td>2 (12.5%)</td>
</tr>
<tr>
<td>School C</td>
<td>1 (6.7%)</td>
<td>4 (26.7%)</td>
<td>7 (46.6%)</td>
<td>0 (0.0%)</td>
<td>3 (20.0%)</td>
</tr>
</tbody>
</table>

Note*: They are often called A-levels or Pre-university levels at Hong Kong.

From Table 10, we can see that more than 70% students responded that Method II should be introduced at F.6/F.7 or the lower forms. Also, the teacher interviewees had made similar responses on this issue. Here are some extracts from the interviews:

Teacher A: It can be introduced at F.2. There is a topic called “rational functions” in F.2, which is about how to find common denominator of two terms. We can teach Method II after learning this topic. We can concentrate on handling the case with linear factors in the denominator. I believe they can handle it.

Interviewer: How about the related concepts of Method II? Is it easy for them?
Teacher A: We can teach how to use this method at F.2, and leave the discussion of its concepts to senior forms, say F.6/F.7.

Teacher B: If we only want to teach how to use Method II, I think F.4 students have enough background to learn this approach.

Interviewer: How about the underlying concepts?
Teacher B: I think it is more suitable for senior secondary students.

Teacher C: It can be introduced at F.6/F.7 level.

Interviewer: Why?
Teacher C: Because the topic of partial fraction decomposition is scheduled in the F.6/F.7 syllabus, but not the lower forms.
Student A3: The topic of partial fractions is easy. Method II could be included in the syllabus of F.6. It can even be included into lower forms, if the applications of partial fraction decomposition (such as integration or summation) are not required.

Student B1: It is okay to learn partial fraction decomposition at F.6. If Method II is taught at junior forms, I think it is also okay. It is not difficult to follow the calculation steps.

Interviewer: What level do you suggest to start learning Method II?
Student B1: It can be at F.4.

Student B2: I think it is possible to teach it (Method II) at F.4/F.5. They can understand how to use Method II. For example, we have already learnt the method of substitution in lower forms.

4.3.5 Interestingness of the method concerned

<table>
<thead>
<tr>
<th></th>
<th>Method I</th>
<th>Method II</th>
<th>No comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>School A</td>
<td>4 (22.2%)</td>
<td>9 (50.0%)</td>
<td>5 (27.8%)</td>
</tr>
<tr>
<td>School B</td>
<td>2 (12.5%)</td>
<td>10 (62.5%)</td>
<td>4 (25.0%)</td>
</tr>
<tr>
<td>School C</td>
<td>3 (20.0%)</td>
<td>7 (46.7%)</td>
<td>5 (33.3%)</td>
</tr>
</tbody>
</table>

From Table 11, we can see that more students perceived that Method II was more interesting than Method I, especially in School B. These results were consistent with those shown in Tables 8-10. In fact, students often appreciate the method more if it is simple and useful in solving problems. Of course, the approach adopted by the teacher was also a crucial factor to influence the students’ responses. For instance, more than 60% of School B students responded that Method II was more interesting to them when compared with Method I. Here are some extracts of the interviews:

Teacher A: The cover-up method (Method II) is more interesting to me. It can relate the theory into practices (applications). It is very good. Although the method of undetermined coefficients (Method I) is more basic, but I won’t say it is interesting.

Teacher B: Compared with Method II, I would say the method of undetermined coefficients (Method I) is rather boring. Comparing coefficients, setting up equations and then solving them, all such operations are very basic and routine. The Heaviside’s
method (Method II) is amazing. The answers can be found by substitutions only. My students also said this method is very special and wonderful.

Teacher C: The Heaviside approach is wonderful. It is more interesting than the method of undetermined coefficients (Method I).

Student A3: Method II is more interesting. It makes us to think more.
Interviewer: What do you mean?
Student A3: I mean it caused me to think why it works like this, without having to compare coefficients and solve equations.

Student B2: Method I is like solving equations to find the unknowns. It is rather simple, but rather routine and boring. The idea of using cover-up (substitutions) in Method II is more interesting to me.

5. Concluding remarks
According to the results of this study, we can see that the Improved Heaviside approach (Method II) has the potential to be introduced to the students who are studying mathematics at the senior secondary level. In general, the students’ performances in using Method II could be comparable to (or even statistically significantly better in School B) that of Method I in the calculation domain, the coefficients domain and the mastery domain in each school. In fact, over 85% of students responded in the questionnaire that Method II could be introduced at the secondary level and over 70% said that it could be introduced at F.6/F.7 or the lower forms. Also, all the three participating teachers and considerable number of students perceived that Method II was more interesting to them (see Table 11). However, there are also drawbacks from the pedagogical perspectives. According to the responses from the questionnaires and interviews, many students responded that Method II was more difficult to understand than Method I. From what we observed in the trial lessons, the teachers concerned had spent too little time (though Teacher B was a bit better) on explaining why Method II works, but rather put more effort on illustrating how it works by examples. Thus, many of the students perceived that Method I was easier to understand than Method II. Most students preferred to learn Method I before Method II because the former one was more simple and straightforward to them. It indicated that the understanding of the concepts behind Method II could significantly affect the perception of its interestingness, as well as the performance in applying it to solve problems, as described in the previous sections.

In brief, we would like to conclude that Method II has potential to be introduced to the senior secondary mathematics curriculum. This study also provides us insights on certain
pedagogical issues to be addressed when teaching Method II, such as the time and effort spent on teaching and learning the concepts behind Method II, as well as the teaching sequence of Method I and Method II. We anticipate that explorations on how to teach or learn this new approach to PFD more effectively would be a further meaningful research topic to pursue.

**Acknowledgments**

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**References**


Appendix A

Problems in Quiz 1
Find the partial fraction decompositions of the following rational functions:

(a) \( f(x) = \frac{2x + 1}{x(x + 1)(x + 2)} \)

(b) \( f(x) = \frac{x^2 + 3x}{(x + 1)(x - 1)^2} \)

(c) \( f(x) = \frac{x + 1}{x(x^2 + 4)} \)

Problems in Quiz 2
Find the partial fraction decompositions of the following rational functions:

(a) \( f(x) = \frac{3x + 1}{x(x + 2)(x + 3)} \)

(b) \( f(x) = \frac{x^2 - 3x}{(x + 2)(x - 1)^2} \)

(c) \( f(x) = \frac{x - 1}{(x + 2)(x^2 + 1)} \)
Appendix B

Self-evaluation after answering each problem
1. Confidence in obtaining the correct answer (Please tick):
   □: Very Strong; □: Strong; □: Medium; □: Low; □: Very Low

2. Self-evaluation of the level of difficulty of the problem (Please tick):
   □: Very Hard; □: Hard; □: Medium; □: Easy; □: Very Easy

Appendix C

Questionnaire:
Please answer the questions below (Please tick as appropriate).
1. Which one of the following methods is easier to understand?
   □: Method of Undetermined Coefficients; □: Improved Heaviside Approach
   □: No comment

2. In general, which one of the following methods is easier to use for finding partial fraction decompositions?
   □: Method of Undetermined Coefficients; □: Improved Heaviside Approach
   □: No comment

3. Do you think the Improved Heaviside Approach should be introduced to students?
   □: Yes; □: No □: No comment

4. If your answer to Q6 is “Yes”, which level(s) do you think is/are appropriate for introducing to the students? (Note: You can choose more than one answer)
   □: AS level; □: A-level; □: Undergraduate level; □: Other level: ______;
   □: No comment

5. Which one of the following methods is more interesting to you?
   □: Method of Undetermined Coefficients; □: Improved Heaviside Approach
   □: No comment

6. Other comments (if any): ____________________________________________
Brief biographies of the authors:

Dr Man obtained a BSc (Hons) degree in Mathematics from the Hong Kong Polytechnic University, a MSc (with Distinction) degree and a PhD degree in Mathematics from the University of London, UK. He also obtained a PGDE (with Distinction) from the Chinese University of Hong Kong. He is currently an Associate Professor at the Department of Mathematics and Information Technology of HKIEd. His publications include books, book chapters, conference proceedings and international refereed journal articles in the areas of Computer Algebra, Geometry, History of Mathematics, Problem Solving, Number Theory, Mathematical Algorithms, Computer Assisted Learning in Mathematics and Mathematics Education.

Dr Allen Leung obtained a PhD in mathematics at the University of Toronto. He is currently an Associate Professor at the Department of Educational Studies, Hong Kong Baptist University. His research areas include the pedagogical and epistemological potentials of dynamic geometry environments, application of the Theory of Variation in teaching and learning, Lesson Study, and language and mathematics. In particular, he pioneered the use of the Theory of Variation as an interpretative tool for dynamic geometry environments. He was a member of the organizing team for Topic Study Group 22: Technology in the teaching and learning of mathematics for the 11th International Congress on Mathematics Education, 2008, Monterrey, Mexico.