CHARACTERIZING PRESERVICE TEACHERS’ MATHEMATICAL UNDERSTANDING OF ALGEBRAIC RELATIONSHIPS

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Abstract: Qualitative research methods were employed to investigate characterization of preservice teachers’ mathematical understanding. Responses on test items involving algebraic relationships were analyzed using within-case analysis (Miles and Huberman, 1994) and Pirie and Kieren’s (1994) model of growth of mathematical understanding. Five elementary and special education preservice teachers were the focus of this study. Analysis showed that preservice teachers demonstrated different levels of mathematical understanding. The nature of the mathematical tasks they completed in class provided contexts for their developing understanding. Findings support the need to engage preservice teachers in mathematical sense-making and reasoning in order to experience what it means to teach and learn for understanding.

Keywords: mathematical understanding, teaching for understanding, preservice teachers, middle school, algebra, relations and functions

Many preservice teachers lack conceptual understanding of mathematics (Ball, 1990; Ma, 1999; Vaiyavutjamai, Ellerton & Clements, 2005). One possible reason identified by different researchers is preservice teachers’ own mathematical experience and lack of knowledge about how to teach for understanding (Brown, McNamara, Hanley, & Jones, 1999; Stump & Bishop, 2002; van Dooren, Verschaffel, & Onghena, 2002). With their investigation of primary preservice teachers’ understanding of mathematics and its teaching, Brown et al. (1999) reported that the mathematical understanding of preservice teachers in the United Kingdom was strongly embedded in preservice teachers’ accounts of their own mathematical experiences, where mathematics was perceived as difficult and threatening. Stump and Bishop (2002) noted that preservice teachers’ views of algebra were typically derived from their experiences in middle school and high school where they developed a conception of algebra as a body of rules and procedures for manipulating symbols. In their study with thirty preservice elementary education majors, more than a third of the preservice teachers had difficulty writing adequate explanations on how to determine a linear relationship from either a table or an equation. More than a third failed to write an equation to match the graph of the line. Only a third adequately described the exponential pattern of change in a given table. Preservice teachers were confused with the concepts of “rate of growth” and the “growth factor.” Such confusion still existed even if the curriculum module they were using distinguished the two concepts.

Another research study supporting this observed lack of conceptual knowledge in mathematics is an investigation of the arithmetic and problem-solving strategies and skills of Belgian preservice primary and secondary school teachers (van Dooren, Verschaffel, & Onghena, 2002). This study found that preservice teachers’ own problem-solving strategies were closely related to how they evaluated students’ arithmetic and
algebraic solutions to problems. Graeber (1999) argued that preservice teachers should have knowledge of instruction which promotes retention, and further, that they should be able to provide alternative representations and to recognize and analyze alternative methods before they can teach with understanding.

Furthermore, several research studies (e.g., Rahal & Melvin, 1998; Brendefur & Frykholm, 2000; Blanton, Berenson, & Norwood, 2001) have investigated the development of preservice teachers’ understanding of mathematics and how different teaching approaches influenced what was learned. Rahal and Melvin (1998) found that, through the modeling of discourse instructional strategies, preservice teachers became more knowledgeable in their understanding and application of these instructional strategies. Preservice teachers’ level of understanding increased on a gradual and consistent basis throughout the investigation period. They stressed, however, that the teacher’s modeling must be employed on a continual basis. In some cases, modeling may not be effective in teaching discourse among preservice teachers.

Given these emphases and evidences from the literature, this current study’s focus on preservice teachers’ mathematical understanding is particularly important. In particular, the purpose of this study is to investigate what characterizes preservice teachers’ mathematical understanding involving linear, exponential, and quadratic relationships. The framework used as a lens in conducting this research is described in the next section.

**Theoretical Framework**

Pirie and Kieren (1994) view “understanding as a whole, dynamic, leveled but non-linear process of growth” and “a constant, consistent organization of one’s knowledge structures: a dynamic process, not an acquisition of categories of knowing” (p. 187). Their model for growth of mathematical understanding consists of eight embedded rings, each representing a level of understanding (see illustration in Figure 2). The first level is *Primitive Knowing*, which includes all previously constructed students’ knowledge. It does not, however, include any prior knowledge about the topic being learned. The second level is *Image Making*, wherein students can make distinctions within previous knowledge. In this level, students are able to use previous knowledge in new ways. At the third level, *Image Having*, students grasp at least one mental image which they utilize when working on a particular mathematical task. This mental image is limited and dependent on the contexts of the mathematical tasks. At the fourth level, *Property Noticing*, students examine and make distinctions and connections between and among their mental images. In this level, they can validate their mathematical knowledge with comprehension. At the fifth level, *Formalizing*, student form generalization of mathematical concepts and no longer need to relate back to specific mathematical contexts that gave rise to their understanding. At the sixth level, *Observing*, students reflect, coordinate, and express their generalizations as theorems. At the seventh level, *Structuring*, students attempt to think about their formal observations as a theory. They are aware of possible inter-relationships among formalized theorems and are able to justify mathematical statements through arguments. Finally, at the eighth level, *Inventising*, students have gained full structured understanding of the mathematical
concept and can create questions that will lead to the creation of new concepts.

This model has different features. First, it involves the notion of “don’t need” boundaries which are represented by thicker lines between Image Making and Image Having, Property Noticing and Formalizing, and Observing and Structuring. This feature means that a student does not always need to be aware of inner levels of understanding in order to move to an outer level. Second, this model has a built-in dynamic “folding back” feature which explains the “non-unidirectional nature” of understanding mathematics. As Pirie and Kieren view it, the development of mathematical understanding is non-linear and allows for the possibility that a student may reconstruct new mathematical understandings. Third, each of the levels in this model beyond primitive knowing has complementary aspects of acting and expressing which are essential in moving from one level to another. Acting includes all previous understanding and provides continuity with inner levels. It encompasses mental and physical activities. Expressing, on the other hand, gives distinct substance to that particular level. It refers to showing to oneself or others the nature of those activities.

I use this model and features of the model to analyze preservice teachers’ mathematical understanding through their responses to sample mathematical tasks given in class. Before the analysis of data, specific details about the study are presented in the succeeding section.

Methodology

I conducted a qualitative study involving five preservice teachers: Leanne, Jessica, Ashley, Rachel, and Zelda. These preservice teachers were either elementary or special education majors. I observed them in class and audio-taped their class sessions to capture their participation during group and whole class discussion. I collected sample work to supplement the audio-recordings and provide contexts to the class discussion. I collected copies of their in-class exams and selected sample test items that pertain to topics on linear, exponential and quadratic relationships. I also administered task-based interviews (article forthcoming) to assess and understand the characterization of their mathematical understanding involving algebraic relationships. For the purpose of this article, I present and analyze preservice teachers’ responses to three test items to characterize their mathematical understanding involving algebraic relationships.

To analyze each preservice teacher’s mathematical understanding, I used with-in case analysis (Miles & Huberman, 1994). I compared responses to three test items to Pirie and Kieren’s (1994) model of growth of mathematical understanding. I identified the levels of mathematical understanding of each preservice as she demonstrated them through her responses to test items administered in class. Again, using the model’s features, I identified features of understanding evident in responses to each test item. These features include acting, expressing, “no need” boundaries, and folding back. I analyzed the responses to test data in terms of Pirie and Kieren’s (1994) eight levels of mathematical understanding: primitive knowing, image making, image having, property noticing, formalizing, observing, structuring and inventising. In the following section, I present in detail the results and analysis of data from three representative test items.
Results and Data Analysis

Test Item 1: Gas Mileage (Linear)

To be able to answer this question (see Appendix) correctly, preservice teachers needed to have a clear understanding of the concept of slope and to interpret slope in the context of the problem. Also, they needed to come up with a question whose answer could be derived from the graph. Results from preservice teachers’ responses are presented in the next paragraphs.

Leanne’s solution. Leanne wrote that “if the scales remained the same, but the line became steeper, the car would be using gas at a greater rate.” Her response showed how she was not clear about the concept of slope in relation to the gas consumed and the distance traveled. She did not understand that a steeper line would mean that Jake would travel a greater distance using less gas. Instead, she associated the steepness of the line with “using more gas for less distance.” Figure 1 shows Leanne’s graph and question with the corresponding answer to the question.

![Figure 1. Leanne’s graph, question, and answer.](image)

Question: After traveling 60 miles how many gallons of gas has Jake used?

Answer: 4 gallons, by extending a line from 60 miles you can see where this meets the line from the graph, then follow that down to see that he used four gallons.”

Figure 2 shows a map of Leanne’s levels of mathematical understanding. Although Leanne knew how to define slope as a mathematical concept (Primitive Knowing shown as (a) in Figure 2), she was not able to define it in terms of the context of the Gas Mileage problem. In class, she was one of the preservice teachers who had no difficulty defining and finding slope using a procedural computation. As evidenced from the question and answer she wrote (see Figure 1), Leanne demonstrated knowledge of a form of representation of slope (Image Making, see (b) in Figure 2). However, she was not able to extend that representation in terms of the meaning of the steepness of the line representing the relationship between the gas consumed and distance travelled. This characterized an indirect movement (represented by a dashed line) from Primitive Knowing level to Image Making level. Leanne claimed that her response to this item was a “shot in the dark” attempt to solve the problem. She realized her mistake but claimed
that she understood the context but not the numbers. She thought that all of the test items given were going to be directly related to what they had discussed and covered in class.

![Figure 2: A map of Leanne’s levels of mathematical understanding.](image)

**Jessica’s solution.** Jessica wrote “steeper the line would mean that the car can go farther on less gas.” Jessica was clear about the car having better gas mileage if the line was steeper. Although she claimed that she did not know much about cars, she correctly identified the relationship of the steepness of the line to “better gas mileage.” Figure 3 shows Jessica’s graph, question, and answer.

![Figure 3. Jessica’s graph, question, and answer.](image)

In this test on linear relationships, Jessica did well on items that required interpretation. She had difficulty with items that required finding equations but was able to interpret the meaning of slope in the context of the *Gas Mileage* problem correctly. This did not seem to interfere with her interpretation of the slope of the line. Figure 4 shows a map of Jessica’s levels of mathematical understanding.
Like Leanne, Jessica had previously learned the definition of slope and the importance of interpreting slope in the context of the problem (Primitive Knowing shown as (a) in Figure 4). Interpreting slope in the context of the problem was taught and emphasized in class activities prior to this test. She had a mental image (Image Having) of what “steepness of a line” means ((b) in Figure 4). When asked what it meant to have a steeper line in a graph showing the linear relationship between distance traveled and gas consumption, Jessica wrote that “if the line were steeper, in this situation it would mean that the car got better gas mileage because the steeper the line would mean that the car can go farther on less gas” (see (c) in Figure 4). This showed how she understood what it meant to be steeper in the context of the problem (Property Noticing), namely that she had developed an image of a steeper line as meaning “having better gas mileage.” This is represented by a solid line in the map shown in Figure 4. Her perceived lack of knowledge about cars did not hinder her from correctly identifying that the steeper the line in the graph, the better the gas mileage (Formalising illustrated as (d) in Figure 4). Her understanding of the concept of slope in relation to the problem was evident in her explanation.

**Ashley’s solution.** In her graph shown in Figure 5, Ashley described that line A showed that Jake drove 15 miles using 2 gallons of gas whereas line B showed Jake drove 40 miles using the same amount of gas. For her, the steeper line meant having better gas mileage.
Question: Did Jake’s car get better gas mileage before or after the tune-up?

Answer: Jake’s car got better gas mileage after the tune-up according to the graph.

Figure 5. Ashley’s graph, question, and answer.

Figure 6 shows a map of Ashley’s levels of mathematical understanding. On her paper, she defined slope as the quotient of the change in $y$ and the change in $x$ (Primitive Knowing shown as (a) in Figure 6). She provided a specific example by creating another line graph (e.g., line B) and identified a point in each line. These representations (Image Making shown as (b) in Figure 6) served as extension of her knowledge of slope given that the second slope is steeper. Ashley further explained on her paper that if the line would be steeper than what was shown on the original graph, the slope would be greater and Jake’s car would burn less gas per mile (Property Noticing shown as (c) in Figure 6). She added that this situation would mean having better gas mileage (Formalising shown as (d) in Figure 6).

Rachel’s solution. Unlike Leanne, Jessica and Ashley, Rachel’s solution was brief and to the point. Rachel briefly wrote “the steeper line would represent more miles per gallon.” She created scales on the graph (see Figure 7) and identified the coordinates of the point (1, 20) on the graph. She identified that 20 miles was Jake’s mileage per gallon. She further verified that using the equation of the line $y = 20x$ allowed her to compute the value of $y$. 
Question: How many miles is Jake getting per gallon on his car?

\[ y = 20x \]
\[ 20 = 20(1) \]
\[ 20 = 20 \]

Figure 7. Rachel’s graph, question, and answer.

Figure 8 shows a map of Rachel’s levels of mathematical understanding. Rachel’s response characterized formalizing what steeper line meant in terms of miles per gallon of gas (Formalising shown as (a) in Figure 8). She went back and supported her response by drawing a graph and identifying its properties. Like Ashley, Rachel used a representation (Image Making shown as (a) in Figure 8) of what steeper slope means using the graph she drew. She used her representation to support her formalized response to the problem. Rachel’s response characterized an understanding that “folds back” from outer level (Formalising) to the inner level (Image Making). In this case, she used her previous knowledge on properties of a graph of a linear relationship to support her conclusion about what a steeper line means in the context of the Gas Mileage problem.

Figure 8: A map of Rachel and Zelda’s levels of mathematical understanding. Although, Zelda’s map has dashed line instead of a solid line from (a) to (b).
Zelda’s solution. Zelda was able to identify that a steeper line means greater slope. However, she was not able to relate that correctly in the context of the Gas Mileage problem. Instead, on her paper, she described that having a line steeper than the one shown on the original graph means “the cost of gas has increased.” She further wrote that “a gallon of gas takes Jake a shorter distance than before.” In her graph shown in Figure 9, she illustrated the number of miles Jake’s car could travel given 7 gallons of gas. She failed to plot points on the original graph. Instead she plotted points on a line that contained the point (7, 70). She defined the \( y \) coordinate of this point as the number of miles Jake’s car travelled given 7 gallons of gas.

![Gas and Distance Graph](image)

Question: How many miles can Jake’s car travel with 7 gallons of gas.

Answer: 70 miles

Figure 9. Zelda’s graph, question, and answer.

Figure 8 shows a map of Zelda’s levels of mathematical understanding. Zelda’s response characterized formalizing what steeper line meant in terms of numerical value of the slope (Formalising shown as (a) in Figure 8). Zelda used a representation (Image Making shown as (a) in Figure 8) to illustrate what steeper slope means in terms of gas usage and distance traveled. Zelda failed to explain her answer correctly in the context of the problem. Like Rachel, Zelda’s explanation demonstrated “folding back” from Formalizing level to Image Having level. However, in the case of Zelda, the supporting example based on her representation was not mathematically correct. Zelda was unable to explain her conclusion clearly (instead of solid line, map should be represented by a dashed line) using the graph she drew. She was not able to make a clear connection between her conclusion and solution.

Test Item 2. Growth/Decay (Exponential)

In this item (see Appendix), preservice teachers were asked two things: (a) to explain the relationship between the growth or decay factor and the shape of the graph by illustrating the relationship with a graph; and (b) to differentiate between growth and decay factors.

Leanne’s explanation. Leanne explained that “the growth factor means that for every unit change in \( x \) there is an increase by a multiple of that in the value of \( y \). The decay factor means that for every unit change in \( x \) there is a decrease by a multiple of that in the value of \( y \).” Figure 10 shows the graphs that she drew to illustrate each concept.
Leanne’s definition is a way of abstracting what she understood as observed properties of growth and decay factors (Formalising shown as (a) in Figure 11). In differentiating between growth and decay factors, Leanne explained that “the growth factor is always larger than 1. The decay factor is a value in between 0 and 1.” During the interview she explained further that “the growth factor means that for every unit change in $x$ there is a constant multiple change in the value of $y$ (Property Noticing shown as (b) in Figure 11). The growth factor is that multiple applied in the value of $y$. For the decay factor there is a constant decrease in the value of $y$ and that decrease is the constant multiple.”

![Figure 10: Leanne’s graphs for growth and decay factors.](image)

Jessica’s explanation. Jessica drew the graphs shown on Figure 12 (Image Making shown as (a) in Figure 13). She explained that “for every change in $x$ there is a constant increase in $y$ in growth factor and for every change in $x$ there is a constant decrease of change in $y$ for the decay factor. She briefly added “growth increases” and “decay decreases” (Property Noticing shown as (b) in Figure 13) but failed to explain further how to differentiate growth from decay factor.

![Figure 11: A map of Leanne’s levels of mathematical understanding.](image)
Ashley’s explanation. Ashley explained that “growth factors are always more than 1. Decay factors are always more than 0 but less than 1. When a table has a decay factor, the $y$ values decrease as $x$ values increase. When a table has a growth factor, the $y$ values increase by a multiple as the $x$ values increase” (*Property Noticing* shown as (a) in Figure 14). In explaining the relationship between growth and decay factors and the shape of the graph, Ashley drew the graphs shown in Figure 15. She explained that the steeper the curve the higher the growth factor. She used the graphs of the equations $y = 5^x$ and $y = 2^x$ to illustrate this relationship (*Image Making* shown as (b) in Figure 14). Unlike Jessica, Ashley used the graphical representation to explain the property of the growth factor.
Figure 14: A map of Ashley and Zelda’s levels of mathematical understanding. Although, Zelda’s map has dashed line instead of a solid line from (a) to (b).

Figure 15. Ashley’s graphs.

**Rachel’s explanation.** To illustrate her explanation, Rachel presented a graph (see Figure 16) showing the decreasing population of whales over several years. She created this graph from the table presented in the third item given on the second test. She thought that this test item number was related to another test question about whales. Her explanation was then based on the graph she created for the whale data.

Using the graph, she explained that “for every unit increase in $y$ (number of years) there is a constant multiple decrease in $w$ (population of whales).”
Rachel further explained that the difference between growth factor and decay factor was related to the “increase and the decrease of the dependent variable. In the growth factor there is a constant multiple growth whereas in the decay factor there is a constant multiple decrease.”

Unlike the other tasks or activities already discussed, this task involved preservice teachers explaining what they knew about the difference between exponential growth and decay, and the exponential growth and decay factor. Figure 16 shows a map of Rachel’s mathematical understanding. As instructed, Rachel used a graph to illustrate her explanation (shown by (a) in Figure 17). In this Image Making level, Rachel referred to a graph of the decreasing population of whales over several years. In order to identify a specific example of a graph that could illustrate her explanation she engaged in “folding back” to previous knowledge (Image Making to Primitive Knowing) ((b) in Figure 17).

Still using the graph but this time as a mental construct (Image Having), Rachel interpreted that, “for every unit increase in $y$ (number of years), there is a constant multiple decrease in $w$ (population of whales).” This is shown as (c) in Figure 17. She differentiated growth factor and decay factor based on the “increase and the decrease of the dependent variable ((d) in Figure 17). Here she was in the Formalising level, where she explained that “in the growth factor there is a constant multiple growth whereas in the decay factor there is a constant multiple decrease.”

![Figure 16. Rachel’s graph of decreasing whale population.](image)
Zelda’s explanation. Zelda explained that “the growth or decay factor determines how steep the graph’s results or line will be. The higher the growth factor the steeper the line” (Property Noticing shown as (a) in Figure 14). In Figure 18, she identified that graph number 1 has a growth factor of plus 2 while graph number 2 has a growth factor of times 2 (Image Making shown as (b) in Figure 14).

She differentiated between growth and decay factors by explaining that “the growth factor increases $y$ by a constant multiple and decay factor decreases $y$ by a constant multiple.” She added that the decay factor “will never be greater than 1 and growth factor will never be less than 1.”
Test Item 3. Congruent Triangles (Quadratic)

In this test item (see Appendix), preservice teachers needed to describe the pattern of change in the number of unshaded congruent triangles and the total number of small triangles from one figure to the next. They were required to draw and identify the fourth figure that would follow the same pattern. They needed to write an equation that would determine the number of unshaded triangles and another equation to determine the total number of small triangles in the \( n \)th figure. In both cases they had to explain their solution. All of the five preservice teachers were able to draw the correct figure which followed the given pattern of congruent triangles.

Leanne’s description. Leanne observed that from one figure to the next, the number of unshaded triangles increased by the number of that figure. She further explained that if it was the 8th figure, eight more unshaded triangles would be added to the number of unshaded triangles shown in the previous figure. She correctly identified the equation \( T = \frac{n(n+1)}{2} \) as the equation that would determine the number of unshaded triangles in the \( n \)th figure. She explained that she noticed that the number of unshaded triangles in the \( n \)th figure was simply the sum of all the previous unshaded triangles. To find the sum of all these unshaded triangles, she explained that she added the first value to the last value then the second value to the second to the last value, and so on. All of these sums yielded the same value from the first to the last which she represented by \( n + 1 \). She continued by adding that this \( n + 1 \) sum should be multiplied by \( n \) or the total number of values to account for all the sums. She explained that the reason for dividing by two was due to the fact that after half the groups of sums there was repetition. This was how she got the equation for the number of unshaded triangles equal to \( \frac{n(n+1)}{2} \).

Leanne presented a good argument but did not algebraically represent her solution. In answering the second part of the question, Leanne explained that the total number of small triangles from one figure to the next was simply the square of the number of the figure. As an example she wrote “in the eighth figure there would be sixty four triangles” and reiterated that the total number of small triangles is equal to \( n^2 \) where \( n \) is the figure number.

In this test item, Leanne drew images to identify the pattern for the succeeding figure. Like all other preservice teachers, Leanne was able to draw the correct figure in the sequence of congruent triangles. In class, Leanne learned (Primitive Knowing) to identify algebraic patterns using geometric figures (shown as (a) in Figure 19). For this task, she identified algebraic patterns (Property Noticing) of the geometric figures (shown as (b) in Figure 19). After being convinced by her observed pattern without using physical manipulation (Image Having), she claimed that that 8th figure contained eight more unshaded triangles ((c) in Figure 19). In examining the figures (Property Noticing), she concluded that the equation \( T = \frac{n(n+1)}{2} \) would determine the number of unshaded triangles in the \( n \)th figure ((d) in Figure 19). She formalized the properties she observed
Formalising) by explaining that the sum of all unshaded triangles was $n$ times the total number of values to account for all the sums which was $n + 1$. She explained that dividing by two accounted for all unique groups (see (e) in Figure 19). Using similar observations, she noticed (Property Noticing) that the total number of small triangles from one figure to the next was simply the square of the number of the figure (see (f) in Figure 19).

Jessica’s description. Jessica explained that the pattern of change was quadratic because there was a “constant change in the second difference in the number of unshaded triangles.” She used the table shown in Figure 20 to justify her solution (Image Making shown as (a) in Figure 21). Although it was not a sufficient explanation, she was convinced that showing the constant second difference made the state that the pattern is quadratic. Like Leanne, Jessica was able to identify the equation $\frac{n(n+1)}{2}$ to represent the relationship. She added that there were $n+1$ groups of $n$ divided by two because the figure was a triangle and not a rectangle. She referred to the triangular figure as half of the rectangular figure with dimensions $n$ and $n + 1$ (Property Noticing shown as (b) in Figure 21).

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Figure 19: A map of Leanne’s levels of mathematical understanding.

Figure 20. Jessica’s table.
Consistent with her explanation on the first question, she explained that the “pattern of change in the total number of small triangles was quadratic because for every time the figure number changes there is a constant change in the second difference in the total number of triangles.” She was able to identify $T = n^2$ as the equation to determine the number of triangles in the $n$th figure but did not explain how she got it (Formalising shown as (c) in Figure 21).

**Figure 21:** A map of Jessica and Ashley’s levels of mathematical understanding.

*Ashley’s description.* Ashley used trial and error and tried many equations which did not all work until she found $u = \frac{n(n+1)}{2}$ where $u$ represented the number of unshaded triangles and $n$ the figure number (Image Making shown as (a) in Figure 21). To justify that this equation was correct, she used the second figure with 3 unshaded triangles and substituted the values in the equation. She explained that “if you put figure 2 into the equation with three triangles you get $3 = \frac{2(2+1)}{2}$ which worked.” Similarly, she did the same thing for the third figure and explained that “if you put figure three in the equation with six unshaded triangles you get $6 = \frac{3(3+1)}{2}$ which again worked” (Property Noticing shown as (b) in Figure 21). She concluded that this would be true for all figure numbers. She used specific examples to show that the equation worked and because it worked for the two examples, she said it would work for any figure number without showing how to justify that the equation worked in general case. For the second question, she was able to identify that the equation $t = n^2$ worked and justified using a table where the values for $t$ were based on her observation that this would be found by squaring the figure number (Formalising shown as (c) in Figure 21). She did not further elaborate on this solution.
Rachel’s description. Rachel immediately identified that the figure numbers were triangular numbers (Formalising shown as (a) in Figure 22). She remembered what was taught in class and described that the number of the unshaded triangles in the $n$th figure was determined by the number of unshaded triangles from the previous figure plus a certain quantity (Primitive Knowing shown as (b) in Figure 22). Her idea made sense but her explanation was not clear. Knowing that the sequence of the number of unshaded triangles comprised triangular numbers she knew right away that the $n$th figure would have $\frac{n(n+1)}{2}$ unshaded triangles. To show that this equation worked, she substituted the values for the fourth figure and showed that this equation would give six as the number of unshaded triangles in the fourth figure (Image Making shown as (c) in Figure 22). For the second question, she identified that the formula would be $T = n^2$ where $T$ is the total numbers of triangles and $n$ is the figure number.

Zelda’s description. Zelda explained that the pattern of change in the number of unshaded triangles from one figure to the next was determined by the equation $\frac{n(n-1)}{2}$. She went on to say that she got the equation by trial and error, and that in the process of figuring out the equation she knew that since the figure was a triangle (which is half a rectangle) then she needed to divide by two (Image Making shown as (a) in Figure 23). She used the relationship between the area of the triangular figure and the rectangular figure (Property Noticing shown as (b) in Figure 23). She just substituted the values shown in the first figure into different equations until she finally identified the equation $\frac{n(n-1)}{2}$. She did not explain clearly her solution. Her solution was not evident from what she presented. Similarly to the second question, she used trial and error. She explained that the number of unshaded triangles is derived by multiplying the figure.
number (i.e., 1, 2, 3,…) by the previous figure number and dividing the product of the numbers by two (Image Making shown as (c) in Figure 23). Zelda did not test if her equation worked for all numbers in the series.

![Figure 23: A map of Zelda’s levels of mathematical understanding.](image)

**Discussion and Conclusion**

Preservice teachers’ levels of mathematical understanding spanned the first five levels (i.e., primitive knowing, image making, image having, property noticing, and formalizing) of Pirie and Kieren’s (1994) model. The order of these levels is not based on the complexity of mathematics involved in the task. None of the preservice teachers demonstrated evidence of understanding that characterized *Observing*, *Structuring*, and *Inventising* levels. These last three levels do not necessarily refer to higher level of mathematics but levels that are characterized by higher mathematical skills such as presenting a theorem, developing formal proofs and creating new mathematical concepts. The nature of the task required in each test item somewhat predefined the five levels spanned by preservice teachers’ mathematical understanding. The items given on tests involved use of mathematics skills that require understanding relationships among variables, making connections, utilizing multiple representations and strategies, and mathematizing observed relationships. Activities requiring these skills have been identified as likely to promote mathematical understanding (Friedlander & Tabach, 2001; Gray & Thomas, 2001; Kaput, 1998; Kleiman, 1998; Knuth Hartman, 2005). All preservice teachers demonstrated the **Formalising** level characterized by abstraction of mathematical methods from previous knowledge. A linear sequence of attaining each level was most common to all preservice teachers. However, Rachel in both *Gas Mileage* and *Congruent Triangles* problems, Zelda in *Gas Mileage* problem and Leanne in *Grown/Decay* problem demonstrated a non-unidirectional form of mathematical understanding. They demonstrated a mathematical understanding that is characterized by abstraction of methods but “folds back” to either support their conclusion with visual or mental representations or an examination of observed mathematical properties.
Preservice teachers’ mathematical understanding did not always demonstrate direct movement between levels of understanding. Instead, some partial or full indirect movements between levels of understanding (i.e., maps represented by a dashed line) were demonstrated by Leanne in the Gas Mileage problem and Zelda in all three problems. In all her solutions, Zelda spent considerable time working within a particular level of understanding. She had difficulty making connections between multiple representations and explaining and making sense of her solutions. Making connections is a significant skill in understanding algebra (Kaput, 1998). Image Making was evident in preservice teachers’ understanding involving tasks that required use of multiple representations. In all cases, the preservice teachers’ understanding was characterized by a change in their ability to think about the mathematical relationships involved in the problems. Simon (2002, 2006) called this “key developmental understanding.” This form of understanding is crucial in their ability to develop conceptual understanding of mathematics.

The dynamic feature of “folding back” was evident in the ways preservice teachers developed their mathematical understandings involving linear, exponential and quadratic relationships. This feature was demonstrated in several examples (e.g., Rachel and Zelda in the Gas Mileage problem, Leanne, Ashley, Rachel, and Zelda in the Growth/Decay problem, Leanne, Rachel, and Zelda in the Congruent Triangles problem). Rachel specifically demonstrated this feature in most of her responses and explanations. “Folding back” was also demonstrated several times by different preservice teachers (for example, Leanne in the Congruent Triangles problem). Pirie (1992; 1994, 2002) argued that “folding back” was crucial to students’ growth of mathematical understanding. In the case of the preservice teachers, “folding back” was crucial in conjecturing, validating and generalizing their solutions to the problems. It was interesting to note that the thirty test item responses discussed in this analysis included “constructing relationships, extending and applying mathematical knowledge, reflecting about experiences and articulating what one knows.” McGaffrey et al. (2001) argued that instructional activities which emphasized problem solving, communication, reasoning, and mathematical connections enabled students to develop complex cognitive skills and processes. These forms of mathematical activities were included in the course taken by these five preservice teachers.

The features of “acting” and “expressing” were also evident in the levels of mathematical understanding of the five preservice teachers. These were characterized by continuity with inner levels of understanding beyond Primitive Knowing (Pirie & Kieren, 1994). All preservice teachers demonstrated understanding that encompassed engagement in mental or physical activities (acting) and the ability to show (expressing) to themselves or others the nature of those activities. Specific mathematical actions such as doing, reviewing, seeing, predicting and recording were demonstrated in various levels of understanding. Although Pirie and Kieren’s model has the “don’t need” boundaries feature (i.e., between Image Making and Image Having, Property Noticing and Formalising, and Observing and Structuring), the levels of mathematical understanding described for each preservice teacher showed that inner levels of understanding always
gave rise to outer levels of understanding. Examples of this can be found for all of the preservice teachers in the study as they worked through the Growth/Decay problem.

Research reports in the literature have emphasized the importance of learning with understanding especially among preservice teachers (Ball 1990, 1991, 2003; Carpenter & Lehrer, 1999; Fennema & Franke, 1992; Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Oliver, & Human1997). An examination of how preservice teachers developed their mathematical understandings as a result of their learning experiences was the main focus of the present study. Data from preservice teachers’ responses to six test items showed how they developed mathematical understanding in handling linear, exponential and quadratic relationships. The results were specific to Leanne, Jessica, Ashley, Rachel, and Zelda’s learning experiences and mathematical understanding. However, results demonstrated commonalities in terms of movement from one level to another. Gaining knowledge of preservice teachers’ levels of mathematical understanding was crucial in helping them develop skills that would hopefully enable them to teach mathematics for understanding (Brown, et al., 1999; Stump & Bishop, 2002, van Dooren, et al., 2002). Results from this data contribute to this knowledge. Data presented and analyzed in this section only focused on test items from classroom instruction. More data from task-based interviews will be presented and analyzed in the next section to provide additional perspectives of preservice teachers’ mathematical understanding in relation to linear, exponential and quadratic relationships.

This study examined preservice teachers’ mathematical understandings through detailed analysis of their responses to tests items and participation in classroom discourse. The richness of the data collected from this study was dependent on the way the class was facilitated and how the mathematical activities were implemented. Preservice teachers were engaged in discursive activities, encouraged to develop understanding of mathematical concepts for which they had previously acquired only procedural understanding, and were expected to make sense of mathematical concepts they were learning for the first time.

In preparing preservice teachers to teach for mathematical understanding, it is imperative that they experience what it is like to be in an environment which fosters that instructional goal. Mathematical tasks that focus not only on mathematical understanding but also on developing favorable dispositions towards mathematics are recommended. This area is not investigated in this study. Thus, future research should examine relationships between mathematical understanding and dispositions towards mathematics.

References


Appendix (Three Test Items)

**Direction.** Solve the following problem situations using the method you think appropriate. Make use of a variety of solution methods if possible. Show all your work and provide a clear explanation when asked.

**Test Item 1.** Gas Mileage (Linear). The graph below shows gallons of gas consumed by Jake’s car compared to distance traveled.

![Graph](image)

a. What would it mean in this situation if the line was steeper than shown and the scales on the graph stayed the same?
b. Put a sensible scale on the axes, and write a question that can be answered from your graph. Show where the answer can be found on your graph.

**Test Item 2.** Growth/Decay (Exponential). Write brief reflective answers to the following:
a. What is the relationship between the growth or decay factor and the shape of the graph? Explain by illustrating your answer with a graph.
b. How can you differentiate between growth factor and decay factor?

**Test Item 3.** Congruent Triangles (Quadratic). The figures below are constructed from small congruent triangles.

a. Draw the next figure in the pattern to the right of those above.
b. Describe the pattern of change in the number of unshaded triangles from one figure to the next. Write an equation for calculating the number of unshaded triangles in the \(n\)th figure and explain how you have gotten your equation.
c. Describe the pattern of change in the total number of small triangles from one figure to the next. Write an equation for calculating the total number of small triangles in the \(n\)th figure and explain how you got your equation.