This paper describes a study that investigated the metacognitive self-monitoring strategies used by
senior secondary school students while working individually on a mathematics problem. After
attempting the problem the students completed a questionnaire that asked them to report retrospectively
on the metacognitive strategies they had used. Examination of the students’ written work and
questionnaire responses revealed some instances of successful self-monitoring, but also occasions when
monitoring was either inadequate or appeared to be lacking altogether. Identifying the characteristic
types of metacognitive failures for each kind of solution strategy highlighted the distinction between two
key elements of effective monitoring: being able to recognise errors and other obstacles to progress, and
being able to correct or overcome them.

In a review of progress in mathematical problem solving research over the past 25
years, Lester (1994) noted with some concern that research interest in this area appears
to be on the decline, even though there remain many unresolved issues that deserve
continued attention. One such issue highlighted by Lester was the role of metacognition
in problem solving – where metacognition refers to what students know about their own
thought processes, and how they monitor and regulate their thinking while working on
mathematical tasks. Although the importance of metacognition is now widely
acknowledged, we still lack an adequate theoretical model for explaining the mechanisms
of individual self-monitoring and self-regulation. This paper reports on preliminary
findings from a larger study that attempted to address some of the limitations of existing
theoretical models of metacognition in mathematical thinking.

The study reported here examined the individual problem solving actions of a
group of senior secondary school students. The students were participants in a two year
research project investigating metacognitive activity in collaborative classroom settings
(Goos, 1997; Goos, 1998; Goos, 2000; Goos, Galbraith & Renshaw, 1999; Goos &
Geiger, 1995) and the results reported here provide glimpses into the mathematical
thinking of individual students as well as suggesting follow up investigations into the
nature of collaborative monitoring and regulation. The specific research questions
addressed in this preliminary study are as follows:

1. What strategies do students use in attempting to solve a non-routine
   problem?
2. How do students recognise and respond to obstacles to progress?
3. How is metacognitive self-monitoring related to problem solving outcomes?
Analysing Metacognitive Processes in Mathematical Problem Solving

In the past there have been two poles of opinion in defining what a problem is: some have labelled as “problems” the routine exercises that provide practice in newly learned mathematical techniques, while others have reserved the term for tasks whose difficulty or complexity makes them genuinely problematic (Schoenfeld, 1992). Yet neither of these schools of thought necessarily acknowledges the role of the problem solver in defining the problem. A task that is an exercise for one person may prove to be much more testing to someone else; thus the problem exists not in the task itself, but in the relationship between the task and the solver (Clements & Ellerton, 1991; Schoenfeld, 1985; Smith & Confrey, 1991). A task is a problem if the person confronting it has to deal with obstacles that hinder progress towards a solution.

Likewise, problem solving involves more than simply enacting effective task strategies. When faced with obstacles and uncertainties, good problem solvers display skill in choosing and testing alternative strategies and the will to maintain their engagement with the problem (Good, Mulryan & McCaslin, 1992). Effective mathematical thinking in solving problems includes not only cognitive activity, such as representing the task and implementing strategies in order to find a solution, but also metacognitive monitoring that regulates such activity and allows decisions to be made regarding the allocation of cognitive resources.

Frameworks for analysing task-oriented mathematical thinking typically identify phases or episodes representing distinctive kinds of problem solving behaviour. For example, Schoenfeld (1985) developed a procedure for parsing verbal protocols into five types of episodes: Reading, Analysis, Exploration, Planning/Implementation, and Verification. This framework specifies the ideal cognitive and metacognitive characteristics of each episode, which can be compared with students’ observed problem solving behaviours. An expanded version of Schoenfeld’s model has been developed by Artzt and Armour-Thomas (1992) to delineate the roles of cognitive and metacognitive processes in small group problem solving; however, these modifications may be applied equally well to individual and collaborative problem solving.

Artzt and Armour-Thomas (1992) separated Schoenfeld’s Planning/Implementation episode into two distinct categories, and included two additional episode types in their protocol coding system: Understanding the problem, and Watching and listening. As watching and listening behaviour was not investigated in the present study, this episode type is not included in the following discussion. The new Understanding episode overlaps somewhat with Schoenfeld’s Reading and Analysis episodes. Characteristics of each episode, as defined by Artzt and Armour-Thomas, are described below.

Reading: Read the problem aloud or silently.

Understanding: Identify task-specific knowledge, paraphrase the problem statement, re-represent the problem by listing facts or drawing a diagram, note the conditions and goals of the problem, recall similar problems, note the presence or absence of important information.

Analysis: Select an appropriate perspective and reformulate the problem in those terms, introduce appropriate mechanisms and principles, seek relationships between the givens and goals.
Exploration: If no progress is being made, search for information that may help the solution process. Decide whether to terminate or continue.

Planning: Select solution steps and strategies for combining them.

Implementation: Follow the plan in a coherent and well-structured series of calculations.

Verification: Check calculations, verify that the solution satisfies the problem conditions and makes sense, evaluate the solution process and confidence in the result.

These episodes need not occur in the sequence above. For example, students may bypass analysis and planning and launch impulsively into implementation, then reach an impasse prompting an exploration episode, return to reading and understanding the problem, identify new information, and finally move through an orderly sequence of analysis-planning-implementation-verification (see Goos & Galbraith, 1996, for an example).

In addition to identifying and categorising episodes of problem solving behaviour, the analysis frameworks of both Schoenfeld (1985) and Artzt and Armour-Thomas (1992) acknowledge the central role of metacognitive processes in keeping the solution process on track, for example, by noting that the solution status or one’s general progress should be monitored and plans modified if necessary. However, a deficiency in both frameworks is the lack of detail in describing the types of monitoring and regulatory activities that would be appropriate and expected in each episode. The suggested scope of these metacognitive activities is detailed in Figure 1, in the columns headed Monitoring and Regulation.

Previous research in this area also has not distinguished between the routine monitoring which merely serves to confirm that all is well, and the more controlled monitoring and regulatory processes triggered when students become aware of specific difficulties. It is helpful to think of these triggers as metacognitive “red flags”, which signal the need for a pause or some backtracking while remedial action is taken. The model in Figure 1 illustrates this distinction: the routine assessment of activity during each episode (for example, assessing one’s understanding of the problem, assessing execution of the strategy) ensures that problem solving stays on track; while metacognitive “red flags” (shown in shaded boxes) warn that something is amiss.

Three types of “red flags” are identified in Figure 1: lack of progress, error detection, and anomalous result. Recognising lack of progress during a fruitless exploration episode should lead students back to analysis of the problem in order to reassess the appropriateness of the chosen strategy and to decide whether to persist, salvage whatever information is useful, or abandon the strategy altogether. In the latter case it is likely that students will need to reassess their understanding of the problem, and search for new information or a new strategy. Error detection during an implementation episode should prompt checking and correction of calculations carried out so far. Finally, if attempts to verify the solution reveal that the answer does not satisfy the problem conditions, or does not make sense, then this anomalous result should trigger a calculation check (assess execution of strategy), followed, if necessary, by a reassessment of the strategy.

Secondary school students’ ability to recognise and act on these specific warning signals was investigated in the study described below. The students attempted a challenging problem, and then completed a questionnaire that asked them to report whether or not they had used specific metacognitive strategies.
Figure 1. An episode-based model of metacognitive activity during problem solving
Method

Subjects

Forty-two mathematics students from three secondary schools in the Australian state of Queensland participated in this preliminary phase of the study. Thirty-one were Year 11 Mathematics B students (nineteen students from a state high school in Brisbane, the state capital, and twelve from a coeducational private school in a provincial city), and eleven were Year 12 Mathematics C students (from a coeducational private school in Brisbane). Years 11 and 12 are the senior years of secondary schooling that precede tertiary entrance. Mathematics B is the mathematics subject taken by all students who seek to enter science based courses at universities. Mathematics C is a second academic mathematics subject taken additionally by some mathematically oriented students.

The decision to focus on senior secondary school mathematics classrooms was prompted in part by the introduction of a new Queensland Senior Mathematics Syllabus which was published in 1992 and progressively implemented over the following three years. The philosophy of the new syllabus is consistent with national and international moves to reform mathematics teaching, as expressed in curriculum documents such as the NCTM Standards in the United States (National Council of Teachers of Mathematics, 1989, 1991) and the National Statement on Mathematics for Australian Schools (Australian Education Council, 1991). In these documents the traditional emphasis on memorisation and basic skills has given way to arguments that students also need to develop reasoning and problem solving capacities. The incorporation of these goals into secondary school syllabuses challenges teachers to re-examine their conceptions of mathematics learning and teaching; hence one of the aims of the larger research study was to explore implications of the new Queensland mathematics syllabus for classroom practice.

Questionnaire

The Self-Monitoring Questionnaire elicited students’ retrospective reports on the metacognitive strategies they had employed while working on a given, non-routine mathematics problem. The questionnaire is based on an instrument used with seventh grade students by Fortunato, Hecht, Tittle and Alvarez (1991). To make the questionnaire more appropriate for older students the original version was modified by deleting some items, rewording others, and including a few new items. In the present study, the questionnaire consisted of twenty-one statements to which students responded by ticking boxes marked Yes, No, or Unsure. All statements included in the original and modified versions of the questionnaire were derived from Schoenfeld’s work on mathematical thinking during problem solving (Fortunato et al., 1991).

The first questionnaire section, titled “Before You Started to solve the problem”, listed six possible strategies concerning reading and understanding the problem, and analysis and planning of a solution method. The second section, “As You Worked on the problem” referred to five strategies for implementing and monitoring the solution method, while the third, “After You Finished working on the problem”, offered four strategies for verifying the solution. The fourth section of the questionnaire asked students to indicate whether they had used any of the six listed heuristics, or “Ways Of
Working” on the problem. (Copies of the questionnaire used in the study and the original instrument developed by Fortunato et al. are provided in the Appendix.)

The Self-Monitoring Questionnaire implicitly investigated students’ ability to recognise and act on metacognitive “red flags” – warning signals, arising during routine monitoring, which indicate the need for regulation or repair. Figure 2 maps the relationship between the questionnaire and the theoretical model of metacognitive activity offered earlier (Figure 1). Each questionnaire statement is identified as a generic type of metacognitive self-monitoring or self-regulatory activity, as used in the framework of Figure 1. In addition, questionnaire statements that target the “red flags” of error detection, lack of progress, and anomalous result are identified.

<table>
<thead>
<tr>
<th>Self-Monitoring Questionnaire Item</th>
<th>Monitoring/Regulation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Before you started</strong></td>
<td></td>
</tr>
<tr>
<td>1. I read the problem more than once.</td>
<td>Assess knowledge</td>
</tr>
<tr>
<td>2. I made sure that I understood what the problem was asking me.</td>
<td>Assess understanding</td>
</tr>
<tr>
<td>3. I tried to put the problem into my own words.</td>
<td>Assess understanding</td>
</tr>
<tr>
<td>4. I tried to remember whether I had worked on a problem like this before.</td>
<td>Assess knowledge &amp; understanding</td>
</tr>
<tr>
<td>5. I identified the information that was given in the problem.</td>
<td>Assess knowledge &amp; understanding</td>
</tr>
<tr>
<td>6. I thought about different approaches I could try for solving the problem.</td>
<td>Assess strategy appropriateness</td>
</tr>
<tr>
<td><strong>As you worked</strong></td>
<td></td>
</tr>
<tr>
<td>7. I checked my work step by step as I went through the problem.</td>
<td>“Red Flag”: Error detection</td>
</tr>
<tr>
<td>8. I made a mistake and had to redo some working.</td>
<td>Assess strategy execution</td>
</tr>
<tr>
<td>9. I re-read the problem to check that I was still on track.</td>
<td>Correct error</td>
</tr>
<tr>
<td>10. I asked myself whether I was getting any closer to a solution.</td>
<td>“Red Flag”: Lack of progress</td>
</tr>
<tr>
<td>11. I had to rethink my solution method and try a different approach.</td>
<td>Assess understanding</td>
</tr>
<tr>
<td><strong>After you finished</strong></td>
<td></td>
</tr>
<tr>
<td>12. I checked my calculations to make sure they were correct.</td>
<td>Assess strategy appropriateness and execution</td>
</tr>
<tr>
<td>13. I looked back over my solution method to check that I had done what the problem asked.</td>
<td>“Red Flag”: Anomalous result</td>
</tr>
<tr>
<td>14. I asked myself whether my answer made sense.</td>
<td>Assess result for sense</td>
</tr>
<tr>
<td>15. I thought about different ways I could have solved the problem.</td>
<td>Assess strategy appropriateness</td>
</tr>
</tbody>
</table>

Figure 2. Metacognitive strategies examined by Self-Monitoring Questionnaire

Task
Because the aim of the questionnaire was to gather data on self-monitoring strategies rather than simply assess mathematical expertise, it was important to supply a genuine “problem” that would challenge the students and call forth the processes of interest, without requiring any specialised mathematical knowledge. These criteria were proved to be satisfied by the MONEY problem:
Divide five dollars amongst eighteen children such that each girl gets two cents less than each boy

Let \( b \) = number of boys, \( g \) = number of girls.

\[ \therefore g = 18 - b \]

Let \( c \) = number of cents each boy receives, and \( c - 2 \) = number of cents each girl receives.

The $5 is therefore divided between \( b \) boys, each of whom receives \( c \) cents, and \( 18 - b \) girls, each of whom receives \( c - 2 \) cents. So:

\[
bc + (18 - b)(c - 2) = 500
\]

\[
bc + 18c - 36 - bc + 2b = 500
\]

\[
18c + 2b = 536
\]

\[
9c + b = 268 \quad (1)
\]

Because we don’t know how to solve an equation with two unknowns (assuming Diophantine equations are beyond our experience), rearrange (1) to give:

\[
c = \frac{268 - b}{9} \quad (2)
\]

Now we know that \( b \) can only have whole number values between 1 and 17, so substitute these values into equation (2) and note those for which whole number values for \( c \) are obtained.

This gives two answers:

\[
b = 7 \text{ and } c = 29 \text{ (7 boys with 29 cents each and 11 girls with 27 cents each)}
\]

\[
b = 16 \text{ and } c = 28 \text{ (16 boys with 28 cents each and 2 girls with 26 cents each)}.
\]

Figure 3. A formal solution to the MONEY problem

It was anticipated that students would attempt a combined algebraic/trial and error solution. A skilled formal approach would resemble that shown in Figure 3.

Students were given the written problem statement and allowed twenty to thirty minutes for working. They were instructed to show all their working and to cross out, rather than erase, any working they thought was incorrect. Only at the end of this time was the questionnaire administered, to avoid cueing students on the strategies it listed.
Results

The results of analysis of the students’ written solution attempts and questionnaire responses are presented in three sections. First, the solution strategies they used are grouped into six categories, which are linked to the outcomes of their problem solving activity – no answer, an incorrect answer, one answer found, or both answers found. The second section summarises responses to questionnaire items that correspond to the metcognitive “red flags” identified in Figure 1. Finally, evidence from the students’ solution scripts and questionnaire responses is drawn together in order to examine their self-monitoring behaviour.

Solution Strategies and Outcomes

Initially students needed to establish the problem’s conditions and goal, introduce symbols to represent variables, and identify the relationships between the variables. The problem has both explicit and implicit conditions. From the problem statement, we know explicitly that there are eighteen children (the number-condition), and that the girls receive two cents less than the boys (the cents-condition). Hence $b + g = 18$, and $(c,c-2)$ represent possible amounts received.

From our real world experience, we also know implicitly that $b$, $g$, and $c$ must be positive integers. From the problem statement, the rule connecting these variables is $bc + g(c-2) = 500$. The ability to recognise and manipulate these relationships was crucial to students’ success in devising a successful solution strategy.

Table 1

<table>
<thead>
<tr>
<th>Strategy</th>
<th>No Answer</th>
<th>Incorrect Answer</th>
<th>One Answer</th>
<th>Both Answers</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1 Random trial &amp; error</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>#2 Incorrect formulation</td>
<td>1</td>
<td>6</td>
<td></td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>#3 Assume $b = g$</td>
<td>1</td>
<td>8</td>
<td></td>
<td></td>
<td>9</td>
</tr>
<tr>
<td>#4 Calculate the mean value</td>
<td>6</td>
<td>1</td>
<td>8</td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>#5 Algebraic reasoning</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>#6 Verbal reasoning</td>
<td></td>
<td></td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Total</td>
<td>12</td>
<td>17</td>
<td>10</td>
<td>3</td>
<td>42</td>
</tr>
</tbody>
</table>

Of the forty-two students who attempted the problem, only three found both answers, while a further ten found one answer. Seventeen students gave incorrect answers, and twelve no answer at all. Because answers alone do not reveal how students approached the problem, their written solution attempts were also analysed to identify the strategies they had used. The six strategy groupings that emerged are described
below, in order of increasing sophistication, and a summary of strategies and corresponding outcomes is given in Table 1. (As many students tried more than one strategy, their work was classified according to the strategy that produced their final answer.)

**Strategy #1: Random Trial and Error.** Two students selected pairs of \((b, g)\) and \((c, c-2)\) values that satisfy the explicit conditions and carried out trial substitutions into the rule. This strategy is clearly inefficient because it is difficult to systematically test all possible combinations of values for all the variables. The students who used this strategy had little chance of finding a correct answer.

**Strategy #2: Incorrect Formulation.** While hardly deserving of the label “strategy”, the approaches used by this group of seven students arose from their misinterpretation of the problem statement or their inability to correctly represent the information it contained. Most students managed to produce an answer, which was invariably incorrect.

**Strategy #3: Assume \(b = g\).** Nine students tried to simplify the problem by fixing the values of \(b\) and \(g\), and allowing only \(c\) and \((c-2)\) to vary. Typically, they assumed (or argued) that \(b\) must equal \(g\), giving nine boys and nine girls. The cents-condition was then applied in one of the following ways:

1. use the rule \(9c + 9(c-2) = 500\) to calculate \(c\) values, giving non-integral answers that violate the implicit problem conditions (\(c=28.7\) cents, \(c-2=26.7\) cents, sum=$5.00);  
2. sometimes round off the answer obtained from the above calculation, giving integral values for \(c\) that no longer satisfy the rule (\(c=29\) cents, \(c-2=27\) cents, sum=$5.04);  
3. test integral values of \(c\), and accept those that come closest to satisfying the rule (i.e. give a sum close to $5).

Because the condition governing the numbers of boys and girls was misinterpreted, this approach cannot give a correct answer.

**Strategy #4: Calculate the Mean Value.** Another way in which students tried to reduce the number of degrees of freedom within the problem was to narrow the search for \(c\) values to those around the mean value; that is, fix the value of \(c\) and allow \(b\) and \(g\) to vary. The mean value is $5 \div 18$, or 27.8 cents per child. Most of the fifteen students who followed this approach chose one or more pairs of \((c, c-2)\) values that were close to the mean, and then tested \(b\) and \(g\) values by substituting into the rule

\[
bc + (18-b)(c-2) = 500.
\]

Finding a correct answer via this mean value cued trial and error approach depended on first guessing a correct \((c, c-2)\) pair, and then conducting a systematic search for pairs of \(b\) and \(g\) values. Although the mean value happens to provide a fruitful starting point in this particular case (the correct \(c\) values are 26, 28 and 27, 29), the success of the six students who chanced upon one of the answers was due to luck rather than a correct formulation of the problem.

A further three students started with a \(c\) value close to the mean and then calculated, rather than tested, values for \(b\) and \(g\). Two found an answer in the manner shown below:
If all eighteen children are given a base level of 26 cents then $4.68 out of $5 is accounted for. The 32 cents left over can be given to the boys, each of whom are to receive two cents more than the girls. Thus there is enough extra money for 16 boys (32 ÷ 2), who get 28 cents each, leaving two girls with 26 cents each.

**Strategy #5: Algebraic Reasoning.** Seven students used algebraic reasoning to reduce the three variables and two equations representing the problem’s conditions and rule to one equation linking two variables. Most realised they could not solve this equation as it stood and used the result to test values of \( b \) or \( g \), accepting those that produced integral values for \( c \). If this rule-based trial and error was complete, then both sets of answers were obtained. Thus, for the algebraic approach to be successful, students had to find a way past the apparent impasse of solving one equations with two unknowns.

**Strategy #6: Verbal Reasoning.** In an interesting variation on the purely symbolic approach of algebraic reasoning, two students derived a verbally expressed relationship between the number of boys and the amount received by each girl:

Give each boy his extra two cents, then divide the remaining money \((500 – 2b)\) equally between all eighteen children to find out how much each girls gets.

As in the algebraic approach, this strategy also must be supplemented by systematic trialing of \( b \) values if both answers are to be found.

**Summary of Strategies.** A successful solution to the problem depends on reducing its complexity. This was attempted by students who used strategies #3 and #4 to impose unwarranted constraints on one of the problem’s conditions – either the numbers of boys and girls \((b = g = 9)\), or the amount of money each receives \((c \quad \$5 \div 18 \text{ for both sexes})\). Values for the remaining variables were then selected and tested to find those that gave the desired sum of five dollars. In contrast, students who used strategies #5 and #6 reduced the problem’s complexity by exploiting its conditions to derive a single equation that represents a relationship between two of the three variables. It was this relationship that then constrained the selection and testing of values for one of the variables. Thus the starting point for trial and error in the algebraic and verbal reasoning strategies was determined through a principled operation on the problem’s conditions, rather than an unprincipled guess at the values of one set of variables.

**Questionnaire Responses**

A high rate of *Yes* responses was recorded for almost all Self-Monitoring Questionnaire statements referring to metacognitive strategies. Response rates for the four statements that might prompt initial recognition of the metacognitive “red flags” described earlier (*lack of progress, error detection, anomalous result* – see Figure 1) are shown in Table 2. While these results seem to suggest that students were immersed in metacognitive activity, it is unwise to accept self-reports of this kind at face value as information relating to regulation of cognition is not necessarily statable (Brown, Bransford, Ferrara & Campione, 1983). The students’ questionnaire responses therefore must be interpreted in the light of their actual problem solving behaviour.

In the next section students’ written solution attempts are examined and, where necessary, compared with their responses to the questionnaire statements in Table 2, to reveal self-monitoring successes and failures.
Table 2

*Questionnaire Responses to Metacognitive “Red Flag” Statements*

<table>
<thead>
<tr>
<th>“Red Flag”</th>
<th>Questionnaire Statement</th>
<th>Percentage of Students Responding Yes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lack of progress</td>
<td>I asked myself whether I was getting any closer to a solution.</td>
<td>81%</td>
</tr>
<tr>
<td>Error detection</td>
<td>I checked my work step by step as I went through the problem.</td>
<td>63%</td>
</tr>
<tr>
<td>Anomalous result</td>
<td>I checked my calculations to make sure they were correct.</td>
<td>77%</td>
</tr>
<tr>
<td></td>
<td>I asked myself whether my answer made sense.</td>
<td>84%</td>
</tr>
</tbody>
</table>

**Successful Self-Monitoring**

Successful self-monitoring is difficult to detect if it merely confirms that satisfactory progress is being made. However, the students’ written work did provide evidence of self-monitoring where difficulties or errors forced a change of strategy. For example, although Table 1 shows that eight of the fifteen students who used a mean value strategy managed to find one of the answers to the problem, it does not reveal that six of these students began working with a different strategy that was subsequently abandoned. In three of these cases, the change of strategy was caused by lack of progress in formulating the problem algebraically. The remaining students discarded their initial strategy because it produced an answer that was either unreasonable (non-integral number of cents per child) or inaccurate (the total did not come to five dollars).

There was also some evidence that other students rejected unreasonable answers, but were unable to identify an alternative strategy. Their frustration is obvious in the following comments, found in their working:

This is not a right answer, because a number of people cannot be a decimal number, they can only be a whole number. (after calculating \( b = g = 9.26 \))

Impossible to calculate because there is no possible way of giving \( 27, \frac{\chi}{2} \text{ cents} \pm 1 \) (after calculating the mean value and trying to give each boy two cents more than each girl)

**Failures in Self-Monitoring**

Examination of Table 1 shows that there were three broad groupings of solution strategies and outcomes:

1. *Inappropriate strategies* (#2 incorrect formulation, #3 assume \( b = g \)) that gave incorrect answers (\( n = 16 \));
2. *Inefficient strategies* (#1 random trial and error, #4 mean value calculation) through which it was possible, with luck and persistence, to find one answer, but that were equally likely to result in no answer being found at all (\( n = 17 \));
3. *Appropriate strategies* (#5 algebraic or #6 verbal reasoning) that had the potential to produce one or both answers, provided that the strategies were
correctly executed and a way was found to solve an equation with two unknowns (n = 9).

Analysis of individual students’ solution scripts and questionnaire responses showed that the above strategy and outcome groupings were associated with corresponding failures to recognise, or act on, the metacognitive “red flags” described earlier:

1. **Anomalous results** were verified and accepted.
2. **Lack of progress** towards obtaining an answer did not lead to a change of strategy.
3. **Errors** in strategy execution remained undetected.

Each of these failures in self-monitoring is described in more detail below.

Table 3  
**Evidence of Self-Monitoring in Users of Inappropriate Strategies (Incorrect Answer)**

<table>
<thead>
<tr>
<th>Evidence from Written Work</th>
<th>Checked Calculations</th>
<th>Checked Sense</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>No evidence of verification</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Faulty verification procedure</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Verified non-integral cents (sum=$5)</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>Verified integral cents (sum$5)</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>11</td>
<td>2</td>
</tr>
</tbody>
</table>

Note: Sixteen of the total of 42 students were categorised as using inappropriate strategies. Of these, fourteen obtained an incorrect answer. (A further two students obtained no answer.)

**Inappropriate strategies.** Of the sixteen students who used an incorrect formulation or assumed $b = 9$ (Table 1), fourteen obtained incorrect answers, that is, an answer that violated the problem conditions. Since an incorrect answer represents a metacognitive “red flag” that should trigger a review of both the accuracy of calculations and the appropriateness of the strategy, it is tempting to assume that these students did not try to verify their answer. However, evidence from their questionnaire responses and written work, summarised in Table 3, suggests otherwise. Eleven students claimed that they checked their calculations, and ten reported that they asked themselves whether their answer made sense (Table 3, Evidence from Questionnaire). In most cases, their written work confirmed that they did indeed carry out some kind of verification procedure (Table 3, Evidence from Written Work); however, many appeared to accept either an integral answer that did not satisfy the problem’s explicit conditions, or a non-integral answer that did not make sense.

**Inefficient strategies.** Seventeen students used inefficient, random trial and error or mean value based strategies (Table 1). Although eight of the students stumbled on
one of the answers, a further eight had found no answer when the time allowed for working had expired. Evidence from the latter students’ written work and questionnaire responses is summarised in Table 4. This lack of progress should have prompted the students to question the usefulness of their strategy and try another approach. In fact, six did claim to have asked themselves whether they were getting any closer to a solution (Table 4, Evidence from Questionnaire). These students may have been poor judges of progress, or simply did not have access to an alternative strategy.

Table 4  
**Evidence of Self-Monitoring in Users of Inefficient Strategies (No Answer)**

<table>
<thead>
<tr>
<th>Evidence from Written Work</th>
<th>Evidence from Questionnaire</th>
<th>Assessed Progress</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Trial and error - no answer</td>
<td></td>
<td></td>
</tr>
<tr>
<td>No progress beyond calculating mean value.</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Total</td>
<td>6</td>
<td>2</td>
</tr>
</tbody>
</table>

Note: Seventeen of the total of 42 students were categorised as using inefficient strategies. Of these, eight obtained no answer. (A further eight obtained one answer, and one an incorrect answer.)

Table 5  
**Evidence of Self-Monitoring in Appropriate Strategy Users (Incorrect Answer)**

<table>
<thead>
<tr>
<th>Evidence from Written Work</th>
<th>Evidence from Questionnaire</th>
<th>Checked Working</th>
<th>Checked Calcs</th>
<th>Checked Sense</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Yes</td>
<td>No</td>
<td>Unsure</td>
<td>Yes</td>
</tr>
<tr>
<td>Incorrect answer caused by undetected errors.</td>
<td></td>
<td></td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Note: Nine of the total of 42 students were categorised as using appropriate strategies. Of these, two obtained an incorrect answer. (A further two obtained no answer, and five obtained one or both answers.)

**Appropriate strategies.** Nine students used algebraic or verbal reasoning strategies (Table 1). Five were at least partly successful, obtaining one or both answers, and another might have found an answer if she had taken more time in systematically trialing b and g values. The other student who failed to obtain an answer was hindered by her persistent, and fruitless, attempts to eliminate one of the two variables from the equation she had derived. Interestingly, this student stated that she was “unsure” whether she had assessed her progress towards a solution (questionnaire response). Despite using an appropriate strategy, a further two students obtained incorrect answers. Evidence from their written work and questionnaire responses is summarised in Table 5. Both these students recognised that their answers were incorrect and/or unreasonable, but they failed to detect simple algebraic errors either while they were working on the problem or later when they checked their calculations.
Conclusion and Implications

The aim of the study reported here was to investigate the metacognitive self-monitoring strategies used by senior secondary school students while working individually on a mathematics “problem” (i.e. a task that presented obstacles to their progress). Students’ self-monitoring activity was inferred from their written work on the problem and questionnaire responses, and interpreted in the light of an elaborated theoretical model of metacognitive processes in mathematical problem solving.

Examination of the questionnaire responses and written working of the students who attempted the MONEY problem revealed connections between solution strategies, outcomes and self-monitoring. Analysis centred on identifying students’ recognition of three metacognitive “red flags”: lack of progress, error detection, and anomalous result. Ideally, each should prompt a reassessment of either the appropriateness of the chosen strategy, or the manner in which it was executed. Thus, expected recognition and response patterns are as follows:

1. students using inappropriate strategies that lead to incorrect or unreasonable answers should check their calculations for errors and, if none are found, consider a change of strategy;
2. students using inefficient strategies that do not lead to an answer at all should review their progress and choose an alternative approach;
3. students using appropriate strategies that, nevertheless, produce an incorrect answer should find and correct their errors.

In practice, only five students used appropriate strategies (algebraic or verbal reasoning, supplemented by principled testing of values for one variable) leading to one or both answers being obtained. As these students were spread across all three participating schools (as indeed were students in all categories of solution strategies), there does not appear to be a connection between successful problem solving and the mathematics teaching that students may have experienced in their schools.

Although there were instances of successful self-monitoring, it was found that many students were either oblivious to the warning signals mentioned above, or were unable to act appropriately if the signals were detected. Even if students do review their progress towards the goal, check their calculations while they work, and attempt to verify the accuracy and sense of their answer, their worthy metacognitive intentions will be foiled if they are unable to recognise when they are stuck, have no alternative strategy available, cannot find their error (or cannot fix it if they do find it), or fail to recognise nonsensical answers.

The problem of recognising difficulties is clearly illustrated in the work of students who “verified” answers that either contradicted the information given in the problem or made no sense in real world terms. Although it is possible that these students had misgivings that they did not record, one wonders whether their years of schooling have engendered a belief that school mathematics tasks need not make sense. Ironically, some of the students who did explicitly reject these kinds of answers could not think of any other way to attack the problem. If we wish to encourage students to monitor and regulate their mathematical thinking, it is important to ensure not only that they are attuned to the signals that alert them to danger, but also that they are well equipped to respond.
In interpreting these results we should not lose sight of the fact that the MONEY problem was chosen for use in this study because of its challenging nature – that is, it was hoped that the task would raise the types of obstacles referred to above so that metacognitive strategies would be called into play. Perhaps, then, it is not surprising that so few students succeeded in obtaining a complete solution, or in effectively monitoring and regulating their problem solving activity. In fact, we have observed similar results with pre-service teacher education students and practising teachers who have tackled this task in professional development workshops. Teachers deserve many such opportunities to analyse their own mathematical thinking and consider implications for classroom practice if they are to successfully implement current curriculum policies promoting reform in mathematics education.

Beyond these immediate findings, the study has attempted to address some of the limitations of existing frameworks for analysing problem solving behaviour. First, a theoretical model of metacognitive processes in problem solving was adapted from Schoenfeld’s (1985) episode-based framework, to identify specific monitoring and regulatory actions that would be appropriate at different stages of the solution process. Second, the notion of metacognitive “red flags” was developed to highlight the difference between routine monitoring of progress and the more deliberate action needed when particular difficulties are recognised. (Further work arising from the present study has also distinguished between circumstances associated with problem solving success and failure; see Goos, 1998).

From a methodological perspective, we acknowledge that questionnaires should not be used in isolation to investigate metacognitive strategy use. For example, the findings reported here deserve follow up via individual student interviews that probe questionnaire responses and seek explanation of the solution strategies they adopted. Further research is also needed to investigate the strategies students apply in regular classroom settings, when peers become an additional resource for tackling obstacles to problem solving progress (see Goos & Geiger, 1995; Goos, 1997 for classroom based studies of collaborative metacognitive activity). Such a research program should also consider implications for teaching – in particular, how mathematics teachers can develop metacognitive abilities and dispositions in their students. The Self-Monitoring Questionnaire, when used in conjunction with a suitably challenging task, is a pedagogical tool that teachers could use with their own classes to extend students’ repertoire of metacognitive strategies, and to gain insights into individuals’ “on line” mathematical thinking.

References


Appendix

SELF-MONITORING QUESTIONNAIRE

Place a tick in the appropriate column to show how you were thinking before, during, and after working on the problem.

**BEFORE you started to solve the problem—what did you do?**

<table>
<thead>
<tr>
<th></th>
<th>YES</th>
<th>NO</th>
<th>UNSURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. I read the problem more than once.</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>2. I made sure that I understood what the problem was asking me.</td>
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<td></td>
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<tr>
<td>3. I tried to put the problem into my own words.</td>
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<tr>
<td>4. I tried to remember whether I had worked on a problem like this before.</td>
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<td></td>
<td></td>
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<tr>
<td>5. I identified the information that was given in the problem.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. I thought about different approaches I could try for solving the problem.</td>
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<td></td>
</tr>
</tbody>
</table>

**AS YOU WORKED on the problem—what did you do?**

<table>
<thead>
<tr>
<th></th>
<th>YES</th>
<th>NO</th>
<th>UNSURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>7. I checked my work step by step as I went through the problem.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8. I made a mistake and had to redo some working.</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>9. I reread the problem to check that I was still on track.</td>
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<td></td>
<td></td>
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<tr>
<td>10. I asked myself whether I was getting any closer to a solution.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11. I had to rethink my solution method and try a different approach.</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**AFTER YOU FINISHED working on the problem—what did you do?**

<table>
<thead>
<tr>
<th></th>
<th>YES</th>
<th>NO</th>
<th>UNSURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>12. I checked my calculations to make sure they were correct.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13. I looked back over my solution method to check that I had done what the problem asked.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14. I asked myself whether my answer made sense.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15. I thought about different ways I could have solved the problem.</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Did you use any of these ways of working?**

<table>
<thead>
<tr>
<th></th>
<th>YES</th>
<th>NO</th>
<th>UNSURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>16. I “guessed and checked”.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17. I used algebra to set up some equations to solve.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18. I drew a diagram or picture.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19. I wrote down important information.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20. I felt confused and couldn’t decide what to do.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21. I used other ways to work on the problem.</td>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>

(If you have ticked YES for Question 21, please write a sentence or two in the space below to explain what you did.)
Metacognitive Questionnaire developed by Fortunato et al. (1991)

(The Self-Monitoring Questionnaire was based on this instrument.)

NO—No, I didn’t do this.
MAYBE—I may have done this.
YES—Yes, I did do this.

BEFORE YOU BEGAN TO SOLVE THE PROBLEM—WHAT DID YOU DO?

1. I read the problem more than once. __ __ __  
2. I thought to myself, Do I understand what the problem is asking me? __ __ __  
3. I tried to put the problem into my own words. __ __ __  
4. I tried to remember if I had worked a problem like this before. __ __ __  
5. I thought about what information I needed to solve this problem. __ __ __  
6. I asked myself, Is there information in this problem that I don’t need? __ __ __

AS YOU WORKED THE PROBLEM—WHAT DID YOU DO?

7. I thought about all the steps as I worked the problem. __ __ __  
8. I kept looking back at the problem after I did a step. __ __ __  
9. I had to stop and rethink a step I had already done. __ __ __
10. I checked my work step by step as I worked the problem. __ __ __ __

11. I did something wrong and had to redo my step(s). __ __ __ __

AFTER YOU FINISHED WORKING THE PROBLEM—WHAT DID YOU DO?

12. I looked back to see if I did the correct procedures. __ __ __ __

13. I checked to see if my calculations were correct. __ __ __ __

14. I went back and checked my work again. __ __ __ __

15. I looked back at the problem to see if my answer made sense. __ __ __ __

16. I thought about a different way to solve the problem. __ __ __ __

DID YOU USE ANY OF THESE WAYS OF WORKING?

17. I drew a picture to help me understand the problem. __ __ __ __

18. I “guessed and checked”. __ __ __ __

19. I picked out the operations I needed to do this problem. __ __ __ __

20. I felt confused and could not decide what to do. __ __ __ __

21. I wrote down important information. __ __ __ __