THE NUMBER LINE: AN AUXILIARY MEANS OR AN OBSTACLE?

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Abstract. The aim of this paper is to investigate the ways in which the number line can function in solving mathematical tasks by first graders (6 year olds). The main research question was whether the number line functioned as an auxiliary means or as an obstacle for these students. Through analysis of the 32 students' answers it appears that the number line functions both as an auxiliary means and as an obstacle with the latter occurring in the majority of cases.

Introduction

Auxiliary means seems to play a very important role in early childhood mathematics. They are crucial elements of high quality mathematics at all levels (Ahmed et al., 2004; Arcavi, 2003; Szendrei, 1996), they are useful for effective mathematics instruction (Fuson & Briars, 1990; Thompson, 1992; Varol & Farran, 2006), they can also assist pupils to develop their own internal representations (Becker & Selter, 1996; Moyer, 2001; Russell, 2000). A kind of auxiliary means is the number line representation.

In the present study we investigated the ways in which the number line can function in solving mathematical tasks by first graders. The main research question was whether the number line functioned as an auxiliary means or as an obstacle for these students.

Theoretical Framework

Many studies report on the several uses of the number line and its crucial role in mathematics education. The number line is used for estimation (Onslow, et al., 2005), for multiplication (Fuys & Liebov, 1993; Kastberg, 2005; Wallace, & Gurganus, 2005), for measuring length (Gravemeijer & Stephan, 2002) and time (Moone & Groot, 2005), for extending students' knowledge (Thompson, 2001: 72) and for giving access to possible solution strategies (Dickinson & Eade, 2004: 46). It also allows the representation of numbers (Chazan & Ball, 1999; Wiegel, 1998), as well as the forming of geometric models for the operations of arithmetic (Herbst, 1997: 38; Kilpatrick et al., 2001: 90).

The important potential of the number line is that it provides a simple way to picture mathematical concepts. Many mathematical ideas and concepts require increasingly complex language to describe and assess, so a representation like the number line can reduce the text that pupils need to be able to interpret in order to access the mathematics in the question (Blinko, 2004: 3). Apart from that, according to Gravemeijer & Stephan (2002: 151), the number line supports the students' performance on counting-type tasks, by offering a way of scaffolding both partial calculations and partial results.

The ways that the number line can be presented are various. It can be structured or semi-structured, with or without numbers or other symbols, modeling the mathematical concept/solution. However, another type of number line representation, which is suggested in the literature, is the empty number line. This type of number line offers the student the freedom to use it in his/her own way for jotting or for working (Thompson, 2004), thus applying his/her own solution strategies.

Although many studies mention the crucial role that the number line plays in organizing thought related to mathematical concepts and ideas, some report difficulties and limitations in its use. Bobis (2007: 410) reported that the use of the number line caused some problems, possibly because it was introduced in a measurement context. Other studies cite that the number line does not model all equations successfully nor is intended to do so (Dickinson & Eade, 2004: 46). According to Ernest (1985 in Thompson, 2001: 72) when addition strategies used by young children when solving simple word problems were translated onto the number line, these strategies appear to be confusing and unlikely to lead to much useful learning. In a study (Skoumpourdi, in press) in which the role that auxiliary means such as the number line play for kindergartners in working out mathematical tasks was investigated, it appears that the effective use of the number line was limited to defining the number sequence.

The potential of the number line for organizing thinking about numbers and operations on the one hand, and the difficulties that arise from its use on the other hand, lead many researchers to propose learning strands and teaching sequences for its use in the teaching/learning process of mathematics. Studies suggest that the use of the number line may have to be taught carefully and systematically as there are many sub-skills that children need to acquire before they can use it with confidence and flexibility (Roushman, 2003). They mention that, once children become proficient in its use, they can use it extremely flexibly.

The most commonly known and used learning strand is the one which is suggested by the Realistic Mathematics Education (RME) approach (Gravemeijer & Doorman, 1999; Gravemeijer & Stephan, 2002). In that approach the number line supports a gradual shift from a *model of* students' informal solution strategies to a *model for* more formal mathematical reasoning. The number line, in this strand, can play a crucial role in supporting the elaboration of the students' informal strategies, and the development of more sophisticated ones. With this approach it is expected that the students gradually shift their attention from thinking about the problem situation to thinking about the mathematical relations between the numbers involved. With the empty number line, the intention is that it be converted by the students according to their thinking. Thus, students determine their actions and decide how they will use the number line (Gravemeijer & Stephan, 2002: 151).

In the new Greek mathematics curriculum (DEPPS) for K-6 students as well as in other countries' curricula (Bobis, 2007), emphasis is given to the development of mental strategies, in the early elementary years, by introducing new auxiliary means like the number line. For this reason many number line representations have been included in the new mathematics textbooks in Greece. But in a study (Skoumpourdi, 2009), in which the role that the number line representations play in these school textbooks was investigated, it was found that, although there is a wide variety of different number line representations, they are not presented in a developmentally correct manner like that which is suggested by the RME approach. More specifically, for the number line representations that are included in the first grade mathematical textbooks, it was found that there were eleven (11) representations in total. Five of these representations were Level 0 which means that the presented number lines were structured and completed and the students were not asked to act on the number line but only to understand it. The other six representations were Level 1 where students were asked to complete the presented number line. There were no number line representations in first grade mathematics textbooks of Levels 2 and 3 in where students had to model their solution on a structured number line or on an empty number line.

However the use of the number line is not yet well established in Greece. Although teachers are given guidance on how to implement its use in the classroom this is still in a very initial stage. In a study that was conducted in Greece (Skoumpourdi, 2008), concerning kindergarten and primary school teachers' views of the role that the number line plays in the teaching/learning process of mathematics, it seemed that, although a large majority of the teachers acknowledged the crucial role that the number line can play in that process, they rarely used it in practice and not in accordance with research results. Research results showed that very few teachers knew the different settings in which the number line can be presented. They seemed to use auxiliary means in the teaching/learning process intuitively and empirically.

Method

The research was conducted in a state primary school in Greece at the end of May, 2008. The participants in the study were thirty-two students (16 girls and 16 boys of 6 years of age). The students in this school represented a broad spectrum of socioeconomic backgrounds. Each student in the sample was given one of the two versions of a written test. One version consisted of word problems and the other version consisted of the same problems accompanied by a number line.

The tasks were designed to be slightly complex in order to engage students in a thinking process and to avoid students' direct answers. Apart from the addition and subtraction tasks, there were tasks which included two operations (addition and subtraction as well as multiplication and addition). More specifically task 'a' required an addition, tasks 'd' and 'e' required a multiplication (or a repeated addition) and an addition, tasks 'f' and 'b' required a subtraction, task 'c' required an addition and then a subtraction.

The tasks were the following (in brackets is described the type of number line that accompanied the tasks):

a. A frog is on Rock 16. After moving 12 rocks forward which rock will he reach? (structured number line with the numbers from 16 to 29 per one).

b. A week ago the temperature was 23^{o}C. Today the temperature is 34^{o}C. How many degrees did the temperature increase? (thermometer with the numbers from 0 to 45 per five).

c. An ant was on the number 9 and he made 7 more steps (each step is one number). His hole is on the number 21. How many steps remain before he reaches his hole? (structured number line with the numbers from 1 to 22 per one with an ant on 9 and a hole on 21).

d. A butterfly is on flower 4. Each day she moves forward 4 flowers. Which flower she will reach in 3 days? (structured number line with the numbers from 1 to 20 per one with a butterfly on 4).

e. A snail is on the number 3. Each day he moves forward 3 numbers. Which number he will reach in 6 days? (structured number line with the numbers from 1 to 21 per two with a snail on 3).

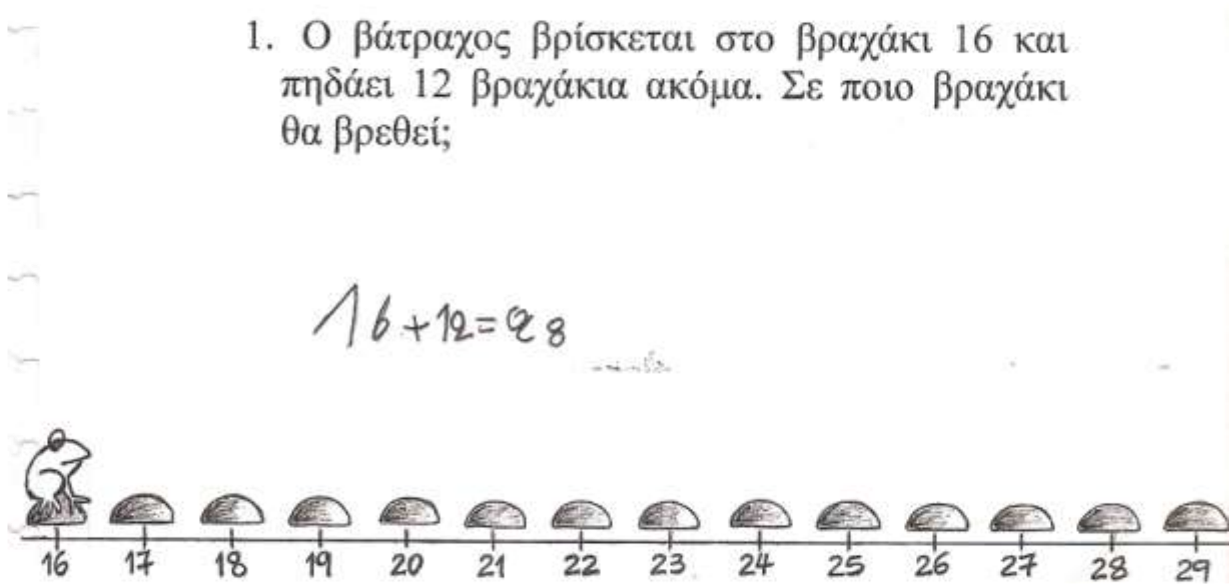
f. A book has 45 pages. John has already read 34 pages. How many pages remain unread? (an empty number line).

The tasks varied in their difficulty. According to the mathematical concept that is included the tasks in order of difficulty could be the following: 'a', 'd', 'e', 'f', 'b' and 'c' with 'a' being the least difficult and ‘c’ being the most difficult. However, according to the mathematical concept in relation to the accompanied number line and its form the order of difficulty could be the following: ‘a’, ‘c’, ‘d’, ‘b’, ‘e’ and ‘f’ task.

The mathematics school textbook for the first grade in Greece includes several tasks with the use of the number line. The tasks concern a sequence of numbers where students have to recognize and to construct certain numbers as well as to order them by placing them in the right location. Also there are addition tasks as well as repetitive addition/multiplication tasks. The strategies that are modeled for using the presented number lines are ‘count by 1’ and ‘count by 2’.
Students that participated in the study had been taught the above and had been familiarized with the several number line representations which are presented in their books.

Results

The data gathered were of two kinds: those which included the answers given to the mathematical tasks that were accompanied by the number line and that which included the answers given to the mathematical tasks that were not accompanied by the number line (Table 1).
From the students’ answers in the mathematical tasks accompanied by the number line (Table 1) we can see that the number line is not used in most of the cases (57 out of 96 times, 59%). In these cases students ignored the number line and answered making the operation or giving an answer immediately (Fig 1).



It is worth mentioning here that these students may have mentally used the structure of the number line for solving the problem, but they did not jot or work on it.

Table 1: Number of students' answers in the mathematical tasks with and without the number line

Mathematical tasks	**Answers**		**a**	**b**	**c**	**d**	**e**	**f**
Accompanied by the number line (n.l.)	*Correct answers*	With the use of the n.l.	10 (62.5%)	1 (6.25%)	0	0	4 (25%)	2 (12.5%)
		Without the use of the n.l.	4 (25%)	1 (6.25%)	2 (12.5%)	0	2 (12.5%)	8 (50%)
	Wrong answers	With the use of the n.l.	0	6 (37.5%)	6 (37.5%)	4 (25%)	6 (37.5%)	0
		Without the use of the n.l.	2 (12.5%)	8 (50%)	8 (50%)	12 (75%)	4 (25%)	6 (37.5%)
Without the existence of the number line (n.l.)	*Correct answers*		12 (75%)	6 (37.5%)	6 (37.5%)	8 (50%)	2 (12.5%)	6 (37.5%)
	Wrong answers		4 (25%)	10 (62.5%)	10 (62.5%)	8 (50%)	14 (87.5%)	10 (62.5%)

When they used the number line (39 out of 96 times, 41%) they did it in two ways. The first way was moving forward from the first given number in the mathematical problem as much as was necessary in order to find the answer. Sometimes they moved forward by 1 (Fig. 2), other times by 2 (Fig. 3), by 3 (Fig. 4), or by 4 (Fig. 5).

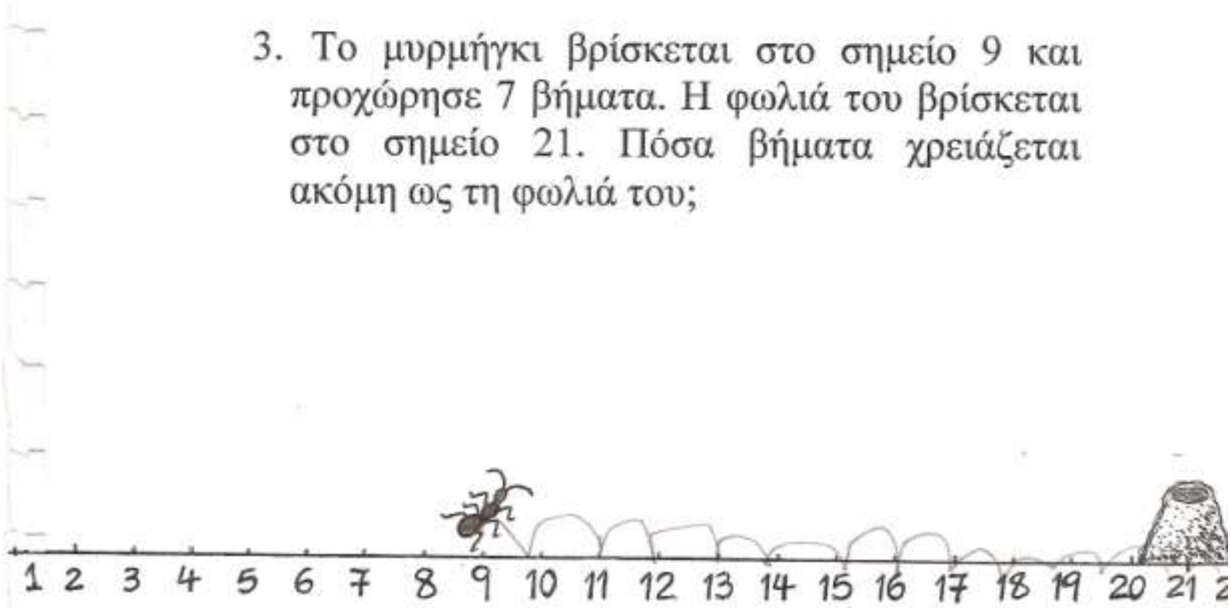


Fig 2 (*The student moved forward by 1)*

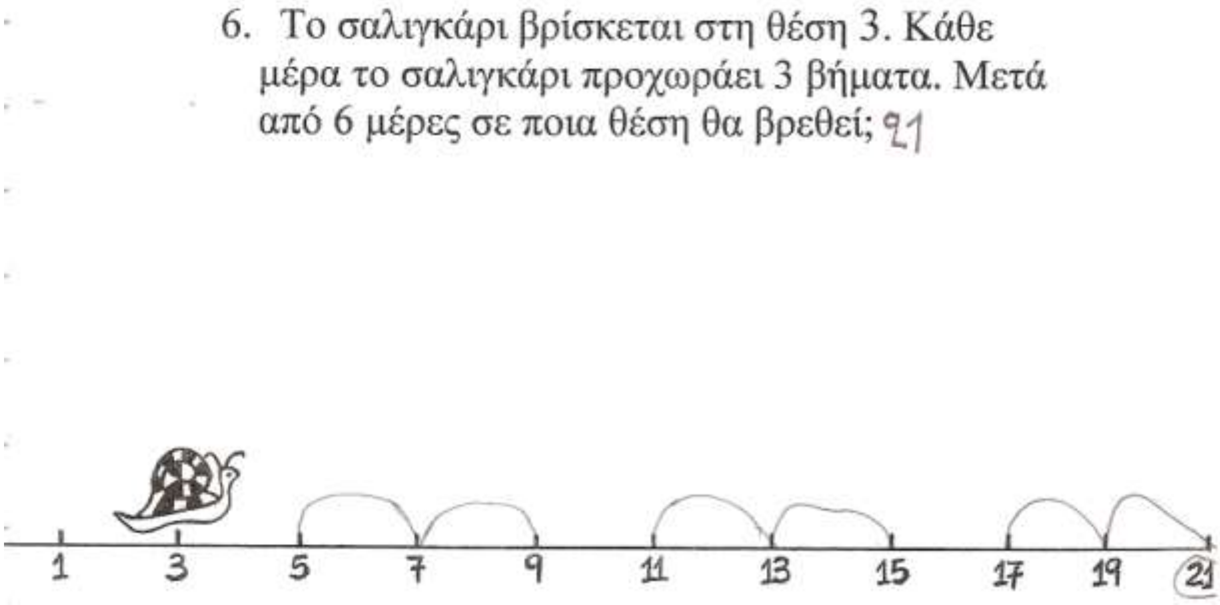


Fig 3 (*The student moved forward by 2)*

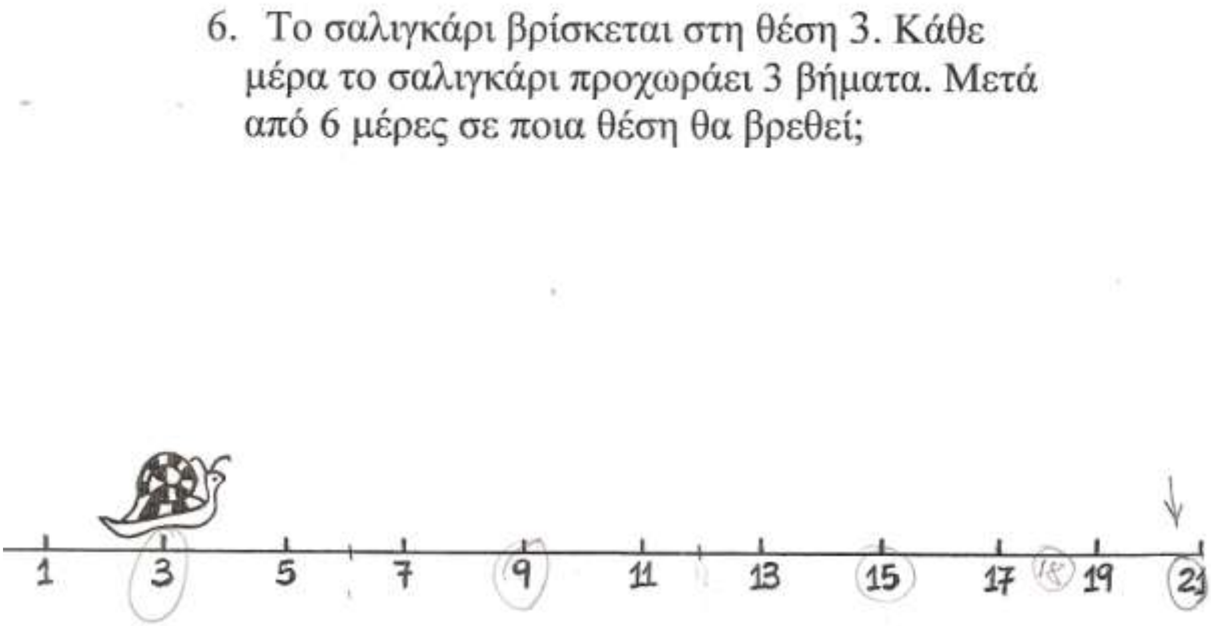


Fig 4 (*The student moved forward by 3)*

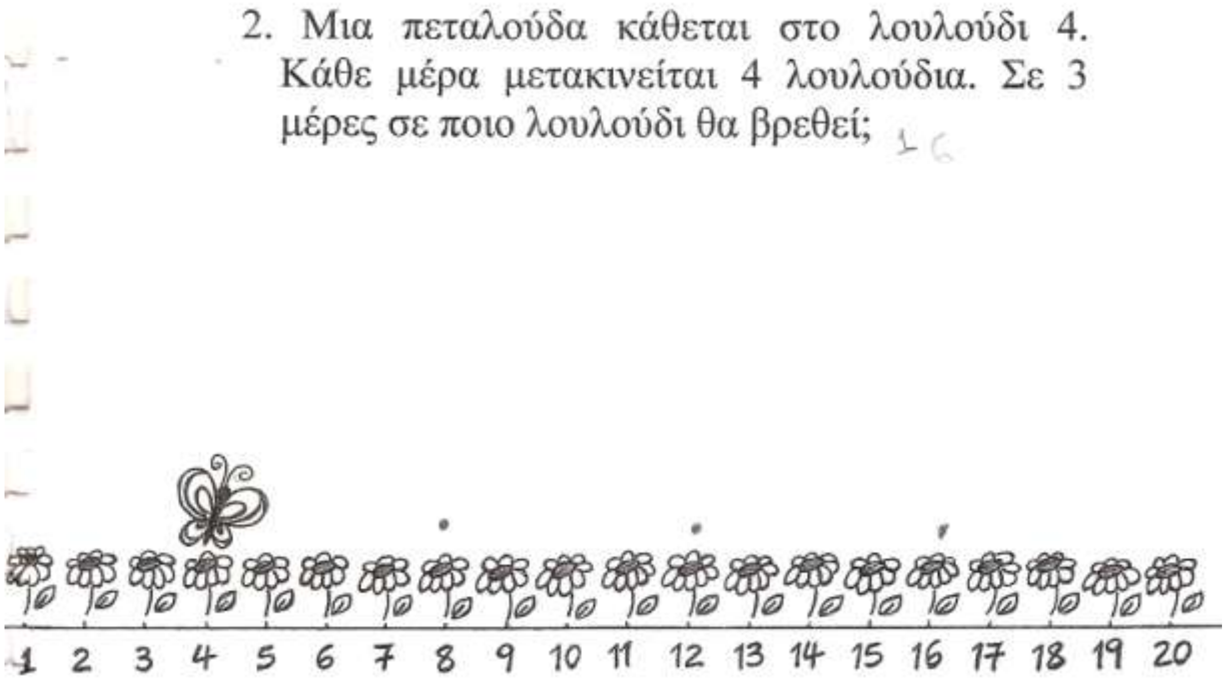


Fig 5 (*The student moved forward by 4)*

The second way was just noting (or circling) the number which represented the answer (Fig. 6).

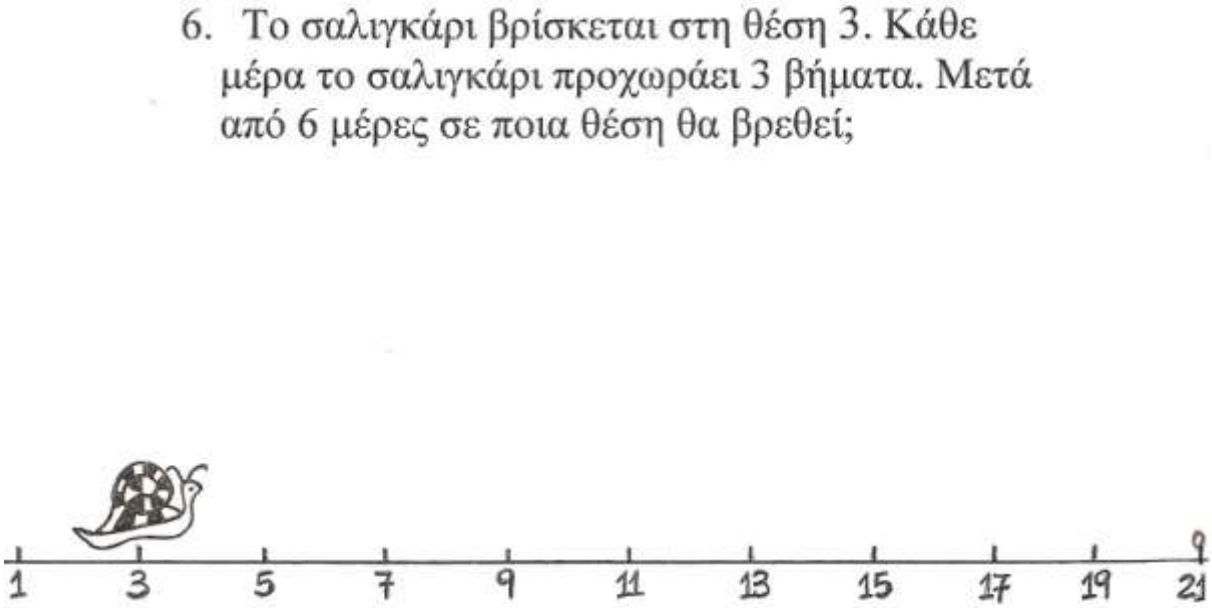


Fig 6 (*The student noted the number 21)*

But most students gave wrong answers when they used the number line (22/39, 56%). When giving wrong answers the main obstacle to using the number line effectively was the inability to calculate accurately after locating the first number. This may have happened because some tasks ('c', 'd' and 'e') had two steps and students often forgot to take into consideration all the information given by the problem. Other mistakes related to the counting of the numbers on the number line and not the spaces between the numbers (for example in task 'a' they start counting from 16 and not from 17).

The thermometer in task ‘b’ and the empty number line in task ‘f’ were special cases. The former was not used at all by any student and the latter, when it was used, was used in a specific manner. This was by ‘dividing’ it either by one on/under the number line (Fig. 7-8) or by five (Fig. 9). This did not always lead to correct answers.

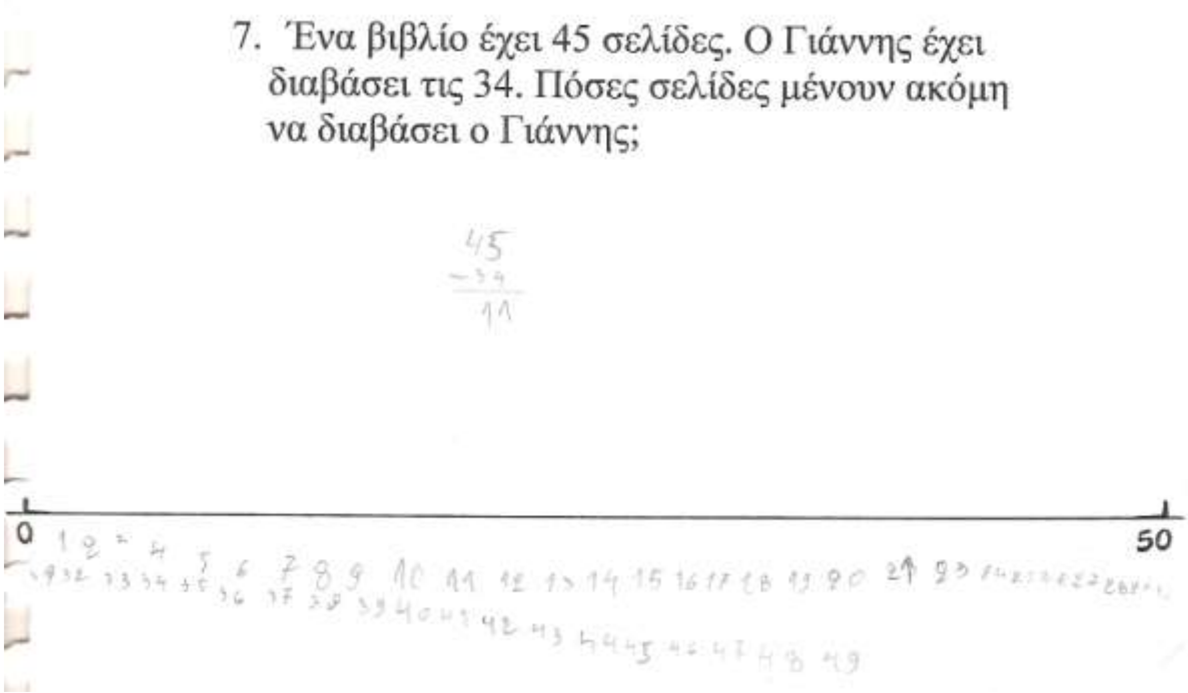


Fig 7 (*The student’ ‘division’ by 1)*

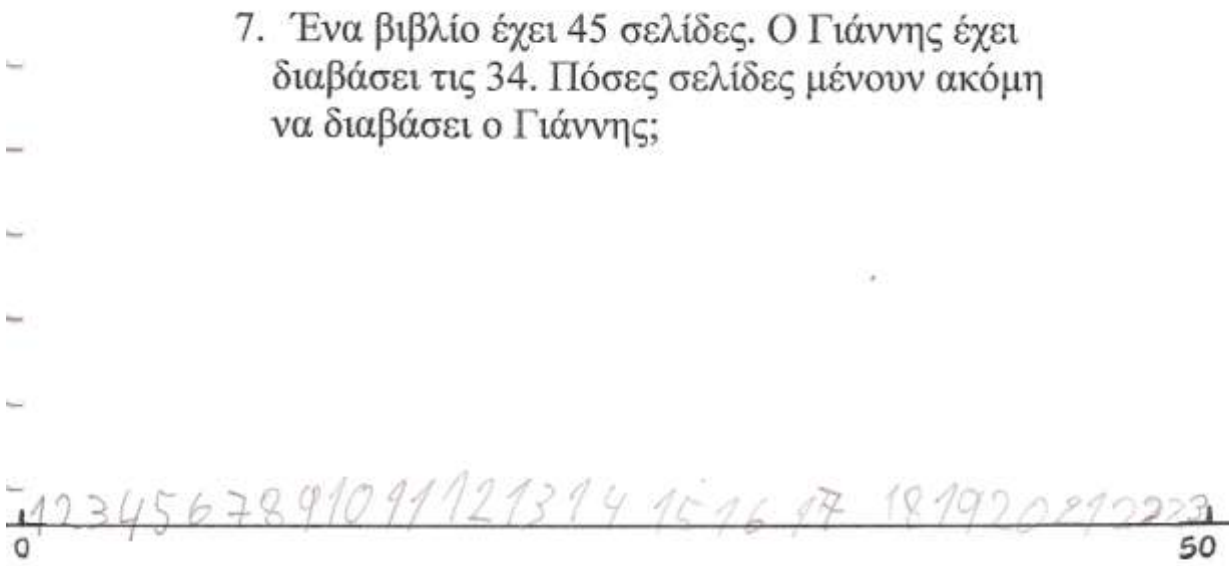


Fig 8 (*The student’ ‘division’ by 1)*

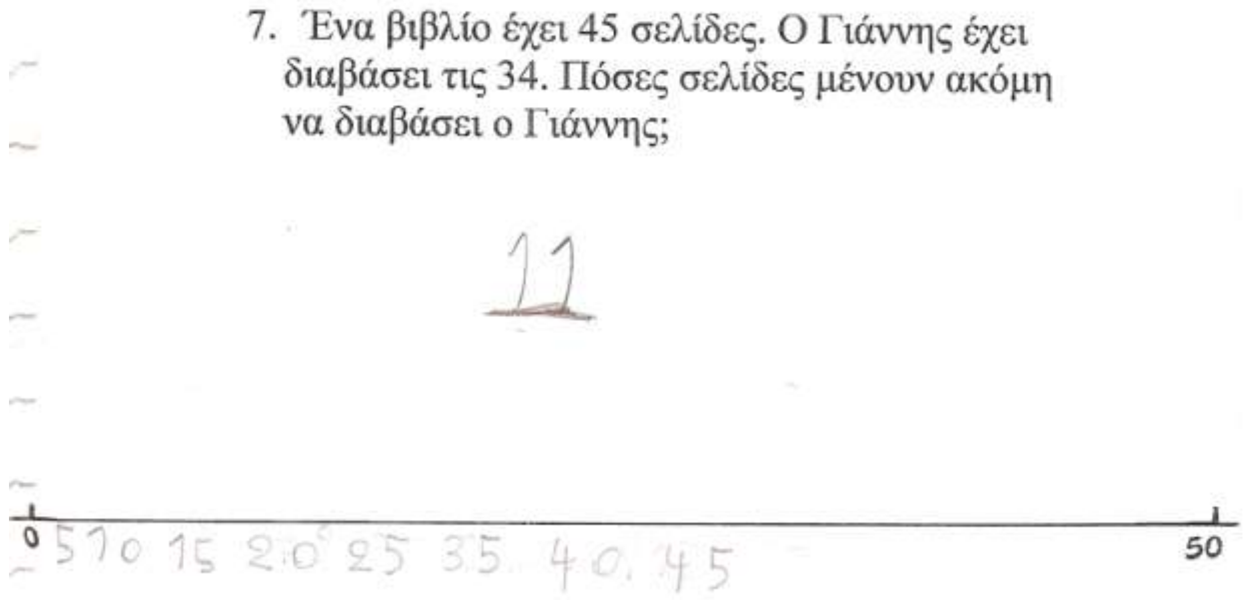


Fig 9 (*The student' 'division' by 5)*

Students' wrong answers in the word mathematical tasks without the existence of the number line (56/96, 58%) were fewer than the wrong answers in the mathematical tasks with the number line (62/96, 65%).

The results were also analyzed according to the difficulty of the tasks. The difficulty of the tasks accompanied by the number line varied (from the easiest to the most difficult: 'a', 'c', 'd', 'b', 'e' and 'f'). However, students' responses were not in accordance with that ordering. Ordering the tasks according to students' right answers when they had the number line, at their disposal, independently from whether they used it or not, we have: 'a', 'f', 'e', 'b', 'c' & 'd'. More specifically, students' most correct answers with the use of the number line were in task 'a' (62.5%) whereas the most wrong answers with the use of the number line were in task 'b', 'c' and 'e' (37.5%). Task 'd' seemed to be the most difficult for these students because none of them answered the task correctly. This seems somewhat strange since task 'e', which was answered by some students (37.5%) had the same form but different numbers from task 'd'. Also the number line which accompanied task 'e' was more complicated than the number line which accompanied task 'd'.

Regarding word problems, ordering the task according to students right answers we have: 'a', 'd', 'b', 'c' & f' and 'e' which also differ from the order of difficulty according to the mathematical concept (from the easiest to the most difficult were 'a', 'd' & 'e', 'f', 'b' and 'c'). Students' most correct answers (75%) derived from task 'a' and most wrong answers from task 'e'. Half of the students (50%) gave right answers in task 'd'. The percentages of students' right answers in the other tasks 'b', 'c' and 'f' were the same (37.5%). According to the results, students seem to find task 'e' more difficult than task 'd', which is contrary to the theoretical order of difficulty.

The above indicates that the type of the representation and the mathematical concept that is included in the task as well as the way task is formulated can function as an auxiliary means or as an obstacle and can influence students' answers.

Discussion – Conclusions

The aim of this study was to provide an answer to the question of whether the number line functioned as an auxiliary means or as an obstacle for solving mathematical tasks by first graders. Although many number line representations have been included in the new mathematics textbooks in Greece and students have been taught the use of the number line it seems from the present study that in most cases it functions as an obstacle since students' wrong answers in the word mathematical tasks, without the existence of number line, were fewer than the wrong answers in the mathematical tasks with the number line. Students had difficulty in interpreting the number line representation and in translating the problem to the number line. Students' inability to use the number line effectively could be due to several factors such as the way the number line is presented in the mathematics school textbooks, the way teachers use it, or the context and the difficulty of the mathematical tasks.

But there were some cases in which the number line functioned as an auxiliary means. In these cases the students used it in order to organize their thought for giving an answer in the mathematical tasks. They used it in several ways either for jotting or for working.

As Dickinson and Eade (2004: 47) mention "in unskilled hands, the number line model remains firmly in the 'cute new ideas' file". Thus, it is neither sufficient to just recommend the use of the number line as an auxiliary means for students' mathematical development nor to just include it in mathematical school textbooks.

For the creative and successful use of the number line it is suggested that a systematic teaching process be designed. This developmental process would support on the one hand the familiarization of students and teachers with the various types of number line representations and on the other hand its use. Thus the number line would become a useful tool for every student and for every teacher supporting the teaching/learning of mathematics.

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