A Study of Students’ Conceptual, Procedural Knowledge, Logical Thinking and Creativity During the First Year of Tertiary Mathematics

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Abstract

This study focuses on students in first year environmental science degree programs, where traditionally mathematical emphasis has been much less than within the strict science or math majors. The importance now placed on applied mathematics, however, means that students need to gain more conceptual and quantitative knowledge of mathematics in not only the environmental degree programs but also in most, if not all, sciences for example, health, education, ecology and biology. In this paper, the authors attempt to gain insight into why many students fail mathematical courses even when the mathematical requirements are not as demanding. This is done by examining students’ conceptual thinking patterns and strategies, as evident in students’ prepared scripts. A total of 133 students were requested to prepare a focus sheet that summarizes their knowledge of topics learned to a point in time. To motivate students to prepare focus sheets, the students were also told that they could use them during exams as notes. This indeed motivated the students to complete the sheets, as they prepared weekly summaries that were later summarized for use in exams. Detailed examination of such sheets allowed researchers to examine and study students’ knowledge, based in terms of procedural work, mathematical skills, strategies and depth of conceptual knowledge as evidenced in the written form. A study of linear and quadratic functions and limit sections led to interesting insights not only regarding mathematical knowledge but also, and importantly, the higher order skills such as problem solving tasks during preparation. Logical and creative competencies were assessed in terms of planning, organisation and linkage; that is, how and which aspects of the knowledge each student focused upon, transferred or linked across topics in order to facilitate use and application of knowledge. The results show average levels of procedural and conceptual competence, but low levels of logical and creative competence in terms of the categories studied. Almost 50\% of students lacked competency in procedural work, while around 54\% lacked conceptual competency. Given the emphasis placed on procedural skills by students, generally the levels were lower than expected. The lack of
structure, planning, preparation and organisation found in the focus sheets was worrying, in that students did not demonstrate deeper levels of understanding of the topics learned. These findings have implications for the first year mathematics teaching teams at universities, especially in the non-specialist mathematical majors, in terms of where the focus of teaching may lie.

KEYWORDS: Tertiary mathematics, learning algebra, conceptual, procedural, logic, higher order thinking, first year university mathematics, environmental science, mathematics education

Introduction

Often students believe that success in mathematics means that they can correctly carry out procedures and use algorithms and formulas (Hidi & Harackiewicz, 2000; Tularam, 2013a). While memorizing rules and procedures is rather important, often this knowledge is simply memorized rather than understood in terms of conceptual links or deeper meanings attached – that is, often students lack a connected structural knowledge base (Tularam, 1998; Tularam, 2013b). Anecdotal and survey evidence gathered by the authors over 20 years of teaching now suggests that many students are still motivated to do little other than memorize to pass university exams, and today this is more to do with time availability. Many students are now working longer hours in paid employment to make ends meet, and they seem to spend more time dealing with general life pressures (Curtis & Shani 2002; Hall 2010; Tularam, 2013).

Students from various backgrounds are now attending universities; that is, students from widely varying mathematical backgrounds seek higher education outcomes (Griese et al., 2011). Of concern, however, is the fact that an increasing number of students are shying away from undertaking quantitative courses, and therefore much lower numbers of students are completing basic mathematics and sciences courses and degrees (http://mathsofplanetearth.org.au/wp-content/uploads/2013/07/MPE-Conference-Report_2013.pdf; AMS-http://www.amsi.org.au/multimedia/pdfs/Maths-Adds-2013-2014.pdf; Tularam & Amri, 2011; Tularam, 2013a). This situation has led mathematicians to question and conduct research into why students are failing in mathematics, or even when successfully passing first year courses, why is it that they are not interested in attempting higher level mathematic courses or degrees (AMS, 2013; 2014).
It is often noted that students try to memorize rules when preparing for mathematics exams (Hidi & Harackiewicz, 2000). However, students often say “I simply don’t know which rules to apply where and there are so many of them I am lost”. Clearly, this becomes problematic later, when students are expected to apply their knowledge to real life or abstract problems in higher quantitative courses. There is evidence that shows students seldom realize the underlying reasons why certain procedures work, or even when they should consider alternative or equivalent methods/procedures that may be more appropriate for the situation at hand – that is, they lack higher order and critical thinking skills (Rach and Heinze, 2011). There is also some research that shows that students lack logical flow in thinking or creativity (Guzman et al., 1998; Mann, 2005; Tall, 2004; Treffinger et al., 2002).

The teaching of mathematics in universities has received some attention in terms of higher order thinking requirements (Guzman et al., 1998; Tall, 2004; Tularam and Amri, 2011; Tularam, 2013a,b; Tularam and Kelson, 1998). Tall (2004) noted that while studying mathematics at an advanced level, students often experience an “abstraction shock”, mainly due to the much more formal nature of mathematics presented at universities than that which they have learned at high school. Similarly, Guzman et al. (1998) noted that “the mathematics is different not only because the topics are different, but more to the point because of an increased depth, both with respect to the technical abilities needed to manipulate the new objects and the conceptual understanding underlying them” (p. 752). It is not surprising then that tertiary lecturers are finding it difficult to lecture, teach or facilitate students even in the less advanced mathematics courses (unpublished survey of mathematicians).

After more than 20 years’ experience in teaching mathematics at university level, the authors note that the previously “deemed” successful “lecture only method” is now considered ineffective this is consistent with Professor Mazur’s findings as reported by Lambert (2012). Although professional development courses exist, it appears that many lecturers are still unaware of alternatives or how to improve the situation, allowing for the increasing numbers of students. University teachers appear to be slow in adapting and/or changing styles, but they are slowly coping with the changes needed to meet the goals of their so called “modern students” (Tularam, 2013; Tularam & Amri, 2011). As noted earlier, the nature of the students entering university has changed considerably over time (Griese et al., 2011). The students of today are no longer only the high achievers, but many are now attending
university courses with an average or lower level of high school mathematics, and from various ethnic minorities, and socioeconomic sectors (Benn 2000). This is indeed an encouraging outcome, however there are other reasons that have been influencing decisions to undertake tertiary study, such as lack of employment opportunities, the “smart state push” by the governments promoting higher degree qualifications, decreasing entry demands in university courses and so on. In addition, there are now lower numbers of students choosing to do advanced mathematics in Queensland’s high schools (Queensland Department of Education) so it is not surprising that fewer students today are progressing to specialist and higher science and mathematical courses within their degree programs.

Ironically today, when the popularity of science and mathematics has once again risen in the public arena, the numbers of students undertaking courses are trending down. Nowadays, mathematics is required, and is even taught more and more in many non-specialist mathematics degree programs (Bibby 1985; Kahn & Hoyles 1997; Noss 1999; Hoyles et al. 200+), because of the need for quantitative skills in modern computer intensive environments. This has also put strain on degree programs in terms of the retention of students in engineering (Taylor & Morgan 1999), and non-specialist mathematics degree programs such as health, finance, business, planning and environment where students are not up the level required in the courses taught (Taylor & Mander 2002). The decline in numbers of students undertaking higher mathematics in Australia and New Zealand has led to advertising promotions, introduction of success advisors, and even the lowering of assessment and entry requirements (Tularam and Amri, 2011). This is not only a Pacific nation’s problem, as the same trend has been noted in the US, the UK (Heard 1998; Kitchen 1999) and Germany (Griese 2011) among other countries, which have highlighted concerns regarding declining numbers (Tularam & Amri, 2011). For example in Germany, Griese (2011) noted large numbers of mathematics students leaving university before graduation.

One of the oft cited reasons for the difficulties students have with mathematics is that “mathematics is too abstract or not immediately useful in everyday life” and this tends to be somewhat true in the students’ world, given their interests. Piaget’s studies have showed that a student’s cognitive level for the acquisition of abstract learning needs to be at the formal operations stage for successful learning of mathematical type concepts. Rach and Heinze (2011) found that many students entering universities now may not have reached the level of sophistication and abstraction necessary for them to cope fully with the demands and levels
of understanding involved in tertiary mathematics Research on conceptual, procedural, skills based competencies of students shows that the levels of these skills are low (Tall, 2004; Tularam, 2013a). In their study of “preparedness” of first year university students, Tularam and Amri (2011) noted that students’ self-preparedness, motivation and persistence played key roles in learning, success in assessment and in problem solving, but clearly more needs to be done to comprehend why many students cannot cope with learning higher mathematical ideas and notions at the first year tertiary level (Sutherland & Pozzi 1995; Hoyles et al. 2001).

The main aim of this study is to examine university level students’ knowledge in terms of the procedural, conceptual, logical and creative aspects of mathematics. The study compared the levels of performance of mathematical knowledge and reasoning at the time of the mid-semester exam. More specifically, the study identified student levels in conceptual and procedural, logical and creative thinking - as demonstrated in the students’ prepared focus learning and revision sheets. A critical examination of students’ written work was conducted to help expose the nature of their knowledge base in mathematics seven weeks after they arrived at university from high school. Categories from education literature were used to assess student work in terms of conceptual, skill and procedural competencies. Logical and creative competencies were assessed, based on details presented which relate to organisation, planning, nature of the links made across content topics, logical flow of steps, evidence of analysis, synthesis and evaluation conducted; and indeed by evidence of transfer of work. The authors were able to gain insights regarding the nature of student knowledge by analysis of a set of sampled topics. Analysis of performance and levels gained in mathematical knowledge and higher order thinking helped identify focus areas where future work is required. Based on the analysis, results and findings, a number of conclusions and implications are presented for first year mathematics teaching teams in terms of student focus on conceptual, procedural, logical, critical thinking and higher order skills.

Background literature

There are a number of studies which have analysed mathematics learning and problem solving (London Mathematics Society 1995; FitzSimons & Godden 2000; Geise et al. 2011; Tularam, 2013, 1997; Tularam and Kelson, 1998; Tularam and Amri, 2011). Griese et al. (2011) used the term *view of mathematics* that was earlier defined by Schoenfeld (1985), to refer to overall experiences in mathematics. Using this definition, Griese et al. (2011) studied
engineering students regarding their view of mathematics – that is, the result of all the experiences the students have had over time while learning. The authors concluded that “mathematical competence is not only about knowledge and skills, but also about disposition to act in productive ways” (p. 85). This disposition refers not simply to affective aspects, but also to the deeper abilities to problem solve, apply logical and creative thinking to address the task or problem at hand - driven by a well-structured knowledge base (Griese et al. 2011).

Lester et al. (1989) advocated the importance of conceptual and procedural knowledge in the development of solution processes. Using this view, this study focuses on the depth of student mathematical cognition and knowledge base; that is, how deep are students’ conceptual structures and procedural competence for learning and problem solving, when students are in their first year of study at universities.

Berger (2006) reported that the student at tertiary level, “adopts the symbol of an improper integral with an infinite limit, makes use of the operations in a template-driven format, without understanding, initially”; that is, the student “is using the mathematical signs in mathematical activities” (incoherent as they are to an outsider) (p. 19). If highly motivated, a student can, over time, develop a better understanding, for example by active reflection upon what he or she is learning from in class activities for example, or by working backwards noting the similarity of related textbook examples presented in the course. This is the so-called socio-cultural effect frame that appears to facilitate the induction of the student into the “cultural symbolic system” of mathematics (Radford, 2000).

Mathematical competence has been difficult to define succinctly. At the most general level, competency in mathematics is characterized both in terms of content (what mathematics students should know) and process (how students should go about doing and understanding mathematics). It does involve a student having the knowledge and understanding to perform procedures based on their mathematical knowledge. It also includes students’ opinions about mathematics, and any mathematical activity they undertake in situations where mathematical knowledge can play a role. Therefore, a student will need factual and procedural knowledge, in addition to a number of concrete skills, but these are not sufficient in themselves to account for mathematical competence. So it is possible to conclude that mathematical competence involves a well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge. In the end, mathematical competence must also include planning, communicating, mathematizing, representing, devising strategies, using
symbolic, formal and technical language and operations, as well as deductive and logical reasoning (Turner, 2011).

Many researchers (e.g. Stone et al. 1966; Weber 1990; Neuman 1997) have analysed students’ written material in depth to infer their knowledge base. Jacobs (2006) identified a number of critical indicators of success in mathematics (Table 1). According to Jacobs (2006), the student’s reality has to do with their textual production as an expression of mathematical thought, in response to instructions given or problems solved or questions set. The above named authors have all examined student written and textual material in depth and examined performance of students using categories and levels of achievement in them. Although Jacobs studied performance during problem solving, the competency indicators defined in his work also appear appropriate for analysis of other written material such as exploring students’ focus learning sheets for example (e.g. Stone et al. 1966; Weber 1990; Neuman 1997), – in addition the task presented to students in this investigation is also considered a problem solving task that may be examined in terms of student performance. In fact students’ focus and their knowledge reality may be revealed in their written work, and in their responses to instructions. Some of the competencies identified are listed in Table 1. Neubrand (2005) stated that: “Different didactical traditions and ways of teaching lead to different ‘inner structures’ of mathematical achievement, made visible by different performance in the types of mathematical activities” (p. 82). This suggests that a particular student’s mathematical achievement levels may be judged by his/her written or verbal performance; that is, insights could be gained from the student performance; the work however must be analysed in terms of what the student was instructed to do; and indeed what s/he has presented as a written and/or verbal response.

Therefore, the research suggests that the level of competence in mathematics may be judged by a detailed examination of the text, response, and written work produced when given a problem solving task. The student’s work can be analysed in terms of whether it exhibits rote learning, real world relevance, application and/or deep and flexible understanding, creativity and higher order thinking (Table 2). In this manner, a student’s focus revision sheet can also expose levels of knowledge such as in the reconstruction of work, copying of work, focus on topics, level of understanding, structure of work/method, level of algebraic writing together with the overall planning, organisation and structure of the presentation. Such analysis can help instructors judge the competence levels of students within the topics taught. Indeed, such
an analysis may be developed from student revision focus sheets. Based on reliability and consistency of categorisation, identification and in depth analysis, students’ inner knowledge may be judged (Table 3).

Table 1: Interpretation of various competencies based on Jacobs (2006)

<table>
<thead>
<tr>
<th>Indicator</th>
<th>Interpretation</th>
</tr>
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<tbody>
<tr>
<td>Procedural competence</td>
<td>This refers to a direct interpretation as revealed by the in-depth analysis and as indicated in the body of knowledge schema revealed in responses.</td>
</tr>
<tr>
<td>Conceptual competence</td>
<td>This indicates that the participant has understood the conceptual basis of the problem.</td>
</tr>
<tr>
<td>Logic</td>
<td>Irrespective of whether the answer was correct or not, the logic indicator refers to logically executed steps in the response.</td>
</tr>
<tr>
<td>Reflection</td>
<td>This was used when it was evident that the participant reflected. Evidence for this may be noted in the text. An example could be “scratching out” work.</td>
</tr>
<tr>
<td>Confidence</td>
<td>This indicator referred to evidence in the text of forthright solutions, preparedness to follow through, with little or no hesitancy.</td>
</tr>
<tr>
<td>Dealing with crisis</td>
<td>This indicator shows how the participant dealt with test items that s/he struggled with.</td>
</tr>
<tr>
<td>Creativity</td>
<td>Irrespective of the correctness of the mathematics used to solve the problem, this indicator labels behaviour in the text, which shows that the participant ventured outside of the norm in dealing with the problem. The norm would be described as constructed response solutions.</td>
</tr>
</tbody>
</table>

Table 2: Nature of learning and application

- Evidence of rote skills
- Evidence of real world relevance
- Evidence of real world application
- Evidence of deep and flexible learning

Table 3: Overall competency scale

- Not competent
- Low competence
- Competent
- Reasonable competence
- High competence

The above framework has been adapted for this study to allow the authors to examine first year students’ competence levels not long after they enter mathematical courses in the environmental science degree at university level.

Study group
The study reported in this paper included 133 students from the first year mathematics course designed for the environmental science degree at Griffith University. This group also included some students from other areas such as education, business and health sciences as an elective. Currently, there exists approximately a 16% per year failure rate with some 2-3% withdrawing before the census date opting to undertake the course later in the program, since the course is compulsory. The course does include some revision of the first year high school work but application and depth of the content is deeper; and complexity of work is at a higher level than high school. Students who have undertaken Mathematics B (algebra and calculus) in Queensland (with higher than B grade) or Math C (Advanced Mathematical Physics and Complex numbers) may find some aspects of the course easier than those of other courses. But those who have gained lower levels in the sound category (40% < x < 60%) in Math B (Introduction to Algebra and Calculus) may still struggle through the course, even when a low sound is considered a pass. Many of the students who have studied Math A consisting of much less algebra at school (a less advanced course with some statistics and only general level maths) find the course difficult (yet these students would have gained basic algebra skills from grades 8 – 12). There are some adult students who have returned to university a number of years after leaving school, but they may have completed a bridging mathematics or an equivalent TAFE (technical colleges) course. Mature students are usually highly motivated and demonstrate diligence and focus in passing first year mathematics courses, even when they find it difficult (Smart & Pascarella 1987. For mature and Math A types of students, the work especially the algebra and calculus sections, may be new but all of the students would have had at least 6 weeks preparation for the topics in this study before the mid-semester exam. As noted earlier, it is important to consider that most students may have done some algebra earlier in high school, but frequently many state their dislike of mathematics and inability to grasp the more abstract mathematical work.

Methodology

At the start of the semester, first year environment science degree students were requested to prepare a focus sheet for their learning in a first year mathematics course. This sheet, which was produced weekly, was marked by students and then was to be revised over time, thus allowing the students to note everything that they believed was critically important for the topic under study; this allowed the researchers to analyse the style and nature of the students’ understanding. To motivate them to complete the sheet, the students were told that they could later use the sheets as notes in their major examinations (totalling 70% of assessment). The
students were requested to revise and re-prepare their sheets for use in examinations prior to exam dates, but all earlier prepared sheets that represented their learning of the topics in the semester were kept and analysed. No other instructions were given and thus the students prepared their focus sheets according to their own style, interest and motivation without any other external directives. Throughout the semester, most of the students were able to develop a rather comprehensive sheet of two sided A4 paper, and from these detailed student notes, the authors were able to gain some interesting insights into student competencies in three introductory topics taught in first year mathematics.

Using the framework from the existing literature on learning, as well as through a process of discussion and reflection, the authors and assistants identified a number of aspects of the focus sheets that represented student competence levels in various topics. One doctoral student and two lecturers had to reach approximately 90% agreement during the analysis of students’ work for the purpose of categorization of the work. The students’ written work was categorised into procedural, conceptual, logical and creative competencies. In this study, a total of 133 students’ scripts were examined in detail, but rather than reviewing all work taught, a sample of three topics were chosen, namely, linear and quadratic functions and limits. These represented a set of important topics often taught in first year mathematics courses, and indeed, some of these may have been learned by the students before undertaking the course at university.

It is clear that students fail to do well in mathematics owing to a number of reasons and this is a complex issue, however this study provided an appropriate opportunity to examine the nature of the students’ content knowledge base and their critical and higher order skills, given that most of them had studied mathematics at high school for a number of years. In this way, the authors could examine the nature of deficiencies in the student knowledge base, such as the conceptual and procedural competencies mentioned in the literature. The opportunity was also available to examine the quality of thinking in the work presented in the focus sheets, and this was evaluated in terms of logical development and creativity. Any evidence of logical steps and connections made in and within the steps was identified by examining each step. Evidence of ways in which the content on the sheet was planned, organised and/or linked with regards to topics, and connections/relations made between topics was judged as creativity in organisation and development of the sheets. This higher level abstracted view of links across topics or examples in the sheets led to inferences regarding a deeper
understanding of content and procedures and higher order skills acquired; that is both critical thinking and knowledge.

More specifically, the students’ knowledge and application in terms of the structure was also studied in depth, based on the definitions in this paper (Table 4). This was done by identifying connections and/or transfer of knowledge shown within the steps of the workings, or within sections of work or across topics, or while demonstrating the application of skills, procedure and content to novel real life applications, or when giving self-constructed examples. The reliability of identification was tested by the three abovementioned personnel, all of whom examined the students’ scripts in detail for consistency in scoring.

As noted earlier, the categories used in this study were developed based on frameworks already existing in the literature. Analysis and refinement of such frameworks led to better definitions, and these in turn led to critical indictors for competency levels. Although the indicators may have been used for the analysis of examination and problem solving task analysis, it is argued here that the whole process of developing a self-constructed focus sheet for the demonstration of knowledge and understanding of a topic or for the use of notes in exams later is also a form of a problem solving task for students. Problem solving in the sense that students should organise information for solving problems in linear algebra, quadratics and limits so that it is readily accessible in an exam, and even relate methods that apply to more than one topic e.g. factorising a quadratic and determining the limit of a more complex quadratic type relation or function. In other words, a student is required to recognise the relevant information necessary to solve a problem, and then organise it such that it can be easily accessed and used to solve a problem under exam conditions. As such, the indicators were deemed appropriate for this analysis. The indicators further suggested the nature and type of evidence and student work required for a certain level of competence. The logical and creative competencies identified are in line with the extant literature on deeper understanding of mathematics (Table 4). In the end, a numerical based criterion was developed to evaluate students’ performance in terms of competence level achieved (Table 4). Examples of how the criteria were applied to student work are shown, with examples in boxes based on topics: Box 1 on linear equations, Box 2 on quadratic equations. Some examples of “typed” student work on linear functions, quadratic functions and limits, as well as scoring in each, are presented in appropriate sections.
The working definitions of competencies used to analyse the student focus sheets are: procedural competence, conceptual competence, logical competence and creativity.

- **Procedural competence** refers to the ability of the student to show the steps taken to solve a problem. There is clear evidence of a step by step method that is deemed appropriate for the solving a problem situation.

- **Conceptual competence** refers to the depth of knowledge of a topic evident in student work with relationships among key ideas being well understood; the student has shown or demonstrated a good understanding of the general idea within a topic and content relationships among the specific subtopics within a content area.

- **Logical competence** refers to the way in which the information regarding content and topics in the preparation sheet were linked or related to each other; also whether the steps in sheets were logical in the order of presentation in terms of deductions made.

- **Creativity** refers to the novel ways in which a sheet was organized, planned and developed; alternate ways in which the student presented the relevant information such as specifically showing the connections between the ways of considering a problem; relating the conceptual understanding to the relevant steps in examples, situations, or in addition to logical steps, using graphs to illustrate the relationships developed; or using self-constructed examples and novel descriptions of applications.

A number based set of criteria for classifying student work in terms of mathematical competencies in specific areas identified is presented in Table 4.
Table 4: Criteria for classifying students’ mathematical competencies

<table>
<thead>
<tr>
<th>Level</th>
<th><strong>Procedural competence</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>No example – no information on a topic but other work presented</td>
</tr>
<tr>
<td>2</td>
<td>No example, some relevant information incomplete</td>
</tr>
<tr>
<td>3</td>
<td>Example – knowledge of a procedure and evidence shown</td>
</tr>
<tr>
<td>4</td>
<td>Example annotated or examples – demonstrates more than 2 different procedures or methods for solving</td>
</tr>
<tr>
<td>5</td>
<td>Examples shown with more than 2 methods annotated and clarity and correct procedures clearly evidenced</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Level</th>
<th><strong>Conceptual competence</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>No example - no information on a topic but other work may be present</td>
</tr>
<tr>
<td>2</td>
<td>Some relevant information regarding the concept</td>
</tr>
<tr>
<td>3</td>
<td>Relevant information linked to an example of the concept</td>
</tr>
<tr>
<td>4</td>
<td>Example clearly annotated with concept knowledge inferred but not clear</td>
</tr>
<tr>
<td>5</td>
<td>Examples - more than 2 methods annotated with concepts well explained in the working</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Level</th>
<th><strong>Logical competence</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>None evident on a topic but other work presented</td>
</tr>
<tr>
<td>2</td>
<td>Some linkage of information of steps in working</td>
</tr>
<tr>
<td>3</td>
<td>Evidence of logical flow, or work and relevant information linked</td>
</tr>
<tr>
<td>4</td>
<td>Clear evidence of step by step development of work; annotation clearly linked to example with all steps explained</td>
</tr>
<tr>
<td>5</td>
<td>Excellent level of logic shown in work, with all annotation clearly linked to examples so the logic of the argument is shown- considering availability of options</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Level</th>
<th><strong>Creativity</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>None evident on a topic but other work presented</td>
</tr>
<tr>
<td>2</td>
<td>Some linkage of relevant information -evidence of planning, organisation</td>
</tr>
<tr>
<td>3</td>
<td>Organisation of relevant information so that it is clearly linked to an example</td>
</tr>
<tr>
<td>4</td>
<td>Well planned, organised and designed sheet, annotated with linkages to applied examples within and across topics</td>
</tr>
<tr>
<td>5</td>
<td>Very clear and well organised, planned and developed sheet; annotated and clearly linked examples across topics such as more than 2 methods linked; a clear transfer of methods to other topics</td>
</tr>
</tbody>
</table>

*Examples of competency in linear functions*

As noted in the Table 4 procedural competence in linear equations was scored as 1 for nothing presented; 2 for a partially worked example, or a complete trivial example: e.g. calculation of the product of two slopes showing their product was “-1” for perpendicular lines; 3 for completely worked example, e.g. \( y - 3 = 4(x - 0), y - 3 = 4x, y = 4x + 3\); 4 for two or more worked examples each using a different method of solution, e.g. the linear equation using two points by determining the slope first; a mark of 4 was given as per a score of 3, but also solving different types of linear problems such as an inequality \(|2x – 3| = 5\): showing the
case of $2x - 3 = 5$, $x = 4$ or $2x - 3 = -5$; $x = -1$; and finally a mark of 5 was given for at least two completely worked examples annotated with the procedural steps (Box 1).

Conceptual competence was scored 2 for concepts presented but not clearly linked to one another, e.g. $m = \text{rise/run} = \Delta y/\Delta x$ not clearly linked to $(y_2 - y_1)/(x_2 - x_1)$, or to the equation $y = mx + b$; or to the point slope equation $y - y_1 = m(x - x_1)$; a mark of 3 was given for some concepts clearly linked to one another via an example, such as $y = mx + b$ where $m = \text{slope}$ and $b = \text{y intercept}$; to find $m$; $\Delta y/\Delta x = (y_2 - y_1)/(x_2 - x_1)$; a mark of 4 was given for concepts that were clearly linked to one another as for a score of 3, but additionally describing a process linking the relevant information to find the equation of a straight line passing through two points; and a mark of 5 was given for concepts that were clearly linked to relevant examples.

Logic was scored as 1 for nothing evident; 2 for some evidence of deduction or logical reasoning in presentation of steps in written work; 3 for evidence of logical steps in the worked examples, e.g. $y = mx + b$ where $m = \text{slope}$ and $b = \text{y intercept}$, and how it can be obtained from $ax + by + c = 0$; or from the point slope $y - y_1 = m(x - x_1)$ equation; also in graphing by finding $x$ and $y$ axis cuts; (i) set $x = 0$, $y = 2$ presented in a logical manner; (ii) let $y = 0$, $x = 3$ etc; the use of logical reasoning to determine the set of points needed to plot a graph; a mark of 4 as for 3 but for some of the steps of the process being made explicit in relation to a worked example; 5 as for 4 but for clear linkage of the steps in the process to worked examples with possible options shown, and how each can be developed using logical reasoning.

Creativity was scored as 1 for nothing evident; a mark of 2 for some spatial connection between conceptual information and worked examples – some planning and organisation of the sheet; a mark of 3 for much more clarity in spatial connection between conceptual information and relationship to the worked example with good planning and organisation of the sheet and a little evidence of transfer across topics; a mark of 4 as for 3 but with other linking of graphs to the worked examples and across topics with a high level of planning and organisation and linking of sheet; and a mark of 5 as for 4 but with clarity in links with the conceptual information, graphs, worked examples and across topics with application examples (novel) included. A very high level of planning organisation and connections in the development of the sheet with appropriate links made across the examples and topics.
Examples of competency in quadratic functions

Procedural competence with quadratic equations was scored as 2 for a partially worked example; a mark of 3 was given for using a specific method to solve a quadratic equation, e.g. example 1, 2 or 3 in Box 2; a mark of 4 was given for using two different methods to solve quadratic equations, e.g. example 1 and 2 in Box 2; a mark of 5 was given for using three different methods for solving quadratic equations (Box 2).

Conceptual competence with quadratic equations was scored as 2 for some conceptual information such as the general form of a quadratic equation $ax^2 + bx + c = 0$, perfect squares and difference of the squares; a mark of 3 for stating the general form of the quadratic equation and clear links to the terms in the quadratic formula and/or linking the perfect square and difference of the squares to solutions by factorising (Box 2); a mark of 4 for the general form of the quadratic equation linked to the quadratic formula as well as some of the steps of the procedure annotated (Box 2); and a mark of 5 was given in addition to the previous category if all of the major steps of the process were clearly annotated with application examples given and a hidden quadratic equation outlined.

In summary, the procedural competence regarding the quadratic equations was scored as:

- 1 for nothing presented;
- 2 for a partially worked example;
- 3 for using a specific method to solve a quadratic equation;
- 4 for using two different methods to solve quadratic equations;
- 5 for using three different methods for solving quadratic equations (Box 2)

The conceptual competence regarding the quadratic equations was

- 1 for nothing presented;
- 2 for some conceptual information such as the general form of a quadratic equation;
- 3 for stating the general form of the quadratic equation and clear links to the terms in the quadratic formula (Box 2);
- 4 for the general form of the quadratic equation linked to the quadratic formula as well as some of the steps of the procedure annotated; and
- 5 as for the previous category but with all of the major steps of the process clearly annotated.
Logical competence was judged in the same manner as in the linear section and based on the definition presented in Table 4. Some students annotated examples with some reasoning behind the steps taken, e.g. in completing the squares with steps clearly specified as to what was done. This was considered to be based on logical reasoning; with creativity also judged in the process of the work but with particular focus on the explicit demonstration of the linkages between process steps and their execution, or considering alternative options of presentation. Average levels of logical and creative competencies were attributed to students who simply listed the steps in the process but did not relate them to each other; or those who listed the steps and then presented a worked example but links could not be clearly inferred from the work. In some sequences, it may be inferred from the work that the student followed the logic in the topic within steps when each line of the example and followed the correct sequence of written steps. Evidence of linkages, alternatives and options posed led to identification of logical and creative thinking in the sheet.

**Examples of competency in limits problems**

Conceptual competence was evident where students left an audit trail as they simplified expressions; for example, where they factorised the expression showing the same terms being cancelled; namely, the term $(x - 3)$ in the following limit problem.

$$\lim_{x \to 3} \frac{x^2 - 3^2}{x - 3} = \lim_{x \to 3} \frac{(x + 3)(x - 3)}{x - 3} = \lim_{x \to 3} x + 3 = 6$$

It is also clear from the example above that the student would have used the expansion: $x^2 - a^2 = (x - a)(x + a)$, then cancelled the common factors to simplify the expression so that the limit can be determined by substituting $a$ as the value of $x$. Thus in general:

$$\lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} \frac{(x + a)(x - a)}{x - a} = \lim_{x \to a} x + a = 2a.$$  

In an example of the type below it is clear that the student understood the rule that 3 cannot be substituted for $x$ because the denominator would become 0 and the fraction meaningless; it is not possible to divide by 0 as the division is undefined.

$$\lim_{x \to 3} \frac{x^2 - x - 6}{x - 3}$$
Therefore, it was necessary to factorise the numerator to obtain factors that may cancel with the expression in the denominator. If it is noted that $x - 3$ and $x + 2$ are the factors of the numerator expression, then student can cancel common factors; after which it is possible to substitute 3 for $x$ to calculate the limit as shown below.

$$\lim_{{x \to 3}} \frac{(x - 3)(x + 2)}{x - 3} = 5$$

Even if a student did not state their conceptual knowledge explicitly, their level of conceptual knowledge can be inferred from the audit trail left in the working of examples. This would not be the case if the example used was copied directly from the text or lecture notes as that does not require conceptual competence, other than to show that the student can recognise a relevant example.

Students who showed strong procedural competence demonstrated that they could use two or more different methods to solve a problem. For example: limits problems could be solved by substitution, factorisation, or dividing by the highest power in the denominator, whichever is appropriate; high in procedural competence meant that the student showed at least two of these methods. Logical and creative competencies were judged in the same manner as stated for topics earlier and based on Table 4. However for a mark of 5, students would have to link the limits to hyperbola graphing and asymptotes; or demonstrate understanding of approaching values of functions; say when $x$ tends to $a$, $y$ tend to $b$ etc.

After conducting the above detailed analysis, it was then appropriate to categorise students into competent and not competent levels. Each level gained in a category can also represent a mark for the student in that category. That is, this mark may be used to represent the level achieved in a particular competence and used for higher level analysis. For the level of competence gained across all of the four categories in each of the topics (linear, quadratic and limit type problems), another grouping was considered: satisfactory or better competence (C) and unsatisfactory or low (LC) level of competence. The condition used for achieving a C was a mark of 3 or above in a category, while for LC was a mark less than or equal to 2 on a scale of 1 to 5. In this way, overall numbers and percentages were developed for students categorised as either a C or an LC.
Any analysis of a qualitative nature may be subject to questions about judgments made in analysis regarding marking and categorisation of students’ work. For this reason a short section on subjectiveness, and thus limitations of the study, is presented after the conclusion.

**Results and discussion**

*Analysis of competencies gained across topics*

There was an average possible score of 15 overall after the analysis of all sheets were completed for each student in the competency level analysis. Figure 1 shows that the students, in general, showed low levels of basic mathematical competencies as defined and judged by researchers. Logical and creative competencies were also found to be lower than the former competencies. In linear functions, the students were taught straight line, graphing, linear expressions and equations, rearrangements of linear forms, inverses, absolute value functions, solving of linear and absolute value equations including applications of the linear models, and so on. In most cases, the students sampled did not demonstrate procedural competence in linear topics stated above and their achievement would be hardly labelled as a satisfactory competence at university level.

Overall about 60% of the students performed at the competent level but the significant minority, around 40%, failed to demonstrate effective procedural capability based on an in depth analysis of their own notes which were to be used in exams. While a lot of linear work is rule based, there are many different rules for different aspects of linear expressions, equations and functions, and students need to be strategic in their use of rules; such as when faced with various sorts of linear work, for example, inequality, absolute value and inverse of linear functions. In this way, even with the use of rules, the students had to be logical and creative in their approach to the writing of work that will be useful in problem solving. In the main, the solving of absolute value equations appeared to be a problem area and many did not note the significance of it in their written work. The solving of absolute value problems does not rely solely on a set formula or fixed rules, but rather students need to logically think through the problem to solve $x$ by considering the meaning of the absolute value symbol and the linear term inside the symbol. Not surprisingly, few students were judged to be logical and/or creative in this area of work.

Although the students appeared to have a comprehension of linear functions of the form $y = mx + c$ type, on average the overall conceptual score in the linear section was rather low, in
that only 59% demonstrated linear competence, but 41%, which is a significant minority, did not. Most students were not proficient with inequalities, absolute value concept, and inverses of linear functions.

**Quadratics**

Procedural competence was lower (46%) in quadratics. In contrast to the linear section, a formula was available for equation solving in quadratics, and it was noted that students could write the formula and work with it initially at least. This appeared to help raise their conceptual development in some manner as well, in that the students demonstrated a more “rounded” conceptual base, with about 63% demonstrating conceptual competence in quadratics. That is, with evidence and examples the students were more easily able to explain their knowledge of other methods of solving problems, such as factorising but to a lesser extent, completing the square.

In comparison to linear functions and the topics related to them, the students appeared to be more in tune with quadratic functions, mostly with the quadratic formula required to solve equations. Also, comparatively higher logical and creative competencies were noted in the quadratics work of students when compared to linear work (Table 11). Either the students did not need to write all they knew about linear topics or the students were more formula focused, realizing that using the formula meant they could at least correctly solve equations for marks in exams; and this may perhaps have provided the impetus of more focus on it. A lot of students wrote the formula, but many did not link or annotate the coefficients in the formula. This may be owing to the fact that they realised the relevance of each of the coefficients and the formula. However, the results in exams showed incorrect use of the formula, particularly when the coefficients were negative; the students used positive values for negative coefficients. Also, the division by $2a$ in the denominator of the formula was applied incorrectly to only the second square root part of the formula. This suggests that formula driven learning is not enough for successful performance, though it seems that this focus may somewhat improve conceptual competency.

**Limits**

In the limits section, the student were also lacking in procedural (49%) and conceptual (41%) competencies, although the limit section was higher in logical competency (40%) when compared to linear (13%); but somewhat similar to logical competence in quadratic
competence (43%). In creativity, the level in the limit section (5%) was similar to that in linear (4%) but lower than that in quadratics (20%). Once again, the overall logical and creative competencies were rather low (Table 11). This is not as surprising, since most students find limits and calculus concepts more difficult to deal with than linear and quadratics for example, however, some basic rules were available. In using limits, the students could provide that information, albeit with low levels of conceptual and procedural competencies; this process also allowed students to demonstrate some logical and creative thinking patterns, but the results were indeed low. It was clearly noted that students lacked a deeper level understanding of limits; it was not often that students realised the approaching a number or asymptotic aspects of limits. The limits section was not related to graphing of hyperbola for example, where the concept of limits was first considered in the course to help students ease into the section on limits, as well as to develop the concept of approach to a number or a value or an asymptotic schema.
Figure 1: Distribution of competencies – procedural, conceptual, logic and creativity

Figure 1 shows that the overall distributions of each of the competencies analysed (max 15 marks). The conceptual scores appear to be more normally distributed when compared with others. The modal conceptual score is 8/15 (approx. 53%). The model and the somewhat lower mean value of the conceptual score (7/15, approx. 46%) are surprising, given that most of the students have completed a number of secondary school mathematics courses with some
work being more or less revision (students’ prior high school information is known to the department). However, anecdotal and literature evidence show that mathematical courses are often disliked by the students entering the introductory mathematics courses. Anecdotal evidence and the literature also show that student persistence and motivation levels are not necessarily high in mathematics courses at high school (Heckman & Rubinstein 2001; Tularam, 2011). There may be other reasons as well, but some research evidence suggests that one of the key reasons is the lack of preparedness of students in the learning of mathematics; that is, they lack the self-discipline to motivate themselves for higher level learning (Duckworth & Seligman 2005; Zimmerman 2010; Tularam & Amri, 2013; Tularam, 2013b). Based on Zimmerman’s (2010) findings one would expect motivation and interest of students, who are not self-regulated learners, to wane when dealing with the more abstract nature of mathematics Interestingly however, when the students had more than enough time to revise previous work, and indeed learn new mathematics up to the examination (6 weeks) just before which this study was completed, their general results were low in both procedural competence and conceptual understanding.

The distribution of procedural competence appears similar to the logical distribution. The mean score of procedural competence is 6.5/15 (43%) while the modal value is 5/15 (approx. 33.3%); the respective values for the logical competence is 5.4/15 (36%) and 5/15 (33.3%). The similarity seems to be related to the nature of judgments made in these two competencies. The students’ procedures were assessed through the written work, and the written work also formed a basis for assessing logical ability –such as conditional and deductive reasoning, or even creativity, though to a much lesser extent. Logical judgment was based on the nature of the procedural presentation in a step by step manner, as well as within an overall presentation. Indeed, students’ written work often shows evidence of not only the procedural work but also the logical nature of their thinking during the development of procedures. It is not surprising then that there is some correlation between these two categories. However, the rather low number of students who achieved higher total scores in procedural competence (Figure 1) is concerning when this is an area in which our students (teachers) place most emphasis when learning. Indeed, the focus on procedural learning is in accord with the literature (Star 2005). Therefore the lower numbers of students in the higher scores in both procedural and conceptual distributions is somewhat surprising, but this seems to be one of the major reasons for high rate of student failure in mathematics courses.
**Higher order skills and creativity**

In contrast, students generally do not score well in higher order thinking skills, according to the literature (Tularam, 1997; 1998) and this was confirmed in this study. Logical and creative competencies appeared significantly lower in overall scores (Tall, 2004). It is noted that students find logical thinking in a mathematics problem solving framework difficult to develop and learn. Figure 1 shows there were rather low numbers in higher scores in both logical and creative thinking. The information presented in student work lacked planning and organisation, and there was little evidence of the many connections and links that could be generated across topics to make their life easier when using the sheets when problem solving. Indeed it was noted in the first semester exam in that few chose to relate their learned knowledge to application examples to help facilitate transfer of their learning, either to theoretical or application examples, or within and across topics.

The lower modal, median and mean values in each competency are somewhat surprising, but the results appear to show that students may indeed reach Piaget’s formal operations at different stages in their mathematical studies. The results show that most of the students have yet to reach the formal operations stage a la Piaget, and seem to suggest that even when over the age of 16 or 17 many students seem to be only just starting to develop a deeper understanding of the abstract notions often identified in mathematics. The Queensland Department of Education notes that there are low numbers of students undertaking higher level mathematical, physics or quantitative based courses in high schools, and this trend is also evident in the university undergraduate and higher degree programs in Australia (see Tularam, 2013a; Tularam and Amri, 2011 for support of this by other authors).

**More specific analyses of the linear section**

Table 5 shows that around 44% (58/133) of students did not present an example or stated any of the methods that could be used to solve linear functions. About 33% (44/133) presented a rule with no examples and about 11% (14/133) did not present anything on the linear topic. Of the 133 students, only 56% presented at least one example; around 28% presented only one example, 20% demonstrated two examples, about 7% presented three examples and 2% showed 4 examples. About 8% (12/133) used substitution, 13.5% (18/133) used the formula for the equation of line, while around 8% (10/133) included an inequality example in the
topic of linear functions. Approximately 35% (46/133) included absolute function and solving of such equations and around 34% (45/133) considered the inverse of linear functions. Further, about 28% (37/133) of the students used one method to solve linear functions, while around 31% (41/133) demonstrated two or more methods to solve linear functions.

Overall around 59% (78/133) of the students demonstrated procedural competence in the linear section; this was higher (as expected) when compared to the quadratic (46%: 61/133) and limit (49%: 65/133) sections (Table 11).

Table 5: Linear section – equation, inequality, absolute value and inverse

<table>
<thead>
<tr>
<th>Substitution</th>
<th>Equation</th>
<th>Inequality</th>
<th>Absolute</th>
<th>Inverse</th>
<th>Frequency</th>
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<td>x</td>
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<tr>
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<td>√</td>
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<td>2</td>
</tr>
<tr>
<td>x</td>
<td>√</td>
<td>X</td>
<td>x</td>
<td>√</td>
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</tr>
<tr>
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<td>x</td>
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<td>x</td>
<td>√</td>
<td>2</td>
</tr>
<tr>
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<tr>
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<tr>
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</table>

The analysis showed that most students did not present an example of any of the methods that could be used to solve linear equations. This may be because they may have thought the work was simpler but evidence in the exam papers does not support this as many incorrect applications of linear work were noted. Some topics taught in the linear section were not related to linear work, and this suggests that students were not able tolerate the linear
subtopics to each other based on similarity of solution processes and graphing. For example; it seems that students failed to decipher higher order patterns that ought to be apparent in the section. For example, there were only a few who delved into the absolute value equations that are taught under the topic of linear functions, but are more demanding to solve in that they require logical reasoning and consideration of various possibilities.

Only a few demonstrated the use of more than one method to solve linear equations of various types. Again, this could be because the linear topics may be perceived to be easier, and therefore students did not feel the need to write all they need to know about them. Also, in the preparation sheet students may want to cover the apparently more difficult topics such as quadratics and limits, but their work overall did not show significantly different results from linear apart from the conceptual competence in quadratics. The students showed similar but lower levels of procedural competence than were noted in limits and quadratics. Table 11 shows that the students’ levels in procedural and logical competencies were similar in both quadratics (46%, 43%) and limits (49%, 40%).

More specific analysis of quadratics section

The quadratic rule played a part in quadratic functions as it seemed to be the focus of student work, rather than applications of the same to real life or theoretical examples.

Table 6: Quadratic formula

<table>
<thead>
<tr>
<th>Formula</th>
<th>Terms identified</th>
<th>Example</th>
<th>Annotation</th>
<th>Frequency</th>
</tr>
</thead>
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<td>x</td>
<td>x</td>
<td>86</td>
</tr>
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</table>

Table 6 shows that around 65% (86/133) of students simply presented the quadratic formula in isolation; that is, not defining the terms in the quadratic formula or relating it in any way spatially to the general form. Only 15% (20/133) of students presented an example applying
the formula to calculate the values of $x$ for which $y$ was equal to zero. None related this solution to graphs and its relation to the $x$ axis intercepts. Only 9.8% (13/133) of students identified the terms before applying the formula to an example. A low 2.3% (3/133) of students annotated (identified $a$, $b$ and $c$) an example with the steps taken to solve the quadratic equation.

Table 7: Quadratic factorising

<table>
<thead>
<tr>
<th>Expansion</th>
<th>Example</th>
<th>Annotation</th>
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<tr>
<td><strong>Total</strong></td>
<td></td>
<td></td>
<td><strong>133</strong></td>
</tr>
</tbody>
</table>

* Annotation of no example provided - setting out the procedural steps

Around 65% (87/133) of the students did not present information regarding factorizing as a possible method to solve a quadratic equation (Table 7). A low 14.3% (19/133) of students provided information about expansion types such as $(a + b)^2 = a^2 + 2ab + b^2$ but the presentation was not typically related to other work. Only 18% (24/133) of students presented a worked example of how to solve a quadratic equation by factorization.

Table 8: Quadratic completing the squares

<table>
<thead>
<tr>
<th>Example</th>
<th>Annotated</th>
<th>Procedural steps</th>
<th>Frequency</th>
</tr>
</thead>
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<td>$x$</td>
<td>$X$</td>
<td>$x$</td>
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</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td></td>
<td><strong>133</strong></td>
</tr>
</tbody>
</table>

Few students recorded the applications of a completed square form of a quadratic equation. About 26% (34/133) of students presented a worked example of completing the squares to solve a quadratic equation (Table 8). Only 6% (8/133) annotated the worked example to explain the process, whereas the majority of students (74% i.e. 99/133) did not use, or even include, the completing of the square as a method of graphing parabolas; such as finding vertices of a parabola or solving any quadratic equation for $y = f(x) = 0$. 

26
More specific analysis of limits section

Table 9: Limit section

<table>
<thead>
<tr>
<th>Method</th>
<th>Example</th>
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<th>Frequency</th>
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</tr>
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</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>133</td>
</tr>
</tbody>
</table>

Table 9 shows that around 17% (23/133) of students wrote an example based on the substitution method to determine the limit of a function. Around 39% (52/133) students used factorization as a method to determine the limit of a function, and only 1.5% (2/133) used the highest power of $x$ in the denominator as a method of determining the limit of an expression or a function. It was possible that there was some transfer of knowledge to the topic on limits from earlier work on quadratics factorisation, in that students gave a number of factorisation examples. Interestingly, only 18% (24/133) considered this method when solving a quadratic equation, when such a method saves much precious time that would then be available to deal with other problem solving related cognitive processes. This evidence may show that students could just be restating the work learned in limits rather than being more creative and relating the method to factorizing in the topic of quadratics. It seems that that procedural and memory based learning approaches may have been the focus for student learning in limits, as mostly such factorized examples were noted, rather than any evidence of deeper insights into the connective nature of topics. In the relation of limits to hyperbola graphing, for example, no students made a connection or a link to hyperbolas and the asymptotic nature of graphing. There was no evidence of student thinking or linking of ideas regarding approaching values such as when “$x$ gets closer to zero $y$ gets closer to infinity type” understanding of limits. Rather, simple rule based work was evident in student work on limits. In a similar manner to the quadratics, where only a few students delved deeper to consider various ways in quadratics could be understood, applied or solved, the students did not demonstrate an approaching concept or idea. The idea of how very large numbers affect the value of a function was not evident in the work on limits; instead the students appeared to think that limit was “a plugging in of a value”—a lower level understanding of limits. An asymptotic
type understanding of limits of functions learned while graphing hyperbolas was not connected to or demonstrated, and this suggests that the transfer or connections of content knowledge was missing. The lack of connections made across topics helps to explain the lower levels in higher order thinking and creativity noted.

Figure 2: Frequency distribution of overall performance for each of the three topics.

Figure 2 shows the distribution of all scores over all competencies attained by students. There are three topics per each score. There is an overall trend downwards towards the higher scores (max 20). Small numbers of students are high (score > 13) in linear and quadratics while there are hardly any above a score of 13 for limits. As expected, the quadratics and limits are higher in frequency than linear at a low scores of 4 and 5; as expected the linear has higher scores overall yet the limits and quadratics are higher at 7, 11 and 12, probably influenced by the relatively higher conceptual noted evident in quadratics.

Summary of findings

Overall 51% of students showed procedural competency and 46% showed conceptual competency (Figure 11). The alternate view is more than 49% of students demonstrated neither procedural nor conceptual competency. Fewer than 32% of students showed satisfactory logical reasoning in the layout of work, such as showing the links between concepts and worked examples, or even grouping worked examples showing different methods of solving particular types of problem – linear, quadratic or limit. Fewer than 10% of students demonstrated a satisfactory level of creativity in their work as well as their presentation of mathematical information regarding a specific topic. The ideas of linking of concepts and relating applications to real life, or indeed using graphical interpretations to illustrate how the problem and solutions as they related to them were seriously lacking.
If logic was considered as a measure of step by step working of a solution, then around 42% of the students would have had demonstrated a satisfactory level of logical competence; that is, still lower than the proportion of students who showed procedural competence even when the examples and work the students demonstrated were used to judge the logical nature of presented work. This somewhat higher level of overall competency in logical thinking is not appropriate and certainly not conducive to mathematical learning.

Another view of the overall data is presented in Figures 6 and 7, and Tables 10 and 11, categorizing students as either competent or of low competency. The data show a high number of students in the low competence (LC) in both conceptual and procedural categories when compared with competent students (C). More concerning is the fact that many more students are in the low level of logical and creative competencies. This should be tested in future research to determine if it explains why students fail to undertake higher mathematical studies or courses. The lack of knowledge of algebra and calculus may deter students to later consider any higher level of general quantitative type courses and may account for why they only attempt the math courses if they are compulsory. There are concerns regarding the self-preparation levels of students, in that students continue to be unconcerned about university learning, in that lecture attendance is now an issue at universities (Liston & O’Donohue 2010). Students who do not do well are poor self-regulated learners (Zimmerman 2010) and this is manifested in their approach to studying for exams, revising just before exams and passing them are key features of time management. Rather than doing more than the minimum required weekly and over the semester for courses, setting themselves proximate goals and evaluating their progress in relation to those goals, less competent students leave much revising to the end of the semester. If this is correct, then it would be dependent on the university personnel or lecturers to guide and often change the assessment methods and styles to engage students into learning over a semester; that is, the university ends up time managing students, when in fact this should a part of student learning during university life. Evidence now suggests that many students still continue to do the minimum preparation required throughout the semester, and do very little even when the mathematics courses are well recognised as being more abstract and difficult to grasp and learn than other disciplines (Tularam, 2013a). All learning in general but especially maths needs consistent and active work throughout the semester. Student preparation and time management for assessment is mostly provided for and guided by the university, however students are not “gently forced” into engagement and acquisition of these abilities. Perhaps a unit should exist in the
university that has appropriate staff to help motivate students (inviting expert motivational speakers etc) and instil in them the need for active reflection time and self-regulation of behaviour possibly explicitly teaching students how to become self-regulatory learners.
The findings may explain why so many of the students dislike mathematics – a well-known fact. The students fail to cope with even the simpler of the algebraic demands and given that their basic algebra and critical thinking skills are both at a lower level, failure may be inevitable. Future research could focus on the learning approaches that competent and not competent students use and how the approach to learning affects the preparation of their worksheets.
Figure 6: Level of procedural and conceptual competency across topics

Figure 7: Levels of logic and creativity across topics

Table 10: Competency level - linear, quadratic and limit

<table>
<thead>
<tr>
<th>Topic</th>
<th>Procedural Competence</th>
<th>Conceptual Competence</th>
<th>Logical Competence</th>
<th>Creativity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LC</td>
<td>C</td>
<td>LC</td>
<td>C</td>
</tr>
<tr>
<td>Linear</td>
<td>55</td>
<td>78</td>
<td>89</td>
<td>44</td>
</tr>
<tr>
<td>Quadratic</td>
<td>72</td>
<td>61</td>
<td>49</td>
<td>84</td>
</tr>
<tr>
<td>Limits</td>
<td>68</td>
<td>65</td>
<td>79</td>
<td>54</td>
</tr>
<tr>
<td>Mean (%)</td>
<td>48.8</td>
<td>51.1</td>
<td>54.3</td>
<td>45.6</td>
</tr>
</tbody>
</table>

* Low Competence (LC) is ≤ 2 and Competence (C) is ≥ 3 on a scale of 1 to 5
Table 11: Overall proportion – competency level versus topic

<table>
<thead>
<tr>
<th></th>
<th>Procedural</th>
<th>Conceptual</th>
<th>Logical</th>
<th>Creativity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LC</td>
<td>C</td>
<td>LC</td>
<td>C</td>
</tr>
<tr>
<td>Linear</td>
<td>0.41</td>
<td>0.59</td>
<td>0.67</td>
<td>0.33</td>
</tr>
<tr>
<td>Quadratic</td>
<td>0.54</td>
<td>0.46</td>
<td>0.37</td>
<td>0.63</td>
</tr>
<tr>
<td>Limit</td>
<td>0.51</td>
<td>0.49</td>
<td>0.59</td>
<td>0.41</td>
</tr>
<tr>
<td>Mean</td>
<td>0.49</td>
<td>0.51</td>
<td>0.54</td>
<td>0.46</td>
</tr>
</tbody>
</table>

**Conclusion**

This study was an investigated university students’ knowledge in terms of the procedural, conceptual, logical and creative aspects of mathematics. The overall results of the analysis showed that the students sampled did not show the required level of competence in procedural or conceptual understanding of some basic topics such as linear functions, quadratic functions and limits. The students’ logical and creative competencies were noted to be rather low for university level mathematical studies, in that higher order thinking skills such as planning, organisation, linking and transfer of knowledge were less evident in their scripts. This low level hinders students’ acquisition of higher abstract notions which are prerequisites for advanced mathematical courses. The results suggest some reasons for why so few students continue with mathematics and sciences majors and this is a problem for the future of the Australian workforce generally.

One of the first, critical requirements of higher level mathematics is an understanding of algebra, linear functions, quadratics and limits. Therefore the findings of this study question the nature of students’ prior knowledge and learning of mathematics. It is a question how students’ prior mathematical knowledge influences the development of higher order and more structured mathematical understanding. It is evident that much work has been done on learning and transfer of mathematical knowledge in recent times, yet it is noteworthy that as far back as 1994, Gates (1994) said “to reach their potential level of transfer they would need to further their mathematical knowledge and understanding, that is, in both depth and clarity” (p. 294). The importance of the nature and conceptual structure of students’ knowledge base is confirmed by their examination results. That is, a good body of knowledge seems to be a critical factor in higher learning but the same knowledge should be easily recalled and demonstrated in terms of concepts’ depth and clarity in both procedure and application. The
students’ basic algebra concepts and skills are at a rather low level. The students showed an inadequate knowledge of limits as used in graphing or when applied to other contexts. In addition, the students also demonstrated low levels of logical and creative competencies critical for higher mathematical and scientific learning. This state of affairs may help to promote the development of negative beliefs and fear of failure in mathematics, but more importantly, this level of knowledge in the end limits the growth of further mathematical knowledge, and so the cycle seems to continue. This investigation of focus sheets did indicate that to improve students need to become more independent and confident self-regulated learners, motivated to develop higher levels of self-preparation.

Limitations

High school teaching of mathematics could be wide and varied and this has not been accounted for in this study, for the students arrive from all over the state and it is difficult to assess the nature of mathematical teaching each student may have received. There are other limitations. As always, in such category and judgment analysis work there is subjectivity in appropriately identifying competencies in students’ work and in apportioning scores or marks to it. The judgments on logical and creative competencies are usually problematic, but in this study judgments were made by an independent marker who has a doctorate in the mathematical sciences but who was independent of the teaching team. Although judgments may be problematic and questionable if strict adherence to the set criteria is applied, the difficulties, problems and questions may be minimized. Commonly in the case of quantitative studies, a 95% confidence interval is presented to demonstrate error in the decisions made. The results should be seen in a similar manner, in that the overall marks developed should be viewed as a range rather than a point estimate, even when no range can be given. However, if results are subjectively understood as low, very low, satisfactory, high and very high, then the categorisation aids the analysis. Allowing for some questionable judgments appear to be significant given the rather low levels achieved by students overall. The findings regarding logical and creative competencies in such a large sample of university students is concerning, but seems to be in line with the current literature.
Box 1 - Linear Example

\[ y = mx + b, \text{ where } "m" \text{ is the slope of the line and } "b" \text{ the y intercept.} \]

\[ m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \]

\[ y - y_1 = m (x - x_1); m_1 \cdot m_2 = -1 \text{ perpendicular lines; } m_1 = m_2 \text{ parallel lines} \]

Finding the equation of a line that runs through the coordinates (2,3) and (1,1)

\[ y = mx + b \]

\[ m = \frac{(3-1)}{(2-1)} = \frac{2}{1} = 2 \]

therefore \[ y = 2x + b \] to find “b” substitute the values of x and y from one of the coordinates on the line e.g. (2,3); or use \[ y - y_1 = m(x - x_1) \]

\[ 3 = 2 (2) + b = 4 + b; \text{ therefore } 3 - 4 = b \text{ thus } b = -1 \]

Therefore, the equation of the line that runs through the two points is

\[ y = 2x - 1 \]
Box 2 - Quadratic Example – graphs not included

y = ax^2 + bx + c = 0

Set out the appropriate expansions such as

(a + b)^2 = a^2 + 2ab + b^2  perfect square

(a – b)^2 = a^2 – 2ab + b^2

a^2 – b^2 = (a – b)(a + b)  the difference of squares.

Solutions

1. Factorising

y = x^2 + 5x + 6 = 0 = (x + 2)(x + 3)

x + 2 = 0  x + 3 = 0

x = -2  x = -3

x^2 + 25 = -10x

x^2 + 10x + 25 = 0

(x + 5)(x + 5) = 0

(x + 5)^2 = 0

x = -5

2. Apply quadratic formula

x = \frac{-b \pm \sqrt{b^2 – 4ac}}{2a}

x^2 + x - 2 = 0, a = 1, b = 1, c = -2

x = \frac{-1 \pm \sqrt{1^2 – 4(1)(-2)}}{2(1)}

= \frac{-1 \pm \sqrt{1 + 8}}{2}

= \frac{-1 \pm \sqrt{9}}{2}

= \frac{-1 \pm 3}{2}

= \frac{2}{2} = 1  or\ \frac{-4}{2} = -2
3. Completing the square  e.g. ¼ x^2 – 2x – 5 = 0

1. take the coefficient of "x^2" outside the brackets   ¼[ x^2 – 8x – 20] = 0
2. Divide both sides by the coefficient of x^2    [ x^2 – 8x – 20] = 0
3. Divide by the coefficient of "x" by 2 then square it  [ ½ (– 8] \]2 = 16
4. Add it to both sides     x^2 – 8x – 20 + 16 = 16
5. Perfect square    x^2 – 8x + 16 = 16 + 20; (x – 4)^2 = 36
6. Square root of both sides    (x - 4) = ±\sqrt{36}
7. Make "x" the subject and simplify    x = 4 ± 6;  therefore x = 10 and – 2

4. Recognition of hidden quadratic in a function
e^{2x} – 7 e^x + 6 = 0
has the same general form as
ax^2 +bx + c = 0 and therefore can be written as (e^x)^2 + 7(e^x) + 6 = 0
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