Opportunities of learning through the history of mathematics: the example of national textbooks in Cyprus and Greece

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Abstract

In this paper, we examine the ways the history of mathematics is integrated in the national textbooks of Cyprus and Greece. Our data-driven analyses suggest that the references identified can be clustered in four categories: (a) biographical references about mathematicians or historical references regarding the origins of a mathematical concept (b) references to the history of a mathematical method or formula containing a solution or proof, (c) mathematical tasks of purely cognitive elements that require a solution, explanation or proof and (d) tasks that encourage discussion or the production of a project that would connect the history of mathematics with life outside mathematics. Furthermore, we employed a framework around the levels of cognitive demands derived from the literature to analyse the identified mathematical tasks.

Keywords: history of mathematics, textbooks, Cyprus, Greece

Introduction

In recent years, a variety of new teaching methods and materials are proposed in order to promote deeper conceptual understanding of mathematics for pupils. The majority of research reported is concerned with the integration of new technologies in mathematics classrooms, such as computers, interactive whiteboards, applets, virtual manipulatives, and so on and so forth. This paper, however, focuses on the utilization of the history of mathematics as an alternative and supplementary means to establish more efficient teaching and learning of the subject. International interest in introducing elements from the history of mathematics is reflected in many relatively recent publications in the field (see for example, Fauvel & van

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Maanen 2000; Siu & Tzanakis, 2004; Thomaidis & Tzanakis, 2009). Exploratory in character, this paper is interested in the opportunities provided by the lower secondary mathematics textbooks of Cyprus and Greece to teachers and pupils in order that their engaging with the history of mathematics will facilitate a deeper conceptual understanding of mathematical topics. The countries studied represent the respective locations of the two authors. We begin this paper by reviewing the literature on the benefits of incorporating elements of the history of mathematics in classrooms. In subsequent sections, we discuss the importance of textbook analysis as well as the findings of our project.

The pedagogical role of the history of mathematics

One of the central roles of the history of mathematics as a scientific area is to provide answers to questions regarding the origins of mathematics, its establishment, its usefulness, its connections to society and other scientific fields, its development, as well as the conflicts and obstacles mathematicians had to overcome throughout the centuries until mathematics took its form as we know it today (Farmaki, Klaudatos, & Paschos, 2004). Consequently, by appropriately incorporating historical references in mathematics lessons, it is more likely that mathematics as a school subject will highlight the human dimensions of this discipline and not present it as disparate to human evolution and as a system of knowledge that arrived to us intact, perfect, with unquestionable truths subject to no alterations, skepticism, reactions, and conflicts (Farmaki et al., 2004). In this way pupils may avoid perceiving mathematics as distant and unfamiliar, while mathematical concepts can be presented to them in a more natural way, leading to a smoother transition to formal methods and proofs (Tzanakis & Thomaidis, 2012).

In general, teachers' practices to integrate history of mathematics in their lessons include, among others, projects on historical issues, etymological analyses of mathematical terminology, examinations of original sources, references to historical facts and incidents, and examinations of problem solving strategies in history (Tzanakis et al., 2002). By studying mathematical problems of antiquity, learners have the opportunity to compare and contrast their contemporary solution strategies with the original solutions of ancient mathematicians. Such a process helps pupils realise the effectiveness of the contemporary mathematical notation in contrast to the ancient one (Fasanelli et al., 2002) as well as the ways in which mathematical knowledge of the past has influenced our modern everyday life (Lawrence, 2008; Jahnke et al., 2002). Furthermore, the history of mathematics may serve as an opportunity for reflection and deconstruction of stereotypical attitudes and beliefs, and offer answers to ontological and epistemological questions regarding the nature of mathematics (Jankvist, 2009b). Utilization of the history of mathematics in classrooms may play an important role in the elimination of racial discrimination. Mathematics as taught in the west is dominated by Eurocentrism; consequently, leading pupils from non-European backgrounds to not feel competent enough to contribute to the field (Strutchens, 1995). References to the important work of mathematicians from other, non-European ethnic origins support recognition and celebration of diversity, and also help learners realise that other societies and cultures have contributed and do contribute to the development of mathematics as a scientific discipline (Michalowicz, Daniel, Fitz Simons, Ponza, & Troy, 2002). From a meta-analysis perspective, Jankvist (2009a) critically reviews the relevant mathematics education literature and distinguishes between the arguments in using history (the "whys") and the different approaches in doing this (the "hows"). With regard to the “whys”, two clusters are identified: the set of history-as-a-tool arguments concerned with students' learning the inner issues of mathematics, and history-as-a-goal arguments concerned with the use of history as a self-contained goal. With respect to the “hows”, Jankvist highlights three types of approaches: the set of illumination approaches (the teaching and learning of mathematics is supplemented by historical information), modules approaches (instructional units devoted to history, which quite often are based on cases), and history-based approaches (indirect incorporation of historical elements in the lessons).

Despite many advantages of the practice of the history of mathematics in classrooms, as outlined above, opposing opinions have also been expressed. Tzanakis and Thomaidis (2012) summarise some of these arguments in the following way: (a) history is not mathematics (b) history may create confusion instead of clarification (c) many pupils find history boring and this might lead them to dislike mathematics (d) teachers do not have much time to dedicate to the history of mathematics (e) lack of necessary teaching sources, (f) lack of teacher training on how to incorporate the history of mathematics, and (g) inappropriate use of the history of mathematics in lessons could encourage and/or maintain stereotypical perceptions on cultures and nationalism. It is, therefore, important that teachers are appropriately prepared to handle historical references in their mathematics teaching in order to avoid these obstacles (Charalambous, Panaoura, & Philippou, 2009; Panasuk & Horton, 2012).

What does research on mathematics textbooks tell us?

Textbooks constitute the main teaching resource for most countries around the world (Mullis et al., 2012), especially for highly centralised educational systems, such as those of Cyprus and Greece. Textbooks play a significant role in the ways mathematics lessons are designed and delivered; their influence over the content of the lessons; the instructional approaches; the quality of activities assigned for pupils, in classrooms and for homework; and, the learning outcomes and achievement (Alajmi, 2012; Törnroos, 2005; Weiss et al., 2003; Robitaille & Travers, 1992). Drawn from the well-known trichotomic distinction of the curriculum in the *intended* (vision and intentions as specified in official curriculum documents and/or materials), the *implemented* (teachers' interpretation and enactment of the curriculum according to their perceptions and knowledge) and the *attained* (learning experience as perceived by learners and their resulting learning outcomes), mathematics textbooks could be seen as a mediator between the first two types (or what Schmidt, McKnight, & Raizen, 1997, call the *potentially implemented curriculum*). From this perspective, Mesa (2004, p. 255–256) talks about "a hypothetical enterprise: What *would* students learn if their mathematics classes were to cover all the textbook sections in the order given? What *would* students learn if they had to solve all the exercises in the textbook?" Rezat and Straesser (2014) take this argument a step further and distinguish between three perspectives on textbooks analyses. The first perceives textbooks as *curriculum materials* that offer supplementary ideas for teaching. From another perspective, textbooks can be approached as *artifacts* that are employed for the preservation and transition of acquired skills. Analyses from this perspective focus only on opportunities to teach and to learn. Lastly, textbooks can be seen as *instruments*. Such analyses take into consideration the actual use of textbooks in lessons.

Many studies make use of various frameworks for analysing mathematics textbooks, which include, among other things, their physical characteristics (i.e. size, number of pages, volumes, and so on), the structure of the lessons, the topics covered, the presentation of particular mathematical concepts, and the level of the cognitive demands of tasks (Bayazit, 2013; Alajmi, 2012; Kajander & Lovric, 2009; Stylianides, 2009). Specific interest has been shared by scholars as far as cross-national comparisons of textbooks are concerned, showing how the cultural expectations and goals of different educational systems are manifested in the instructional materials produced (Haggarty & Pepin, 2002). Cross-cultural textbook analyses typically fall under three broad categories, namely *horizontal*, *vertical*, and *contextual*

(Charalambous, Delaney, Hsu, & Mesa, 2010). Horizontal analysis focuses on textbooks' general characteristics (i.e. content, structure, etc.), while the textbook is examined as a whole. Such examples are Stevenson's and Bartsch's (1992) analyses of Japanese and American textbooks, as well as the work of Campbell and Kyriakides (2000), who have investigated Cypriot and English elementary textbooks as part of the national curriculum of the two countries. In vertical analysis, the interest is on how the textbooks under examination treat certain mathematical concepts. From this perspective, Ding and Li (2010) discuss the ways the distributive property is presented in US and Chinese elementary textbooks, while Alajmi (2012) examines how elementary textbooks in the US, Japan, and Kuwait address fractions. Finally, contextual analysis is concerned with the role of the textbooks in classroom activities, either by the teacher or the pupils (see Remillard, 2005; Rezat, 2006). Only few comparative studies employ more than one of the types of analysis above, as, for example, Haggarty's and Pepin's (2002) work with English, French and German textbooks, and their function in classrooms, and Charalambous et al's (2010) framework developed for analysing learning opportunities provided by Cypriot, Irish, and Taiwanese textbooks.

This study

In this paper we analyse references included in the lower secondary textbooks of Cyprus and Greece in regards to the history of mathematics. Both countries have highly centralised educational systems (Charalambous et al., 2010; Saiti & Eliophoto-Menon, 2009) in which schools are considered segments of the domain of government, and not of the community. The national curricula and textbooks are prepared by each country's respective Ministry of Education, more specifically, by the Pedagogical Institute, a department of the Ministry of Education. In both the Cypriot (MoEC 2010) and the Greek intended curriculum (MoERA, 2002), explicit references are made to the importance of the history of mathematics. Nevertheless, the purpose of this paper is to examine how the general curricular references are actually transformed into learning opportunities in the Cypriot and Greek textbooks for both teachers and pupils.

In our study, textbooks are treated as artifacts (Rezat & Straesser, 2014) since we examine their content in relation to the history of mathematics, and not in relation to other teaching material, or the ways by which they are utilized in classrooms. Furthermore, our approach could be regarded as a comparative vertical analysis (Charalambous et al., 2010), which focuses on how the two sets of textbooks, Cypriot and Greek, treat a particular concept. For the purposes of this project, we analysed the national textbooks of Cyprus and Greece for lower secondary education (grades 7, 8, and 9). The educational system of Cyprus is currently reforming its curriculum for all subjects and grades; consequently, only the textbooks for lower secondary education were available to us once our study began; therefore, we chose to focus our analyses on the three lower secondary grades. The Cypriot textbooks were launched between 2012 and 2013 by the Pedagogical Institute of Cyprus, and the Greek ones were introduced in 2007 by the Pedagogical Institute of Greece. At the first stage of data collection, all references to the history of mathematics in both textbook series were identified. We were interested in both encyclopedic pieces of information and mathematical tasks inviting pupils to interact with them and provide solutions or answers. We worked independently with the two data sets (Cypriot and Greek), trying to find ways of clustering the references identified. Due to long distance, in person meetings were not possible; however, after individual progress we had online meetings to discuss our ideas, which were, interestingly, similarly handled.

By employing a data driven analysis (Kvale & Brinkmann, 2009) as well as the constant comparison process outlined by Strauss and Corbin (1998), the references identified were initially clustered in two broad categories. Further analyses resulted in the division of the two categories into two subcategories each. The first category was concerned with references that did not pose questions to pupils; they'd rather provide historical pieces of information. In this category, we could find (a) simple biographical references about mathematicians or historical references concerning the origins of a mathematical concept and (b) references to the history of a mathematical method or formula including a solution path or proof. Figure 1 shows an example of the first subcategory and provides biographical information about the life and work of the German mathematician Georg Cantor (Cypriot textbooks, grade 7, part A, page 27). Figure 2 shows an example of a historical reference to a method and its solution process. In particular, it explains how Ancient Egyptians used a piece of rope with 13 knots and 12 equal line segments to create right angles, a method called in the book, "the reverse of the Pythagorean theorem" (Greek textbooks, grade 8, page 128).

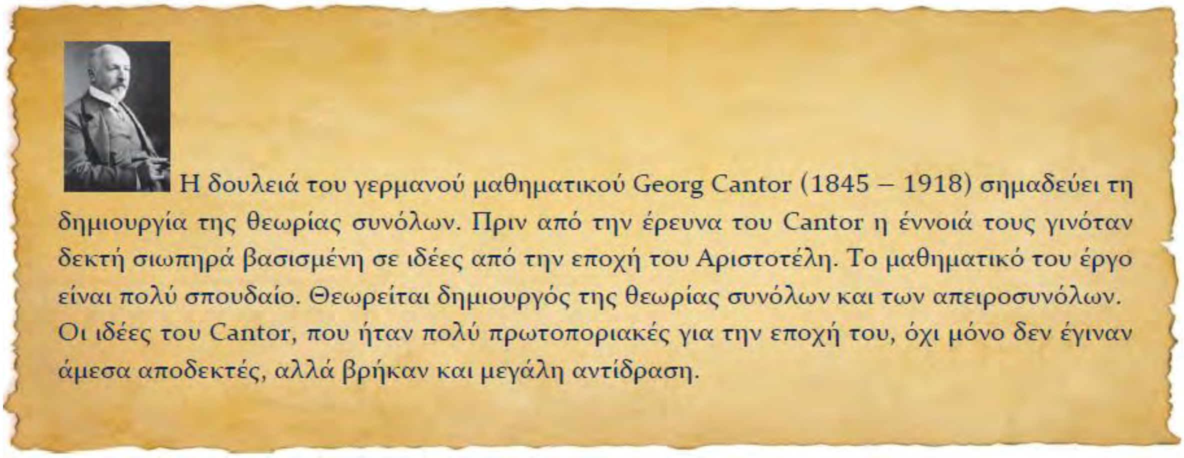
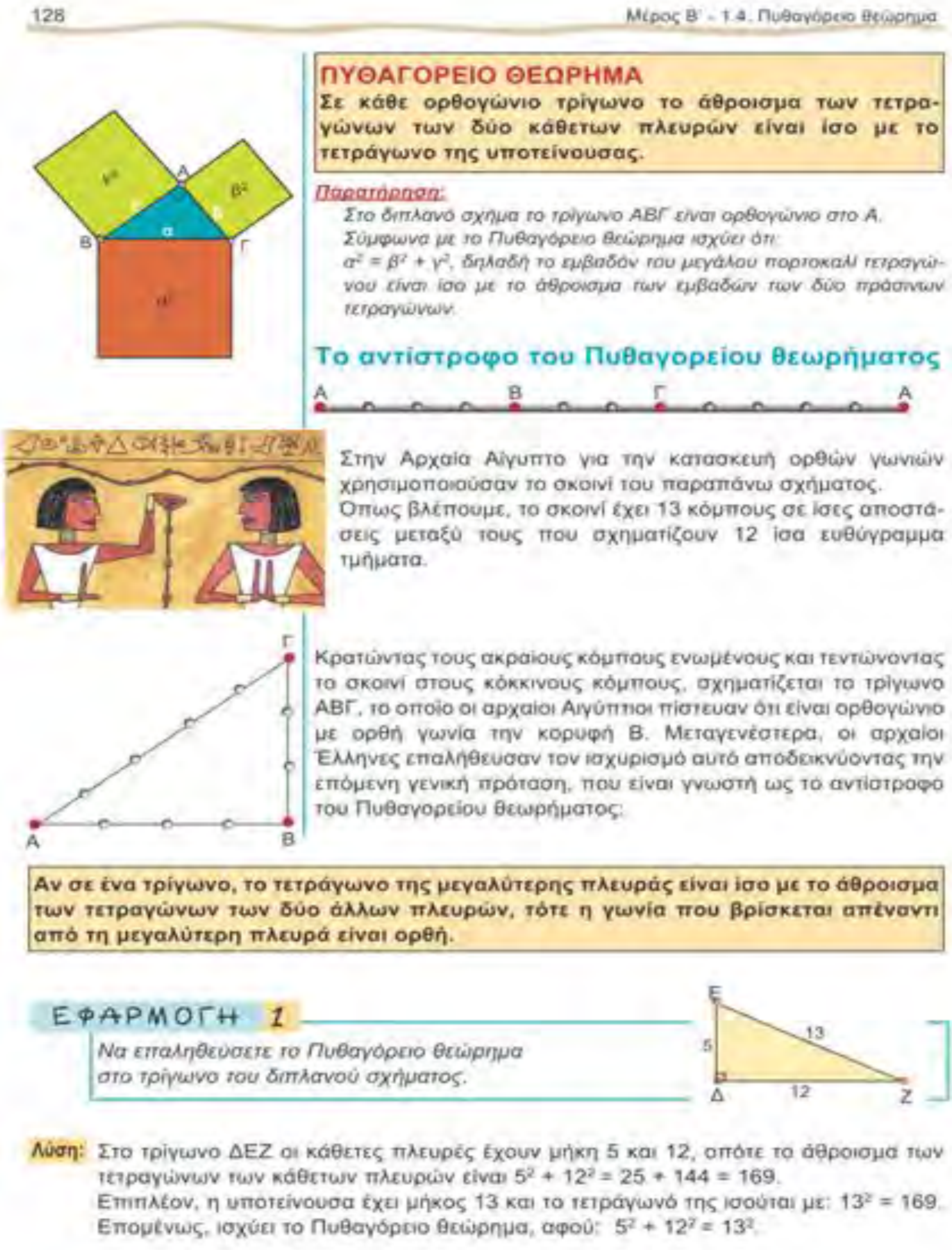

Η δουλειά του γερμανού μαθηματικού Georg Cantor (1845 – 1918) σημαδεύει τη δημιουργία της θεωρίας συνόλων. Πριν από την έρευνα του Cantor η έννοιά τους γινόταν δεκτή σιωπηρά βασισμένη σε ιδέες από την εποχή του Αριστοτέλη. Το μαθηματικό του έργο είναι πολύ σπουδαίο. Θεωρείται δημιουργός της θεωρίας συνόλων και των απειροσυνόλων. Οι ιδέες του Cantor, που ήταν πολύ πρωτοποριακές για την εποχή του, όχι μόνο δεν έγιναν άμεσα αποδεκτές, αλλά βρήκαν και μεγάλη αντίδραση.

Figure 1: Biographical information about the life and work of the German mathematician, Georg Cantor (Cypriot textbooks, grade 7, part A, page 27).

The second category comprised tasks that invited pupils to interact with them and provide some sort of answer. This included (a) mathematical tasks of purely cognitive elements that require a numerical solution, explanation or proof and (b) tasks that encourage discussion or the production of a project that would connect the history of mathematics with life outside mathematics. Examples of these subcategories are illustrated in Figures 3 and 4 respectively. Figure 3 shows the method applied by the ancient Greek mathematician, Thales, for the calculation of the distance between a ship and the coastline. Furthermore, pupils are asked to prove Thales' method and explain how he could be sure that the distance was right (Greek textbooks, grade 9, page 197).

Figure 4 provides an example of a task that requires pupils to carry out a small project comparing and contrasting the number systems of Mayans and Babylonians-Sumerians, as well as discussing the difficulties of these two systems, which eventually led to a need to establish the positional decimal system (Cypriot textbooks, grade 6, part A, page 85). It is worth stating that there are no previous references to the two ancient number systems in the textbooks. Pupils are asked to conduct their own research and find information about them, since the textbook implies that there is no further instruction about these systems.



128 Μέρος Β' - 1.4. Πυθαγόρειο θεώρημα

ΠΥΘΑΓΟΡΕΙΟ ΘΕΩΡΗΜΑ

Σε κάθε ορθογώνιο τρίγωνο το άθροισμα των τετραγώνων των δύο κάθετων πλευρών είναι ίσο με το τετράγωνο της υποτείνουσας.

Παρατήρηση:

Στο διπλανό σχήμα το τρίγωνο ΑΒΓ είναι ορθογώνιο στο Α. Σύμφωνα με το Πυθαγόρειο θεώρημα ισχύει ότι:

$\alpha^2 = \beta^2 + \gamma^2$, δηλαδή το εμβαδόν του μεγάλου πορτοκαλί τετραγώνου είναι ίσο με το άθροισμα των εμβαδών των δύο πράσινων τετραγώνων.

Το αντίστροφο του Πυθαγορείου θεωρήματος

Στην Αρχαία Αίγυπτο για την κατασκευή ορθών γωνιών χρησιμοποιούσαν το σκοινί του παραπάνω σχήματος. Όπως βλέπουμε, το σκοινί έχει 13 κόμπους σε ίσες αποστάσεις μεταξύ τους που σχηματίζουν 12 ίσα ευθύγραμμα τμήματα.

Κρατώντας τους ακραίους κόμπους ενωμένους και τεντώνοντας το σκοινί στους κόκκινους κόμπους, σχηματίζεται το τρίγωνο ΑΒΓ, το οποίο οι αρχαίοι Αιγύπτιοι πίστευαν ότι είναι ορθογώνιο με ορθή γωνία την κορυφή Β. Μεταγενέστερα, οι αρχαίοι Έλληνες επαλήθευσαν τον ισχυρισμό αυτό αποδεικνύοντας την επόμενη γενική πρόταση, που είναι γνωστή ως το αντίστροφο του Πυθαγορείου θεωρήματος:

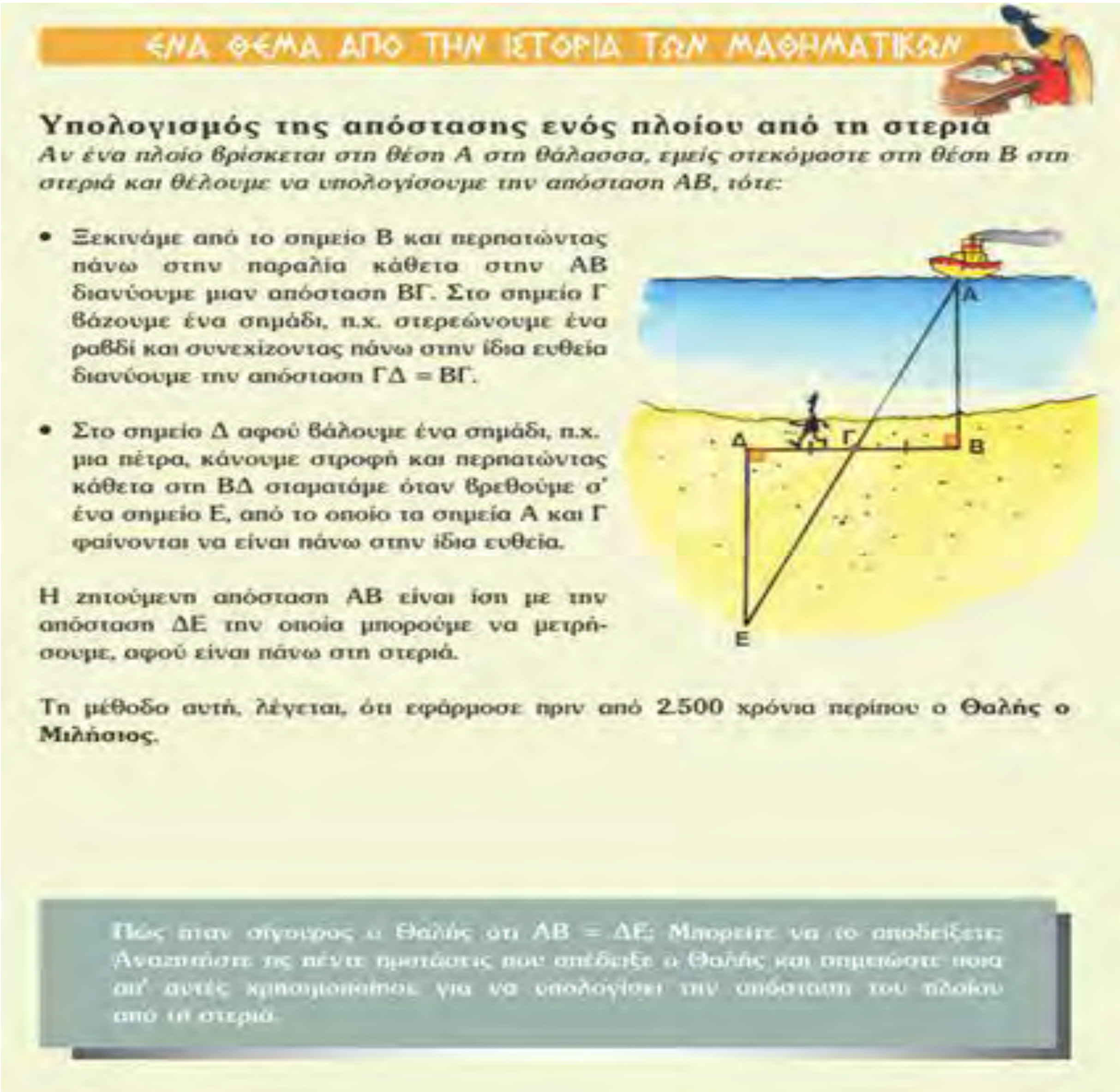
Αν σε ένα τρίγωνο, το τετράγωνο της μεγαλύτερης πλευράς είναι ίσο με το άθροισμα των τετραγώνων των δύο άλλων πλευρών, τότε η γωνία που βρίσκεται απέναντι από τη μεγαλύτερη πλευρά είναι ορθή.

ΕΦΑΡΜΟΓΗ 1

Να επαληθεύσετε το Πυθαγόρειο θεώρημα στο τρίγωνο του διπλανού σχήματος.

Λύση: Στο τρίγωνο ΔΕΖ οι κάθετες πλευρές έχουν μήκη 5 και 12, οπότε το άθροισμα των τετραγώνων των κάθετων πλευρών είναι $5^2 + 12^2 = 25 + 144 = 169$.
Επιπλέον, η υποτείνουσα έχει μήκος 13 και το τετράγωνό της ισούται με: $13^2 = 169$.
Επομένως, ισχύει το Πυθαγόρειο θεώρημα, αφού: $5^2 + 12^2 = 13^2$.

Figure 2: "The reverse of the Pythagorean theorem" from Ancient Egypt (Greek textbooks, grade 8, page 128).



ΕΝΑ ΘΕΜΑ ΑΠΟ ΤΗΝ ΙΣΤΟΡΙΑ ΤΩΝ ΜΑΘΗΜΑΤΙΚΩΝ

Υπολογισμός της απόστασης ενός πλοίου από τη στεριά

Αν ένα πλοίο βρίσκεται στη θέση Α στη θάλασσα, εμείς στεκόμαστε στη θέση Β στη στεριά και θέλουμε να υπολογίσουμε την απόσταση ΑΒ, τότε:

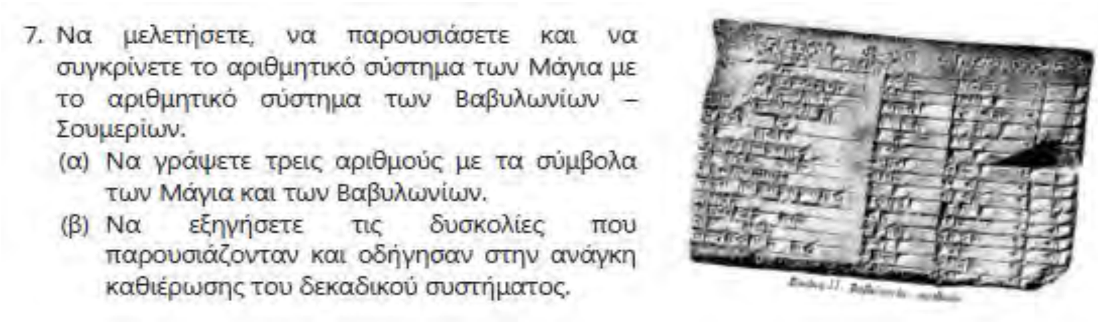
- **Ξεκινάμε από το σημείο Β και περπατώντας πάνω στην παραλία κάθετα στην ΑΒ διανύουμε μιαν απόσταση ΒΓ. Στο σημείο Γ βάζουμε ένα σημάδι, π.χ. στερεώνουμε ένα ραβδί και συνεχίζοντας πάνω στην ίδια ευθεία διανύουμε την απόσταση ΓΔ = ΒΓ.**
- **Στο σημείο Δ αφού βάλουμε ένα σημάδι, π.χ. μια πέτρα, κάνουμε στροφή και περπατώντας κάθετα στη ΒΔ σταματάμε όταν βρεθούμε σ' ένα σημείο Ε, από το οποίο τα σημεία Α και Γ φαίνονται να είναι πάνω στην ίδια ευθεία.**

Η ζητούμενη απόσταση ΑΒ είναι ίση με την απόσταση ΔΕ την οποία μπορούμε να μετρήσουμε, αφού είναι πάνω στη στεριά.

Τη μέθοδο αυτή, λέγεται, ότι εφάρμοσε πριν από 2.500 χρόνια περίπου ο Θαλής ο Μιλήσιος.

Πώς ήταν σίγουρος ο Θαλής ότι ΑΒ = ΔΕ; Μπορείτε να το αποδείξετε; Αναζητήστε τις πέντε προτάσεις που απέδειξε ο Θαλής και σημειώστε ποια απ' αυτές χρησιμοποίησε για να υπολογίσει την απόσταση του πλοίου από τη στεριά.

Figure 3: A mathematical task illustrating Thales' method of calculation for the distance between a ship and the coastline (Greek textbooks, grade 9, page 197).



7. Να μελετήσετε, να παρουσιάσετε και να συγκρίνετε το αριθμητικό σύστημα των Μάγια με το αριθμητικό σύστημα των Βαβυλωνίων – Σουμερίων.
   (α) Να γράψετε τρεις αριθμούς με τα σύμβολα των Μάγια και των Βαβυλωνίων.
   (β) Να εξηγήσετε τις δυσκολίες που παρουσιάζονταν και οδήγησαν στην ανάγκη καθιέρωσης του δεκαδικού συστήματος.

Figure 4: Small project task about comparing and contrasting the number systems of Mayans and Babylonians-Sumerians (Cypriot textbooks, grade 6, part A, page 85).

Table 1 shows the distribution of the references per country. No significant differences can be observed in regard to the two broad categories and the references of the history of mathematics from each country, χ^2 (1, N = 137) = 0.986, p = 0.32.

	No question(s) for pupils		**Asks pupils to interact with it**	
	Simple historical/biographical references	*Show solution/proof of a method/formula*	*Mathematical tasks*	*Encourage discussion/project*
Cyprus	27	2	28	2
Greece	41	4	26	7

Table 1: The distribution of the references per country.

Analysing the mathematical tasks

At the second stage of our analyses we turned our focus on the mathematical tasks of the two textbook series. For the purposes of our analyses, we employed the framework developed by the QUASAR project team (Stein & Lane, 1996; Smith & Stein, 1998; Stein et al., 2000). After analysing hundreds of tasks that were used in project classrooms, the QUASAR researchers created the Levels of Cognitive Demands (LCDs), which constitute sets of criteria that ‘‘when applied to a mathematical task (in print form) ... can serve as a judgment template (a kind of scoring rubric) that permits a ‘rating’ of the task based on the kind of thinking it demands of students’’ (Stein et al., 2000, p. 15). Based on the LCD criteria, tasks can be clustered in two main categories, lower-level demands and higher-level demands. Moreover, each includes two sub-categories. In lower-level demands, there are *memorisation tasks* and *procedures-without-connection tasks.* Memorisation tasks require simply recalling or reproducing rules, definitions and factual knowledge previously learned. Procedures-without-connection tasks are algorithmic and are presented in a simple and familiar form. They pose very limited cognitive demand, but even this much is required for the successful completion of procedures being practiced. The higher-level demands include *procedures-with-connections tasks* and *doing-mathematics tasks*. Procedures-with-connections tasks could engage pupils with the procedures; however, their goal is to help pupils develop a deeper level of understanding. They are usually represented in multiple ways (e.g. tables, graphs and symbols), and pupils are required to establish connections between the representations to develop meaningful learning. Doing-mathematics tasks involve non-algorithmic thinking and considerable cognitive efforts. Resolution of such problems requires doing research, making discoveries and using metacognitive skills that include self-monitoring and self-regulation. For more details about the analytical criteria of each category, readers can find more information in Smith and Stein (1998).

Initially, we worked independently trying to characterize each task with the use of the four LCDs. Later, we tried to explain to each other the reasons why each task was assigned to a particular level. For most cases, agreement was observed. Yet, as Stein et al. (2010) report, identifying the LCD of a task was not always a straightforward process. Nevertheless, by closely examining those tasks and presenting our arguments together, we managed to reach an agreement for all of them. Below we show some examples of tasks for each LCD.

Figure 5 shows an example from Cypriot textbooks (grade 7, part A, page 57). The illustration shows an ancient Egyptian writing numbers in the hieroglyphic number system. Two questions are posed for pupils, which were seen as two separate tasks. The first asks pupils to find the number, which is presented in the image. This task was seen as one of memorization simply because pupils reproduce facts learnt from a table in a previous page showing the correspondence between the hieroglyphic symbols and their value in the decimal system. The second question asks pupils to calculate how many hieroglyphic symbols are required for them to write the year of their birth. This task was seen as a procedure without connection to meaning because it is phrased in such a way that is does not encourage a deeper conceptual understanding between the two number systems.

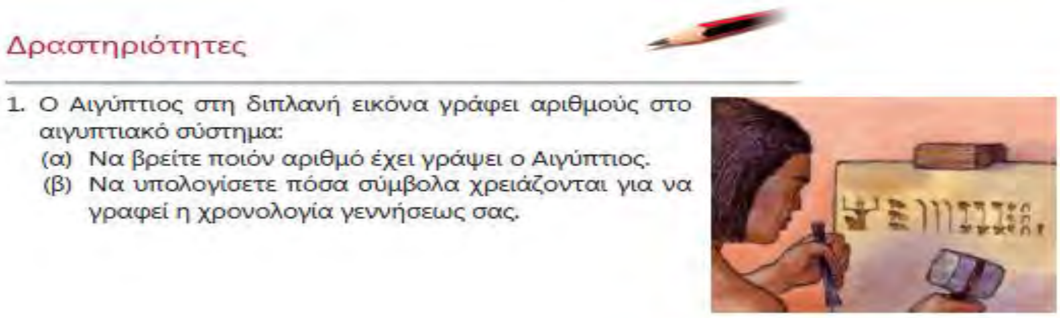
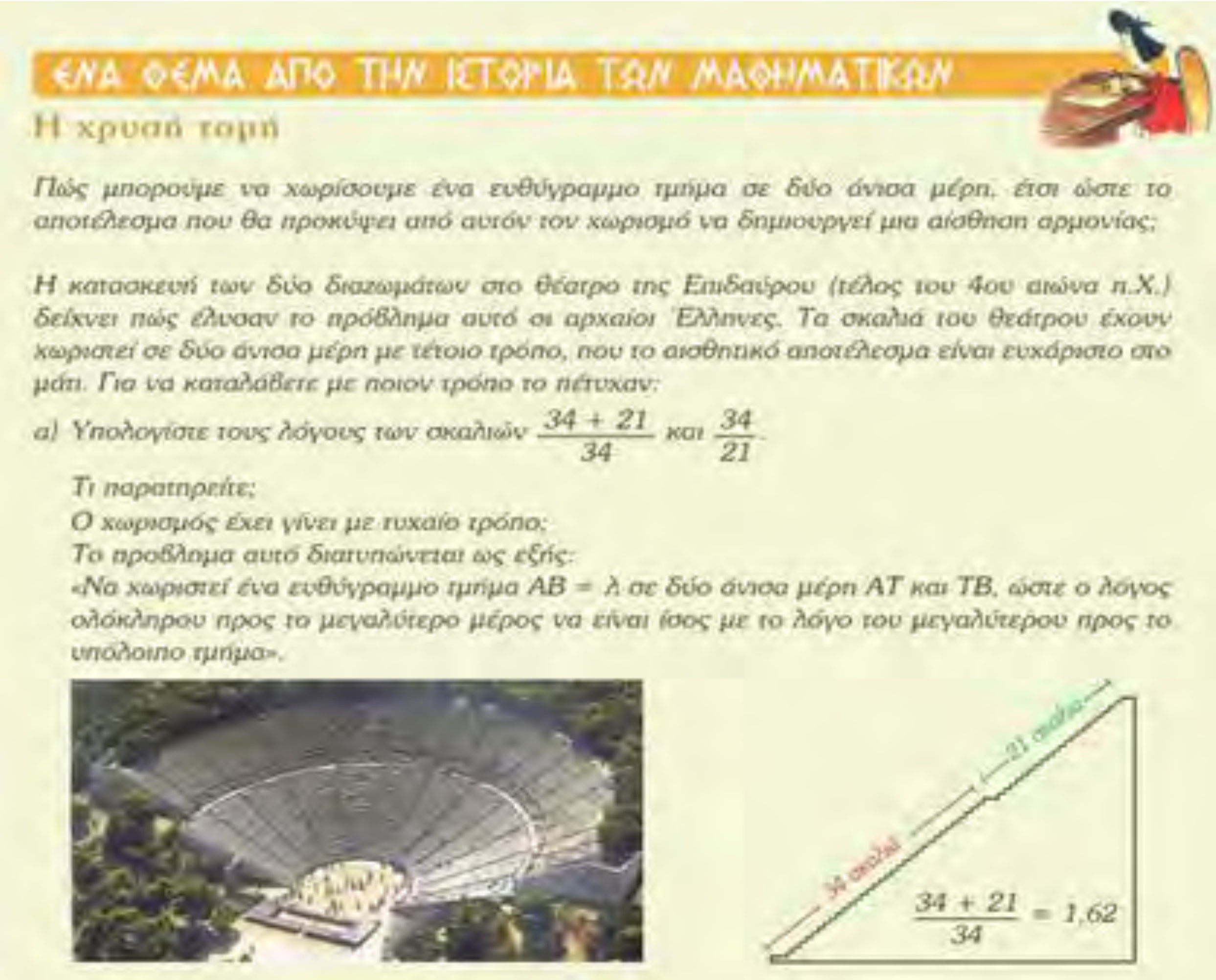


Figure 5: An example of a memorization task (sub-question 1a) and a procedures-without-connection-to-meaning task (sub-question 1b), from the Cypriot textbooks (grade 7, part A, page 57).

Below figure 6 is an example of a task that was labelled as a procedure promoting connection to meaning. It provides an example of the golden ratio in the construction of the ancient theatre of Epidaurus. Pupils are asked to calculate two ratios concerning the stairs of the theatre (both of which result in 1.61) and make statements on whether this is random (Greek textbooks, grade 8, page 109). Finally, figure 7 demonstrates an example of a task that was perceived as doing mathematics, due to its complexity and the fact that no further information is provided in the textbooks. The task presents the well-known Zeno's paradox about Achilles and the Tortoise. Pupils are asked to investigate, confirm or refute Zeno's argument that, despite Achilles moving ten times faster than the tortoise, he won't be able to reach it considering the tortoise is one stadion (approx. 192m) ahead (Greek textbooks, grade 7, page 136).


ΕΝΑ ΘΕΜΑ ΑΠΟ ΤΗΝ ΙΣΤΟΡΙΑ ΤΩΝ ΜΑΘΗΜΑΤΙΚΩΝ

Η χρυσή τομή

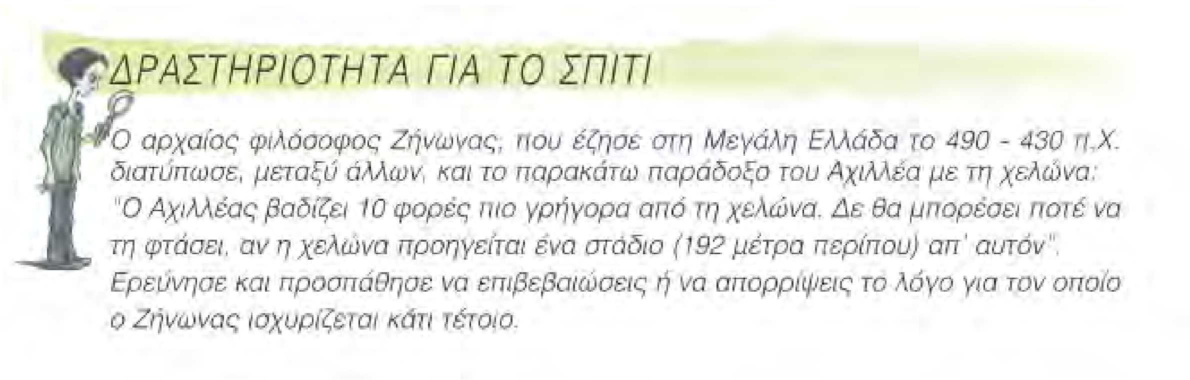
Πώς μπορούμε να χωρίσουμε ένα ευθύγραμμο τμήμα σε δύο άνισα μέρη, έτσι ώστε το αποτέλεσμα που θα προκύψει από αυτόν τον χωρισμό να δημιουργεί μια αίσθηση αρμονίας;

Η κατασκευή των δύο διαζωμάτων στο θέατρο της Επιδαύρου (τέλος του 4ου αιώνα π.Χ.) δείχνει πώς έλυσαν το πρόβλημα αυτό οι αρχαίοι Έλληνες. Τα σκαλιά του θεάτρου έχουν χωριστεί σε δύο άνισα μέρη με τέτοιο τρόπο, που το αισθητικό αποτέλεσμα είναι ευχάριστο στο μάτι. Για να καταλάβετε με ποιον τρόπο το πέτυχαν:

α) Υπολογίστε τους λόγους των σκαλιών $\frac{34+21}{34}$ *και* $\frac{34}{21}$.

Τι παρατηρείτε;
Ο χωρισμός έχει γίνει με τυχαίο τρόπο;
Το πρόβλημα αυτό διατυπώνεται ως εξής:
«Να χωριστεί ένα ευθύγραμμο τμήμα ΑΒ = λ σε δύο άνισα μέρη ΑΓ και ΓΒ, ώστε ο λόγος ολόκληρου προς το μεγαλύτερο μέρος να είναι ίσος με το λόγο του μεγαλύτερου προς το υπόλοιπο τμήμα».

Figure 6: An example of a procedures-with-connections-to-meaning. “The Golden Ratio” (Greek textbooks, grade 8, page 109).



ΔΡΑΣΤΗΡΙΟΤΗΤΑ ΓΙΑ ΤΟ ΣΠΙΤΙ

Ο αρχαίος φιλόσοφος Ζήνωνας, που έζησε στη Μεγάλη Ελλάδα το 490 - 430 π.Χ. διατύπωσε, μεταξύ άλλων, και το παρακάτω παράδοξο του Αχιλλέα με τη χελώνα: "Ο Αχιλλέας βαδίζει 10 φορές πιο γρήγορα από τη χελώνα. Δε θα μπορέσει ποτέ να τη φτάσει, αν η χελώνα προηγείται ένα στάδιο (192 μέτρα περίπου) απ' αυτόν". Ερεύνησε και προσπάθησε να επιβεβαιώσεις ή να απορρίψεις το λόγο για τον οποίο ο Ζήνωνας ισχυρίζεται κάτι τέτοιο.

Figure 7: An example of a doing-mathematics task. Zeno's paradox, "Achilles and the Tortoise" (Greek textbooks, grade 7, page 136).

Table 2 shows the mathematical tasks for each LCD in the textbooks series of the two countries. No significant differences were observed as regards the lower and higher LCD of the two sets, χ^2 (1, N = 54) = 1.187, p = 0.28.

	Level of Cognitive Demands			
	Lower-level demands		Higher-level demands	
	Memorization	*Procedures without connection to meaning*	*Procedures with connection to meaning*	*Doing mathematics*
Cyprus	1	11	8	8
Greece	3	12	7	4

Table 2: The LCD of the tasks in the two countries.

Discussion and conclusions

Clearly, both countries value the history of mathematics, a fact not only expressed in their curricula, but also reflected in their national textbooks. This could be attributed to a common cultural-historic heritage and the profound role of ancient Greek mathematics in the development of contemporary mathematical thinking. Nevertheless, the two countries' educational traditions share much in common, historically speaking (Koutselini-Ioannidou, 1997). In the case of Cypriot mathematics textbooks – launched a few years after the Greek ones – they include several slightly modified examples. For instance, in the Greek textbook of grade 8 (page 10) we find a mathematical riddle from the tombstone of the ancient Greek mathematician Diophantus, inviting passers to solve it and calculate the age of his death. In the textbook, an equation showing the solution to the riddle follows. In the Cypriot textbooks (grade 7, part A, page 173) although the same riddle is included, no solution is presented and pupils are asked to calculate Diophantus' age. In our analyses, we considered the task from the latter series as one of high cognitive demands (doing mathematics) due to the complexity of the riddle.

About half of the references in the Cypriot series and more than half in the Greek one constitute biographical/historical information that does not ask pupils to interact with it in any way. From our experiences as former pupils in the two educational systems, such information typically remains unexploited in classrooms, along the lines of those arguing that the mathematics curriculum is overloaded with too many topics to be covered that teachers do not have much time to dedicate to the history of mathematics (Tzanakis & Thomaidis, 2012). Also, despite comments of authors like Lawrence (2008) and Jahnke et al. (2002), who see advantages of the history of mathematics in lessons as an opportunity for pupils to realise how mathematical knowledge of the past has influenced our modern everyday life, both series include very few project-based tasks. Once again, from our experiences, Cypriot and Greek secondary mathematics teachers perceive such tasks as not "mathematical" and often ignore them, as a result. It is worth mentioning that both series are not restricted to ancient Greek mathematics; they include references from many civilizations, and this is an opportunity considered important for the elimination of racial discrimination (Michalowicz et al., 2002; Strutchens, 1995).

As far as the mathematical tasks of cognitive elements are concerned, a balance between lower and higher cognitive demands is observed for both textbook series. Of course, this doesn't necessarily say much about how each textbook task is enacted in classrooms (Mesa, 2004; Schmidt et al., 1997). In their recent work, Stein and Kaufman (2010) comment on the various phases that an instructional task goes through, from its initial appearance on the pages of a written curriculum to its actual enactment in classrooms by pupils and teachers. Simply put, even tasks that appear to be of low cognitive demands may turn out to be employed in ways that promote higher order mathematical thinking. Also, tasks that are seen as requiring high levels of cognitive demands may be exploited in teacher-centered ways that encourage memorization and no deeper connection to mathematical meaning. Future research could investigate teachers' beliefs and attitudes about the historical references provided by the textbooks, as well as on how these are exploited during their planning and teaching.

To return to Jankvist's (2009a) arguments about the "whys", it can be argued that the distribution of the references to the four categories of our framework (simple historical/biographical references; solution/proof of a method/formula; mathematical tasks; discussion/project tasks) implies an intentional balance between the utilization of the history of mathematics as a tool and as a goal, on behalf of the textbook writers of both series. In this sense, mathematical tasks can be perceived as promoting history-as-a-tool, while the other three categories contribute to the promotion of history-as-a-goal. However, as already argued above, our anecdotal experiences indicate that Greek and Greek-Cypriot mathematics teachers often ignore references other than purely mathematical tasks and do not actually incorporate them in their teaching. We believe that, despite the two series' potentials for focusing "on the developmental and evolutionary aspects of mathematics as a discipline" (Jankvist, 2009a, p. 239), the way teachers use the textbooks in classrooms does not allow this to happen. This echoes the argument by Charalambous et al. (2009) and Panasuk and Horton (2012) about the necessity of appropriate teacher preparation on how to handle historical references in their mathematics teaching.

In closing, we cannot but emphasize the significance of collaboration between researchers from different countries in respect to comparative enquiries. Scholars must be aware that things with the same name might have a different meaning and function across nations, and they ought, therefore, to avoid assuming that colleagues elsewhere share the same understanding. Furthermore, we would like to invite colleagues from other countries to draw on our emerged framework of analysis about the types of historical references in mathematics textbooks, and to examine the ways in which the history of mathematics is included in other curricula, textbook series, and classrooms around the world.

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