Creating Cube Nets by using Educational Tools in Pre-school

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Abstract. Our research programme, a part of which is presented in this paper, looked at four year old children’s ability to use manipulatives in the construction of cube models. We looked at how pre-school children managed the creation of cube nets as a mathematical problem and whether graphical representations of solutions could become a useful tool in a child’s quest for solving the nets of a cube mathematical problem. We followed the work of twenty four-year old children who were asked to solve a cube creation geometry mathematical problem. In order for the problem to be successfully solved, graphical representations of solutions had to be developed. The data analysis showed that, during the course of the programme, children developed the ability to use manipulatives in order to solve the mathematical problem. The development of strategies for graphically representing the solutions also seemed to have played an important role in solving the specific mathematical problem.

Keywords: early childhood mathematics, cube nets, geometry tasks, 3-D shapes, manipulatives.

Introduction An important aspect of mathematics education is children’s occupation with mathematical problems concerning geometrical solids. Clements and Sarama (2009) recommended that as early as pre-school, children should become familiar with 3-D shapes and their properties, manipulate developmentally appropriate materials for the creation of solid shapes and pair solids with corresponding solid nets. Van de Walle
(2007) reported that the integration of the above aspects of stereometry within a teaching context did not have the typical characteristics of any school routine with the use of repeated practices or the memorization of rules and methods since their solution process was not known in advance. According to Robertson (2001), the children’s interest to investigate an unknown area to them, acted as an internal motive which activated them to acknowledge the agreements accompanying each problem, and triggered their invention of solution strategies. The study presented in this paper was a part of a broader teaching intervention programme which aimed at four-year old children becoming familiar with the mathematical problem solving process. In this paper we look at the children’s (i) strategies while creating a cube through its nets and (ii) ability to detect nets which can create a cube.

Theoretical framework

3-D shape construction

According to Gutiérrez (1996) geometry could be considered as the origin of spatial thinking in mathematics. Many researchers defined spatial ability as the mental process used to perceive, store, recall, create, predict, edit, communicate spatial images, or judge the relationships among figures or objects in different spatial contexts (Linn and Petersen 1985; Elia et al. 2014). Nowadays, pre-school children’s occupation with solid shapes is considered an important aspect of their mathematical education and young children show special interest in working on spatial mathematical problems (Davis and Hyun 2005). Constructions using 3-D material give children the opportunity to discriminate between 2-D and 3-D shapes, to become familiar with the manipulation of material which can help the synthesis of geometrical solids while, at the same time, help in the discovery of solid shapes’ properties (Clements and Sarama 2009; Elia et al. 2014).

A study by Markopoulos and Potari (2005) concerning children’s manipulations of physical materials in creating 3-D geometrical solids, indicated a development from a holistic to a relational consideration of geometrical solids but not necessarily in a linear order. In addition, the context of dynamic transformations promoted the development of children’s thinking. Thus, within this framework, during the process of spatial thinking, interaction between spatial images and external representations was considered essential. Spatial ability and mathematics achievement have been related (Wheatley, 1990). Moreover, spatial sense has been related to mathematical skills and competencies (Wheatley et al. 1994). Most researchers defined spatial ability through describing its consisting factors. Lohman (1988) identified three factors:

(a) Spatial visualization: the ability to comprehend imaginary movements in a 3-D space or the ability to mentally manipulate objects. Although the elements composing spatial ability differed among researchers, there seemed to be an agreement in the fact that spatial visualization functioned as an important factor in spatial thinking (Markopoulos and Potari 2005; Ryu et al. 2007). For researchers, spatial visualization consisted of the ability to mentally manipulate images. Kosslyn (1983) defined four sets of image processes: (i) activating information stored in long-term memory and constructing a representation in short-term memory, (ii) inspecting the object in the image by reinterpreting it, (iii) maintaining the image over time, and (iv) transforming the image by rotating it, adding or deleting parts, or changing the color.

(b) Spatial orientation: remaining unconfused by the changes in the orientation of visual stimuli that requires only a mental rotation of configuration. For Clements, spatial orientation (Clements and Sarama 2009) included the ability to act upon and select relative information and interpret enigmatic information,
Spatial relations: the speed of manipulating simple visual patterns such as mental relations and the ability to mentally rotate a spatial object effectively. Clements and Sarama (2009) noted that young children can learn rich concepts about 3-D shapes when (i) provided with a variety of examples and nonexamples and (ii) encouraged to talk about their observations and experiences. They pointed out that observing and talking about 3-D shapes should coexist with corresponding experiences of 3-D manipulatives in order to ensure the formulation of mathematical mental images of the shapes.

Collaborative activity, during which children addressed each other, communicated their ideas and, tried to understand a different perspective of a situation, were found to be educationally useful process for the familiarization of complex geometrical concepts, such as paring 3-D shapes with their corresponding nets (Davis and Hyun 2005). Moreover, discussions should encourage children’s description while encouraging the development of mathematical language (Davis and Hyun 2005).

3-D shape construction as a mathematical problem solving process

The construction of 3-D shapes with the use of educational manipulatives, as well as the reverse process of finding all possible nets of 3-D shapes, have been considered mathematical problem solving processes. Every mathematical problem, in relation to its contents, should combine the three following characteristics. Firstly, it should be close to the children’s existing knowledge, it should be appealing and within the sphere of their interests while at the same time it should develop their mathematical thinking (Van de Walle 2007). Secondly, the emergence of the problem’s content should be the central focus of the solving process. It has been found that the use of manipulatives could, within a realistic context, support memory processes and facilitate data recollection giving young children the possibility to validate the truth of their assumptions. But manipulatives alone did not contribute to learning. Therefore it was the use of manipulatives within a certain context of activities that empowered the possibilities to facilitate learning (Uttal et al. 1997; Martin et al. 2007; Mix 2010). Thirdly, suggested problems should accept multiple answers and, at the same time, give the children the opportunity to check these answers. It has been noted that children tended to stop the mathematical problem solving process when they found one solution (Shiakalli and Zacharos 2014). Thus, the encouragement of continuing to detect more possible solutions to a problem that had already been solved was suggested (Shiakalli and Zacharos 2012; Kapa 2002; Fortuno et al 1991). The search for new solutions to a mathematical problem should lead to the development of children’s autonomy, defense of their choices, discussion concerning strategies developed and used, classification of solutions, and strategy assessment. Shiakalli (2014) studied the possibility of the mathematical problem solving process becoming the object of teaching in early childhood education and whether children’s engagement in problem solving processes could lead to the development of skills and conceptual understanding. The sample consisted of twenty-five children aged between four-and-a-half and five-and-a-half years old attending public preschool in Cyprus. The children were asked to construct different sized squares using specific manipulatives. Findings showed that children responded positively to the problem and were successful in solving it. During the problem solving process children demonstrated development of skills and conceptual understanding. Children’s consistent involvement with the mathematical problem solving process, as
well as teacher-children and children-children interactions played an important role in the positive outcome of the activity.

The contribution of graphical representations

Research studies (Shiakalli and Zacharos 2014; Davis and Hyun 2005) have shown that teachers can help children manipulate complex mathematical meanings and ideas by creating rich educational environments. Such environments included appropriate educational materials which triggered children’s interest for exploration as well as appropriate tools which mediated and facilitated learning. Graphical representations of ideas, data and information were considered to contribute to the construction of mathematical knowledge. Moreover, researchers (Clements and Sarama 2009; Shiakalli and Zacharos 2012) noted that mathematics education as early as pre-school should encourage children to record data with notes, symbols and illustrations. Children should often be invited to record data dealing with quantity or space and solve problems using graphical representations. Children should also be expected to interpret symbols and have the ability to use them during mathematical practices. This means that children should have the ability to reconstruct, interpret and transform related data in forms found within the science of mathematics. Initial pictorial and later symbolic representations have been found to lead children to incorporate the concepts involved in mathematical activities (Tall et al. 2012). This process related to forms of mathematising, which Freudenthal (1983), described as the ability to organize and imprint experiences into typical mathematical language.

Representations in mathematics education have been distinguished between internal representations and external representations (Goldin and Kaput 1996; Goldin 2002). Internal representations have been defined as a set of intellectual pictures, ideas and expressions a person uses to enable them to connect information with situations and to assess and distinguish between principal and secondary information (Goldin and Kaput 1996). External representations have been described to include symbols, diagrams, and figures; amongst others. They have been noted to (a) enable us to observe the child’s understanding of a specific notion and, (b) stimulate the transformation of knowledge (Shiakalli and Zacharos 2012). Netz (1999) noted that the function of graphical representations in education, as well as the function of geometrical shapes during the process of proof, showed that their role was not limited only to pedagogical goals, but they involved an essential component in the formation of thought. Moreover, as researchers (Piaget 1970; Davis and Hersh 1981; Netz 1999) pointed out, the formulation of abstract mathematics should not be considered as something fixed or given; it should be constructed during the course of the child’s cognitive development. The framework of this study is limited to external representations. More specifically, we present the pedagogical representations used by the teacher and children, as group members and carriers of knowledge and thinking tools, in order to explain a concept, a relation, a connection or the process of solving a mathematical problem.

Sinclair et al. (1983) found that in the early stages of education, children rarely resorted to the use of written symbols and pictures in order to record their thoughts. Children did not understand why they should use this singular set of mathematical symbols. Pre-school and early elementary school children, although familiar with arithmetic symbols, usually avoided using them when recording quantities. Thus, in order to encourage children to use symbolic representations during the mathematical
problem solving process, the teacher should aim at helping children make relations between personal and formal representations (Cai and Lester 2005), within a framework giving sense to educational practices (Van Oers 1994; 2001; Zacharos et al. 2011). Shiakalli and Zacharos (2012) studied whether eighteen Cyprus pre-school children’s graphical representations of solutions to the pentomino problem (see below in Figure 1) could contribute to their success in solving the specific mathematical problem. Findings showed that graphical representation of the solutions, as well as the forms of teacher-children and children-children interactions, played an important role in the positive outcome of the activity. They noted that “graphical representation activities in pre-school education need to be embedded within a context which enables us to expect children’s positive response. Such a context is functional for children’s learning when it encourages children’s interactions and also when the problem to be solved is within the children’s ‘zone of proximal development’ (Shiakalli and Zacharos, 2012, p. 329).

As presented above, research literature provides definitions of spatial ability and outlines the ways in which young children should engage with concepts of stereometry. It also describes the ways in which graphical representations could become a useful tool within the mathematical problem solving process. However, there is no research evidence concerning the skills and strategies young children use in order to solve such mathematical problems. Research (as discussed in both previous sections) has shown the positive outcomes of young children engaging with (a) 3-D geometry mathematical problems and (b) activities involving the creation and use of graphical representations of solutions as part of the mathematical problem solving process. But there is no research evidence of the possibilities of tying the two together; what outcomes would young children’s engagement with a stereometry problem engaging the creation of graphical representations have?

**Research questions**

Our aim was to describe the involvement of four year old children with the mathematical problem solving process involving graphical representations of the solutions. The questions we set out to answer were:

- How do pre-school children manage the creation of cube nets as solutions to a mathematical problem? Which skills and strategies are used?
- Can graphical representations of solutions become a useful tool in children’s quest for solving the nets of a cube mathematical problem? If yes, how?

**Method**

Research design, presented in this paper, was based on the principles of naturalistic research while using elements from design-based research. Naturalistic research (Cohen et al 2007; Lave and Kvale 1995; Bogdan and Biklen 1992; Lincoln and Guba 1985) is research being carried out in the subject’s natural environment. It is based on the principle that the subjects’ natural environment (e.g classroom) is the main source for data collection. Therefore, it acknowledges that data are influenced by social factors and need to be culturally and socially defined. Environment limits and detailed descriptions are essential in order to ensure external validation and possible research replication. Participant evaluation is also essential in order to ensure internal validation. During naturalistic research the researcher is part of the environment under study- during the research process researchers cannot be
subjective and detached from the classroom processes since every person/group participating in this process (teachers, students, parents, administrators) play a role in what happens in the classroom. This is why the “key” (Cohen et al 2007, pp 134) to the whole research process is the researcher and how he/she will interpret data and results (which are descriptive and analyzed inductively) - records and descriptions are through the eyes of the participants since understanding the meanings and intentions plays a central role. Lastly, during naturalistic research the emphasis is on the processes rather than the outcomes.

Design-based research (DBR), a research methodology commonly used in the learning sciences, falls into the sphere of naturalistic research since within DBR, interventions are conceptualized and then implemented in natural settings in order to test the ecological validity of a theory and to generate new theories and frameworks for conceptualizing learning, instruction, design processes, and educational reform (Barab and Squire 2004; Collins et al 2004; Burkhardt and Schoenfeld 2003). DBR begins by addressing complex problems in real learning contexts, through the collaboration of researchers and practitioners. Analysing the problem leads to the development of innovative solutions which are informed by existing design principles. These principles are then put into practice through iterative cycles of testing and refinement leading to reflection in order to produce new design principles and enhance solution implementation (Reeves 2006). Edelson (2002) gives the definition, the basic characteristics and the objective of DBR by using two words: usability and innovation. According to Barab et al (2005, p.15):

“Design-based research is a collection of innovative methodological approaches that involve the building of theoretically-inspired designs to systematically generate and test theory in naturalistic settings. Design-based research is especially powerful with respect to supporting and systematically examining innovation. In part, this is due to the fact that conducting design-based research involves more than examining what is. It also involves designing possibilities and then evolving theories within real-world contexts”

The study, presented in this paper, was based on a research team: two in-service pre-school teachers, the classroom teacher, the researcher (a pre-school teacher herself) and an observer who was the scientific director of the study. The research team designed the content of the study and assessed the teaching interventions. When and where necessary the implemented activities were improved and re-implemented. The teaching interventions were carried out by the classroom teacher. The researcher visited the classroom during all interventions as an observer.

This study formed a part of a broader educational programme which extended throughout the school year. It included the development of educational activities aimed at the development of investigative learning practices through the mathematical problem solving process. Additionally, our central methodological approach for the study of young children’s geometric thinking was based on investigative learning. The investigative learning approach gave us the opportunity to closely study the ways in which children develop and construct their knowledge (Clements and Battista 1992).

The educational programme consisted of two phases:

(a) structured teaching programme. A three week structured programme was created by the research group and implemented by the classroom teacher aiming at familiarizing the children with aspects of the mathematical problem
solving process within an investigative learning framework through structured classroom activities (Table 2).

(b) semi-structured programme. A four month semi-structured programme was created by the research group and introduced by the classroom teacher. After the completion of the structured programme, the semi-structured programme was introduced being materialized through the children’s work at the Mathematics Table during Free Activity Time (details given below).

More specifically, the educational programme (structured and semi-structured) aimed at the children (Table 2):
- becoming familiar with the mathematical problem solving process through their work on specific mathematical problems,
- using graphical representations as (a) tools for representing data and (b) tools for solving mathematical problems,
- becoming familiar with mathematical problems with multiple solutions.

This paper analyses data collected during the implementation of the semi-structured programme where children worked applying the mathematical problem solving process by choosing among five different mathematical problems. The number analysis problem, which was the first to be introduced to the children, was presented during Circle Time by a hand puppet. The hand puppet read the problem from a sheet of paper, discussing it with the children and left the classroom asking the children to post the solutions to it (posing the need for finding ways of graphically representing the solutions of the problem). Then the teacher discussed the mathematical problem with the children and presented the corresponding material. Children proposed their ideas and demonstrated them using the manipulatives. When a child demonstrated a solution that was not according to the problem guidelines, the teacher would read the problem again and discuss the guidelines to ensure the child understood. When a child would propose a solution that had already been created the teacher posed the need of having to find original solutions every time. This discussion led to the necessity of solution recording through graphical representations. At the end of the lesson, the teacher said that the children could work on the problem (individually, in pairs or in groups) during Free Activity Time every day. The formulation of working pairs and groups was purely decided by the child-he/she could choose the child/children he/she wanted to work with in order to solve the mathematical problem during Free Activity Time.

The Cyprus National Curriculum (Ministry of Education and Culture 1996) states that from 7:45 am to 9:05 am daily, children participate in “Free Activities” (see Note). During Free Activity Time children have the opportunity to choose among a variety of activities aiming at the development of different social, cognitive and sensory skills. During Free Activity Time in February 2012, five months after the beginning of the school year, the teacher and children organized a Mathematics Table which consisted of a round table seating four children at a time situated in the main area of the classroom. At the beginning, the classroom teacher spent more time at the Mathematics Table introducing the mathematical problems and offering the children help and support. This was gradually reduced and the teacher divided her time between all corners and tables organized at Free Activity Time.

During the course of the semi-structured programme, the children had the opportunity to work on five different mathematical problems (details given below). After having worked on the first mathematical problem several times, the teacher would introduce the second mathematical problem to the child. After having worked on the second mathematical problem more than once, the teacher would introduce the
third mathematical problem to the child and so on (Table 1). Thus at the Mathematics Table different children could be working on different mathematical problems at any given time. The children were not obliged to solve all problems - when the teacher introduced the new mathematical problem to a child she would ask whether he/she would like to try to solve it. If the reply was negative the teacher would introduce the following mathematical problem (Table 1). After the introduction and their first time of solving each problem, the children were free to choose among the mathematical problems they had already worked on. Each child could work on a chosen problem as many times as he/she wanted. The classroom teacher would always introduce the new mathematical problem to the child at the mathematics Table individually. During the child’s first attempt to solve a mathematical problem the classroom teacher sat next to him/her encouraging his/her efforts and scaffolding his/her learning mainly through posing questions. During the child’s following attempts to solve the same mathematical problem again, the classroom teacher was not continuously present. She would come to the Mathematics Table if the child asked her to. When choosing to work on a mathematical problem already solved by the child, the teacher would encourage interaction with the other children sitting at the Mathematics Table at the same time.

The children were not obliged to find all the solutions of the mathematical problem in one session. If they felt tired or wanted to engage in another activity, they could stop and return to the problem the following day. Each child could work on a mathematical problem for three or four consecutive days. All problems were gradually introduced and remained at the Mathematics Table until the end of the semi-structured programme (Table 1). During the course of the programme, and before being introduced to the 3-D mathematical problem, the children had worked on four mathematical problems:

- a number analysis mathematical problem: in how many different ways can x raindrops fall from two clouds? (where x the number of raindrops set by the child) (Shiakalli and Zacharos 2014)
- a combinatorial mathematical problem: in how many different ways can I place a red, a yellow and a blue button in the spaces set on my snowman’s tummy?,
- two geometry mathematical problems : (a) in how many different ways can I create a hexagon using other shapes?, (b) in how many different ways can I connect five identical squares, so that every square has at least one of its sides common with, at least, another square?- the pentomino problem (Shiakalli and Zacharos 2012) (Figure 1).

All four problems were multiple solution problems, thus, in order to solve them the children had to graphically represent their solutions. All problems were accompanied by corresponding manipulatives as well as answer sheets.

![Image of pentominoes](image_url)
After having worked on some or all the above mathematical problems, as many times as each child wanted, the 3-D (nets of a cube) mathematical problem was introduced to the Mathematics Table (Table 1).

<table>
<thead>
<tr>
<th>Number analysis problem</th>
<th>Combinatorial mathematical problem</th>
<th>Hexagon creation mathematical problem</th>
<th>Pentomino mathematical problem</th>
<th>Nets of the cube mathematical problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>19/1/2012 until 18/5/2012 (every day throughout free activity time/90 min session)</td>
<td>26/1/2012 until 18/5/2012 (every day throughout free activity time/90 min session)</td>
<td>13/2/2012 until 18/5/2012 (every day throughout free activity time/90 min session)</td>
<td>5/3/2012 until 18/5/2012 (every day throughout free activity time/90 min session)</td>
<td>16/3/2012 until 18/5/2012 (every day throughout free activity time/90 min session)</td>
</tr>
</tbody>
</table>

Table 1: Introduction of mathematical problems and duration of children’s opportunity to work on each mathematical problem.

The net of a cube mathematical problem

The teacher would present the manipulatives (Figure 3) and pose the problem: “In how many different ways can you connect these squares so that when you close your construction it will form a box?”. Solutions created by flipping or rotating existing solutions were not considered new solutions. Having worked on the pentomino problem the children were familiar with the idea of flipped and rotated solutions not being accepted as original solutions (an idea on which the classroom teacher would spend individual time with each child when encountering such issues during the child’s work on the pentomino problem). The total number of solutions was eleven (Figure 2) and children had to detect at least five in order to have solved the problem successfully. The children were given manipulatives (five identical plastic squares - Figure 3) in order to create their constructions and squared paper in order to record their solutions. One of the aims of the broader educational programme was for children to graphically represent solutions of the mathematical problems posed to them (Table 2). Thus, the children were familiar with the idea of graphically representing solutions as a part of the mathematical problem solving process.

Figure 2: The eleven cube nets
Figure 3: Manipulatives for the creation of a cube mathematical problem

All mathematical problems introduced to the Mathematics Table included elements of the criteria mentioned in the mathematical problem solving section: (1) children who chose to work on a problem, at the Mathematics Table, would begin the process having chosen to. The children also had the opportunity to choose the mathematical problem they wanted to work on. During the mathematical problem solving process it was anticipated that children would develop and apply new skills and develop their mathematical thinking. (2) the mathematical problems’ content was of central focus - the children would have the opportunity to solve the problems through repeated trials and verifications. As described above, the children were free to solve the same problem as many times as they wanted using corresponding manipulatives and answer sheets. When deciding to move on to a new mathematical problem, they could return to mathematical problems they had already solved and work on them again. (3) all mathematical problems accepted multiple answers. Through graphically representing their solutions the children had the opportunity to (i) find original solutions and (ii) check their answers. In this way, it was anticipated that, through the development of these skills the children would be guided to construct new ideas.

Participants and data collection

Twenty children aged between four and four and eight months (mean 4.4) attending a pre-school public class in Cyprus participated in the broader study, sixteen of which chose to work on the net of the cube mathematical problem. Codes S1 to S20 were given to the participating children (code S1 (age 4) was given to the youngest child and code S20 (age 4.8) was given to the oldest child). For data collection, ethical principles were complied: anonymity of research subjects, parental consent for subject participation in the study and teacher consent.

Data was collected through videotaped classroom interventions (all sessions during the structured and semi-structured programme were videotaped), the researcher’s notes while observing the children applying the problem solving process and children’s answer sheets. During the implementation of the structured programme the members of the research team would individually view the videotaped classroom interventions and fill out a semi-structured observation grid. Where the team members felt that the aim of the classroom activity was not fulfilled the activity would be re-assessed and re-designed by the team and re-implemented by the classroom teacher. During the implementation of the structured and semi-structured programme the
researcher visited the classroom daily in order to (a) observe the children’s work, collecting data through a semi-structured observation grid and (b) videotape classroom activities and the children’s work at the Mathematics Table. During the implementation of the semi-structured programme the researcher was always present at the Mathematics Table observing children’s work (Table 2).

<table>
<thead>
<tr>
<th></th>
<th>Structured</th>
<th>Semi-structured</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Aims</strong></td>
<td>For the children to</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- become familiar with the mathematical problem solving process through their work on specific mathematical problems,</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- use graphical representations as (a) tools for representing data and (b) tools for solving mathematical problems,</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- become familiar with mathematical problems with multiple solutions</td>
<td></td>
</tr>
<tr>
<td><strong>Duration</strong></td>
<td>7/11/2011 until 18/11/2011 (four 40 min sessions per day every day) and</td>
<td>19/1/2011 until 18/5/2012 (every day throughout free activity time/90 min session)</td>
</tr>
<tr>
<td></td>
<td>11/1/2012 until 18/1/2012 (two 40 min sessions per day every day)</td>
<td></td>
</tr>
<tr>
<td><strong>Participants:</strong></td>
<td>Classroom teacher, researcher, all children attending class on the day.</td>
<td>Classroom teacher, researcher, children working at the Mathematics Table (between one and four children at a time).</td>
</tr>
<tr>
<td>(a) in the classroom</td>
<td>Research group</td>
<td>Research group</td>
</tr>
<tr>
<td>(b) outside the classroom</td>
<td>Videotaped classroom interventions (structured teaching interventions),</td>
<td>Videotaped full sessions from Mathematics Table, researcher field notes, children’s answer sheets, research team members’ individual semi-structured observation grid (filled out after having viewed videotaped classroom interventions).</td>
</tr>
<tr>
<td>Data collection</td>
<td>researcher field notes (semi-structured observation grid), research team members’ individual semi-structured observation grid (filled out after having viewed videotaped classroom interventions).</td>
<td></td>
</tr>
</tbody>
</table>

*Table 2: Procedures, instruments and participants.*

**Data Analysis**

Children who chose to work on the nets of the cube mathematical problem were initially presented with the problem, the corresponding manipulatives and a squared answer sheet for solution representation. It is important to note that:

(a) although the teacher would tell the children that by finding five different solutions the problem was considered to have been successfully solved, six children...
(S3, S7, S12, S17, S18, S19) did not stop the process when having detected five solutions. One child (S17) continued the process managing to detect six solutions, three children (S3, S7, S18) continued the process managing to detect seven solutions and two children (S12, S19) continued to detect eight solutions. This data shows evidence that the children were developing the skills of patience and persistence in solving a mathematical problem. Moreover, five children (S6, S7, S12, S14, S18) showed increased interest for the problem by choosing to work on it twice (Table 3).

(b) the nets of the cube problem was a cognitively challenging mathematical problem for four-year-old children. All children who chose to solve this mathematical problem were observed to have worked on the problem for at least forty minutes before asking to work on a different activity during a Free Activity session. Children’s persistence and patience were noted by all members of the research team while viewing the videotaped episodes of the semi-structured programme. Of course, not all children managed to solve the problem in a single ninety minute Free Activity session- most children returned the following days in order to complete the process. In order to solve the nets of the cube mathematical problem the children had to firstly create a structure with the manipulatives and “close” it in order to see whether it formed a cube, secondly “open” the structure in order to check their answer sheet ensuring their structure was original and thirdly to graphically represent their structure. This is why most children needed three or four days in order to detect the minimum of five solutions of the problem.

(c) all children who chose to work on the nets of the cube mathematical problem managed to solve the problem successfully. Their success was a result of different factors: (i) mainly during the structured interventions the teacher had created a learning context encouraging investigative thinking, creativity, critical thinking, verbal interaction and collaboration. Within this context the children were encouraged to pose questions, suggest, try out and assess ideas and thoughts. All ideas were valued and discussed. Moreover, the teacher acted as a group member scaffolding children’s learning rather than guiding it, (ii) during the structured interventions all activities were meaningful to the children and created a need for finding multiple solutions. Thus when working on the mathematical problems at the Mathematics Table (during the semi-structured programme) the children were familiar with the idea that they had to detect multiple solutions. During the semi-structured programme, the teacher’s role was important in the child’s success to solve the mathematical problem during his/her first attempt since she (the teacher) was the one encouraging the child to continue trying, discuss ideas and interact with other children in order to find new ideas and continue detecting original solutions.

<table>
<thead>
<tr>
<th>Child code</th>
<th>Age</th>
<th>Times chosen to work on the problem</th>
<th>Number of solutions graphically represented</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>4</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>S2</td>
<td>4,3</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>S3</td>
<td>4,1</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>S4</td>
<td>4,1</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>S5</td>
<td>4,2</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>S6</td>
<td>4,2</td>
<td>2</td>
<td>5 first time / 5 second time</td>
</tr>
<tr>
<td>S7</td>
<td>4,2</td>
<td>2</td>
<td>5 first time / 7 second time</td>
</tr>
<tr>
<td>S8</td>
<td>4,3</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>S9</td>
<td>4,3</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>
Table 3: Number of times each child worked on the problem and number of solutions detected and graphically represented.

<table>
<thead>
<tr>
<th></th>
<th>4.2</th>
<th>1</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>S10</td>
<td>4.2</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>S11</td>
<td>4.3</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>S12</td>
<td>4.4</td>
<td>2</td>
<td>5 first time / 8 second time</td>
</tr>
<tr>
<td>S13</td>
<td>4.4</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>S14</td>
<td>4.5</td>
<td>2</td>
<td>5 first time / 5 second time</td>
</tr>
<tr>
<td>S15</td>
<td>4.6</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>S16</td>
<td>4.7</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>S17</td>
<td>4.7</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>S18</td>
<td>4.7</td>
<td>2</td>
<td>5 first time / 7 second time</td>
</tr>
<tr>
<td>S19</td>
<td>4.8</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>S20</td>
<td>4.8</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>

For data analysis two main categories were created: basic skills used by the children in order to solve the problem and strategies developed by the children in order to solve the problem. Sub-categories were then created for each of the two categories. This categorization emerged after all research team members had viewed the videotaped data and had noted the basic skills and strategies used by the children in order to solve the nets of a cube mathematical problem. The notes were then compared and discussed. Four categories were created based on the most dominant skills demonstrated by the children (Table 4): (a) verbalization (Episode 1), (b) course planning (Episode 2), (c) mathematical problem solving steps application (Episode 3), (d) strategy development and improvement (Episodes 5, 6, 7). Five categories were created based on the strategies the children had been observed to develop during the course of their work: (a) trial-and-error, (b) analogical reasoning (Episode 4), (c) global organizational principle- creating a line-of-four squares, “closing” it and detecting possible positions for the remaining squares (Episode 5), (d) global organizational principle - placing two squares on the opposite sides of a four-square line (Episode 6), (e) reverse course (Episode 7)

<table>
<thead>
<tr>
<th>Skills</th>
<th>Verbalization</th>
<th>Using words to express thoughts.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Course planning</td>
<td>Planning the course of action before beginning to work.</td>
<td></td>
</tr>
<tr>
<td>Mathematical problem solving steps application</td>
<td>Defining a problem =&gt; planning a course of action =&gt; executing =&gt; assessing.</td>
<td></td>
</tr>
<tr>
<td>Strategy development and improvement</td>
<td>Developing a plan and consistently applying it.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Strategies</th>
<th>Trial-and-error</th>
<th>Repeated, varied attempts which are continued until success.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analogical reasoning</td>
<td>Thinking based on a comparison between two systems of objects or situations, highlighting similar aspects.</td>
<td></td>
</tr>
<tr>
<td>Global organizational principle- creating a line-</td>
<td>Creating a central reference point (a four-square line) and adding to it.</td>
<td></td>
</tr>
</tbody>
</table>
of-four squares, “closing” it and detecting possible positions for the remaining squares

Global organizational principle - placing two squares on the opposite sides of a four-square line

Creating a core assumption (two squares should be placed on the opposite sides of a four-square line in order to create a net of the cube).

Reverse course

Trying to find out how the problem (the cube, in our case) was created.

Table 4: Skills and strategies used

Using mathematical skills

Some children were observed to be orally repeating the problem before starting to work on it, showing that they had grasped one of the key elements of the problem solving process: the first step in solving any mathematical problem is to realize its facts and demands (Episode 1).

Episode 1: Verbalizing the problem
S2 (age 4.3) had worked on all four mathematical problems several times. This time the teacher introduced the net of the cube problem for the first time. After having carefully heard the introduction of the problem S2 said:
1.1 S2: So what I need to do is to take these squares, connect them and then close them. If they make a box it’s a solution and I will write it down. If they don’t make a box I must try another idea.
1.2 S2: So, I am going to take the squares, connect them and close them. Yes this is what I am going to do.

S2 chose to work on the problem once managing to detect five solutions. Another interesting aspect of the above monologue was that S2 showed a positive attitude towards mistakes when stating that a construction that did not formulate a cube was a prompt to try a new idea.

During their work, children were observed to be applying the mathematical problem solving stages of plan-execution-evaluation (Episodes 2 and 3)

Episode 2: Planning before executing
S17 (age 4.7) had worked on the number analysis problem once, the hexagon creation problem once and the combinatorial and pentomino problems several times. She had now heard the introduction of the net of the cube problem.
2.1 S17: So, I am going to take the squares, connect them and close them. Yes this is what I am going to do.
2.2 S17 chose to work on the problem once, managing to detect six solutions.

Episode 3: Applying the basic steps of the mathematical problem solving process
S9 (age 4.3) chose only to work on the hexagon creation problem and the pentomino problem several times (not having tried to solve the number analysis and the combinatorial problems).

3.1 S9: *I will put them all in a row* (connected all pieces forming a line), *now I will close it*. . . *it doesn’t make a box. So that was not a good idea. Another idea now*

As in Episode 1, S9 seemed to use his mistake as an indicator of having to try another idea rather than a setback.

**Developing a strategy**

All children were observed created and applied a strategy in order to solve the net of the cube problem. The strategies developed by the children are presented and analyzed below.

- **Trial-and-error**. Nine children developed the trial-and-error strategy (S1, S6, S9, S10, S11, S14, S17, S18, S20): the children began their work by randomly connecting the manipulatives. When having connected all the pieces they would “close” their construction in order to see whether it would form a cube. If the construction was successful they would represent it on their recording sheets. If not they would try another way of connecting the squares.

- **Analogical reasoning**. One child (S3) developed the strategy of creating constructions based on the pentomino problem: the child would recall pentomino solutions (Figure 1) and based on those, then create hexaminoes. She then would “close” her construction to see whether it formed a cube. If the construction was successful she would record it on her answer sheet and continue by creating a different hexamino (Episode 4).

**Episode 4: Analogical reasoning-Constructing hexaminoes**

S3 (age 4.1) had solved the pentomino problem twice before choosing to work on the net of the cube problem. S3 started working on the problem by linking all pieces to formulate a line.

4.1 S3: *But this is like the solution of the other problem with the five squares.*
4.2 *Does it make a box?*
4.3 S3 “closed” her construction and saw that it did not form a cube
4.4 S3: *The line does not make a box. I shall try with other solutions.*
4.5 S3 created solution U (Figure 1) placing four squares in a line instead of three (as she would do for a pentomino solution).
4.7 S3: *This doesn’t make a box either. Another now.*
4.8 S3 produced solution F1 (Figure 1) placing four squares on the vertical lime instead of three)
4.10 S3: *This makes a box*
4.11 S3 recorded the solution on her answer sheet (Figure 4)
4.12 S3: *Another one now.*
4.13 S3 created the pentomino solution Z (Figure 1) using four squares in a line instead of three).
4.15 S3: *This also makes a box.*
S3 continued working in the same way and managed to create hexaminoes based on the pentomino solutions (Figure 4):
- T and X (Figure 4) placing four squares in a line instead of three
- M₁ (Figure 4) with three squares on the vertical line instead of two
- M₂ (Figure 4) with the sixth square placed above the fifth.
- F₂ (Figure 4) with the sixth square placed next to the line of three horizontally.

In the above Episode, S3 developed and systematically applied her strategy based on analogical reasoning. Her work reflected a sophisticated recognition of similarities between the mathematical structure of two problems separated by time and context. We noticed that the core of this strategy was based on a “local creation” (Tall et al, 2002), which was drawn by the pentomino problem and completed by a trial-and-error strategy to the degree where the first strategy proved inadequate to lead to a solution.

Moreover, by working in this way, S3 showed that she was able to manipulate the pentomino problem successfully, allowing us to conclude that the child had gained understanding of the properties of the pentomino as well as the net of the cube mathematical problems.

![Figure 4](image)

*Figure 4: Managing to detect seven solutions of the net of the cube problem by linking it to the pentomino problem*

- **Global organizational principle—Creation of a line using four squares and closing the construction in order to locate new possible positions.** Two children (S3, S18) developed the strategy of locating the empty spaces on a basic construction of four squares connected in a line: the children began by connecting four squares in a row and closing their construction (Episode 5). They were observed to be carefully locating the empty spaces, opening their construction and placing the two remaining squares in those positions. They then recorded their answer, removed the two squares from their construction (leaving it again with four squares in a line), “closed” the line and studied the empty spaces. When opening the construction they were observed to be avoiding placing the two remaining squares in positions they had already placed them. This was accomplished by careful observation of their answer sheet.
Episode 5: Global organizational principle - Creating a line-of-four squares, “closing” it and detecting possible positions for the remaining squares

S18 (age 4.7) had worked on the number analysis problem twice, the combinatorial problem three times and the pentomino problem three times. He started working on the nets of the cube problem by randomly connecting the squares and “closing” his construction in order to see if it formulated a cube. After three failed attempts, S18 linked four squares in a row, “closed” his construction and noted where the empty spaces were. He then opened his construction again and placed the two remaining squares in such positions as to formulate solution 2 (Figure 2). After placing the two remaining squares on his construction, S18 “closed” it again to make sure that it formulated a cube.

S18 connected four squares in a row and “closed” his construction:

5.1 S18: There is a hole here and here. Now I will open it and put the square there.

S18 continued working in the same way: he would connect four squares “close” the construction and detect where to place the remaining two squares in order to “close the holes”. This led to the detection and representation of five solutions of the problem (solutions 1, 9, 10, 3, 4 of Figure 2).

In conclusion, a developmental course was observed, by the specific children who applied the strategy (S13, S18): while at the beginning we saw a pure trial-and-error strategy, during the course of their work we observed a more mentally developed strategy which combined the trial-and error strategy with a “local classification strategy” (Tall et al, 2002, p.27), which is not distinguishable from the implementation of a general solving plan.

- Global organizational principle - Creation of a line using four squares and placement of the two remaining squares in different positions along the line: Two children (S7, S12) developed the strategy of placing two squares in different positions on opposite sides of a basic construction created by connecting four squares in a line: the children created a basic construction of four squares in a line. They then would place the two remaining squares on opposite sides of the line and close their construction in order to see whether it formed a cube. After having recorded their answer the children would remove the two squares (leaving the basic four-square line intact) and place the two squares in other positions. Again the children were observed to be consulting their answer sheets before placing the two remaining squares in certain positions. It was noted that both children began their work by creating a basic construction of four squares connected in a line and placing the remaining two squares randomly in different positions (Episode 6).

Episode 6: Global organizational principle - Placing two squares on the opposite sides of a four-square line

S12 (age 4.4) began by creating a line of four squares and placing the two remaining squares on the line randomly. He then “closed” his construction and saw that neither of his ideas formed a cube. When he placed the two remaining squares on opposite sides of the line (forming a letter “T”) and “closed” his construction seeing that it did form a cube he stated:

6.1 S12: Oh, a good idea. . . place my squares differently now but not on the same side because they make a couch –not a box.
S12 applied his strategy systematically managing to detect four more solutions (solutions 9, 10, 3, 4- Figure 2)

Apart from the ability to develop and systematically apply a strategy, Episode 6 revealed that S12 had also developed the ability to formulate and use a conclusion. Episode 6 also revealed evidence of assessment during the application of the mathematical problem solving process.

The strategy developed here seemed to be the most cognitively advanced strategy, since it was based on an organizational principle, where the placement of four squares created the “hull” of the cube, a fact that facilitated the placement of the two remaining squares.

- **Reverse course.** Three children (S2, S16, S19) developed the strategy of reverse course: the children began by connecting the squares in a shape of a cube. They then “opened” their construction revealing a net and recorded it on their answer sheet. They continued their work by connecting the manipulatives into a cube, again. They then would “open” their construction again and study their answer sheet to verify whether they had already recorded that answer (Episode 7).

**Episode 7: Reverse course strategy**

S2 (age 4.3) carefully listened to the problem presented to her by the teacher. She then took all six squares and started connecting them forming a cube. She touched the vertexes and faces of her construction and ran her fingers across the edges. She then carefully “opened” it revealing a net:

7.1 S2: *This is a solution. I will write it down*
7.2 R (Researcher): *And what will you do next?*
7.3 S2: *I will make my box again and open it, but not in the same way. If I open it in the same way I will have the same solution. I don’t want that. I need a different solution.*
7.5 S2 applied her strategy managing to detect five different solutions.

The strategy described in Episode 7 resembles the proof “*method of analysis*” suggested by Pappus of Alexandria (Polya, 1973), according to which, in order to solve a mathematical problem one begins by hypothesizing what is wanted to be found, and by working backwards, discovers a series of actions for the course needed to be followed. Since the educational programme to which the children were exposed during the course of the study (details given during method description above) consisted of no reverse thinking activities, the strategy developed by the three children (S2, S16, S19) was considered to be a sophisticated cognitively processed strategy.

**Conclusions and discussion**

Our first research question concerned the way four year old children manage the creation of cube nets as a mathematical problem. Our findings showed that all children, who chose to work on this problem, applied the mathematical problem solving process following the steps set by Deek, et al (1999) and Kapa (2002). More specifically, all children began by experimentation in order to become familiar with the problem. Although the mathematical problems were given to the children, we noted that the step of problem formulation was evident in their work: at the beginning of the process the children showed that they were trying to understand the basic
notions of the problem before trying to solve it (Episode quotes: 1.1-1.3, 2.1-2.2). They then organized their action plan and carried it out by constructing solutions (Episode quotes: 2.1-2.2, 3.1-3.3, 5.1, 7.3-7.5). They proceeded in graphically representing the solutions (Episode quote: 4.11). Finally all children assessed their work in order to ensure the problem was successfully solved. Close observation of children’s work showed that they applied assessment not only in order to ensure completion of the process but in order to evaluate their work during the process as well (Episode quotes: 3.2, 6.1-6.2).

Through the study of the mathematical problem solving process, as applied by the children, we noted that children did not consider the problem solved after finding only one solution. Instead they persisted in their quest until they managed to detect a minimum of five solutions. Fortuno, et al (1991) noted that children tended to stop the mathematical problem solving process after finding one solution. In this study, the fact that the children had been working on mathematical problems with multiple solutions for a long period of time, seemed to play an important role in them not believing they had completed the process after having found one solution. Moreover, strong teacher encouragement to continue the quest for more solutions (as suggested by Kappa, 2002), during the early stages of this study, seemed to have had a positive effect on the children.

Findings showed that four year-old children not only effectively applied strategies but also developed and refined them in future applications of the mathematical problem solving process. Some strategies were similar (S2, S16, S19 applied the reverse course strategy) while others were original (S3 applied the analogical reasoning strategy). It is important to note that S2, S16 and S19 (who applied a similar strategy during the course of their work) were not sitting at the Mathematics Table together at any given time. Although the strategy was similar, each child developed it individually.

The observation of the strategies developed by the children showed a strategy developmental continuum: the pure trial-and-error strategy evolved in a strategy combining trial-and-error with local classification strategies (S18, Episode 5). Additionally, we observed strategies based on the analogical use of models deriving from practices developed in the solution process of other mathematical problems (S3, Episode 4). We considered the analogical reasoning strategy a powerful and sophisticated strategy since the development of analogical reasoning seems to compliment the overall development of mathematical thinking (Papic et al., 2011). We also located sophisticated mentally developed strategies, such as the strategy based on a global organizational principle of creating a cube “hull” on which new solutions were detected (S12, Episode 6) and the reverse course strategy (Episode 7). We consider the fact that out of the five children (S6, S7, S12, 14, S18) who chose to solve the problem twice, three (S7, S12, S18) evolved their strategy during the second time they worked on the problem: all three children used trial-and-error the first time they solved the problem. The second time they began with trial-and-error but proceeded to develop it into the global organizational strategy using the latter consistently throughout the mathematical problem solving process.

Concerning the second research question, looking into the children’s use of graphically represented solutions as tools for solving the mathematical problem, our data showed that the process of graphical representation of solutions was crucial (a) in enabling children to detect as many solutions as possible and (b) in allowing children to verbalize their observations concerning the problems, showing that the mathematical problem solving process had become a complex procedure consisting of
sophisticated forms of thinking. Moreover, our findings support Elia et al.’s (2014) claim in the role of language in formation of spatial ideas.

Conclusively, our study supports the idea (Davis and Hyun 2005) that external graphical representations can help children to create new meanings as well as their ability to express mathematical ideas.

In order to solve the net of the cube mathematical problem all children used the manipulatives for constructing their solutions and the squared paper sheet for graphically representing successful constructions. Having worked on other mathematical problems using different manipulatives (as already described in this paper), the children were accustomed to using manipulatives as tools for solving mathematical problems. All children were observed to study their answer sheets before proceeding in creating a construction (showing that they were using it as a guide for new ideas). All children were observed to study their answer sheets after having completed a successful construction and before recording it. These behaviors showed that the children used their graphical representations as (a) guides for new ideas, (b) guides for avoiding to record same solutions, (c) useful tools during the problem solving process.

In looking at the conditions under which creating the nets of the cube mathematical problem solving was successful in pre-school, our findings seemed to detect that the implementation of the structured programme, aiming at familiarizing the children with the mathematical problem solving process, played an important role in the children successfully solving the nets of the cube mathematical problem. The introduction of appropriate mathematical problems and the creation of a developmentally appropriate classroom environment, aimed at nurturing investigative learning, seemed to have been the two most important factors in children’s later success. Our evidence showed that children as young as four years old had the ability to successfully solve complicated mathematical problems (such as the nets of the cube mathematical problem) under three conditions: (a) they were familiar with the mathematical problem solving process, (b) they were familiar with the external representation of solutions process and use it as a guiding tool, and (c) worked within a context where investigation, questioning and communication are encouraged and valued. All mentioned conditions are included in Perry et al’s (2007) list of “powerful mathematical ideas”.

Our data also pointed out that in order for the children to acquire and use skills connected to the mathematical problem solving process, familiarization with the process through mathematical problems was essential. In other words, expertise in subjects, and in our case mathematics, is not a spontaneous consequence of a child’s occupation with geometrical objects but a process of social transfer (Vygotsky, 1978; Zacharos, et al 2011), and in the case studied here, systematic occupation and teaching.

The time element also seemed to have played an important role in the successful acquisition and use of the mathematical problem solving process. Our data showed that children needed to be given the opportunity to work on mathematical problems repeatedly and through a long period of time. We believe that the elements of time and repetition were the two factors that helped the children develop and demonstrate skills such as analogical reasoning and strategy development and improvement during the nets of the cube mathematical problem solving process.

Lastly we found that when the mathematical problems presented to the children (a) triggered their interest, (b) encouraged the use of external representations and (c) had multiple solutions, children, as young as four years old, not only could apply the
mathematical problem solving process but could successfully solve complicated mathematical problems as well.

Note
The Cyprus National Curriculum (Ministry of Education and Culture, 1996) states that from 7:45 am to 9:05 am daily, children must participate in “Free Activities”. During free activity time children have the opportunity to engage in different activities (organized in “corners”) aimed at the development of different social, cognitive and sensory skills. Children choose the “corner” they would like to work/play in. Some corners are designated by the National Curriculum (dollhouse, construction corner, drawing corner, storytelling corner, puppet playing corner, play dough corner) and can be found in all pre-school settings throughout the year, while others can be organized by the teacher depending on the children’s interests or a specific curriculum topic and can be temporary.

References


