Strand A *NUMBER* Introduction

Introduction to Topics

This is the starting point for enhancing your mathematical skills and knowledge. We will assume that you have a basic understanding of number, including place value, but the initial units of work on decimals, fractions and percentages will cover all the key concepts in number work.

We also include some optional material on Scientific Notation (Unit 5) and Number Systems (all but the first section on Roman Numerals) that cover important mathematical concepts not directly related to the Key Stage National Curriculum. Such material is starred (*) and you can choose to omit these sections and units as they will not count in the online assessment.

These are important topics in mathematical progression.

You can use the initial audits to check whether or not you need to study particular sections of each unit. If you obtain full marks (normally 5 out of 5), then just move on to the next section; if you obtain 4 out of 5, then see if you understand your error. If you achieve 3 or fewer marks on a particular audit, then we suggest that you read carefully the text (or use the interactive text) for this section and then try the assessed components, that is, the multi-choice questions and revision test.

Your tutor has various suggested approaches and solutions to these problems so if you need advice please contact your local tutor. We will also provide this information online later in the course.

Historical Context

Place Value

The familiar place value concept used in our number system, and usually referred to as the Hindu - Arabic system, had its origins in India and brought to the West by means of the Arabs. The Babylonians had a place value system, but it was based on 60, whilst the Chinese, from the earliest times, had a system based on 10.

Around the 7th Century, the Indians dropped symbols for numbers higher than 9 and began to use symbols for 1 to 9 in our familiar place value arrangement (with a dot used for the zero). The first reference to this is in fact attributed to a Siberian priest in 662. Others began to use systems in which words stand for numbers, for example

Number	*Word*
0	sky
1	moon
2	eye
3	fire

Thus fire - sky - moon - eye is in fact 2103, the place value beginning on the left with the units.

Whilst the exact sequence of development is not at all clear, certainly by 870, a decimal place value system for integers existed both in India and China.

Decimals

Our current way of writing (and manipulating) decimals is, largely, attributable to the Belgian mathematician, *Simon Stevin* (1548-1620), who moved to Holland and worked for the Dutch government as quartermaster-general of the army. In this role, he organised a school of engineering at the University of Leiden, to meet the growing need of the Dutch nation for trained engineers, surveyors and navigators.

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Prior to his work, a notation for decimal fractions had been used, particularly in the Islamic world, but not manipulated in the way with which we are now familiar. Stevin made the significant jump of designing a system in which arithmetic operations (i.e. $+$, $-$, $\times$, $\div$) used with whole numbers, worked in the same way with decimals.

For example, he first wrote 364.3759, as

$$364 \textcircled{0} 3 \textcircled{1} 7 \textcircled{2} 5 \textcircled{3} 9 \textcircled{4}$$

meaning $364 + \frac{3}{10} + \frac{7}{100} + \frac{5}{1000} + \frac{9}{10\,000}$ or $364 \frac{3759}{10\,000}$. He made the point that no fractions were used in his notation, and his article went on to show how all the basic operations can be performed.

It should also be noted that, in a second major article, Stevin made the point that firstly 'unity' is a number (previously it was not regarded as a number, rather a generator of numbers), as are decimal squares, square roots, etc.

In principle, he defined what we now refer to as the set of 'real numbers'.

Fractions

It is of interest to note that, from earliest times, two systems of fractions have been used. One was employed by the ancient Egyptians as early as 1700 BC. Indeed, in about 1850 BC, King Amememhat III came to the throne and he initiated an extensive system of irrigation, necessitating a knowledge of mathematics never before developed (except perhaps in Mesopotamia). There is good reason to believe that in this reign, about 1825 BC, there was written the original of the oldest elaborate manuscript on mathematics still in existence, the *Ahmes treatise*; if this conjecture is correct, the unit fraction (with 1 as the numerator) was already known in Egypt at this time. All their fractions had 1 as the numerator, and other fractions were expressed as the sum of fractions with 1 as the numerator, e.g.

$$\frac{4}{9} = \frac{1}{3} + \frac{1}{9}$$

Of course, for some fractions, an approximation is needed in order to express it as a series of fractions with 1 as the numerator, e.g.

$$\frac{1}{4} + \frac{1}{7} + \frac{1}{9} + \frac{1}{10} + \frac{1}{20}$$

is used as an approximation for $\frac{17}{26}$. You can use a calculator to check its suitability.

The second system, used by the Babylonians (c 200 BC) and brought to Europe by the Greeks, used fractions with $\frac{1}{60}$ as the base, e.g. $1 \cdot 22'\ 7''\ 42'''$ represents $1 + \frac{22}{60} + \frac{7}{3600} + \frac{42}{21\,600}$ etc.

The number 360 was probably an early guess at the number of days in a year; even after it was clear that the number of days was *not* 360, this magical number was still used as the angular measure round a circle (just note how many factors 360 has to see its wondrous nature!). It is fascinating to note that the earliest dated event in human history is the introduction of the Egyptian calendar of 12 months of

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30 days each, plus five feast days, in the year 4241 BC. Such an achievement is remarkable, given that it was a better calendar than that used in Europe from the time of the Romans until the reform of Gregory XIII (1582), and might be regarded as better than the one used to the present day – more than 6000 years later!

Percentages

The background to percentages is rather more obscure, however, but they are thought to have originated in Italy in about 1425, when a symbol was used for 'per cent', i.e. per 100. Over time, this was modified, and eventually the % symbol was established. The symbol ‰ meaning 'per mille' was also used, but never extensively.

The % symbol to represent percentages, and what it means, has become important in today's society, and regularly expresses changes in contexts such as wages, inflation, interest rates and VAT, in addition to its normal usage. So it is vitally important that all our students both understand the concept, and competently put it into practice.

Factors

Prime numbers and their properties were first studied extensively by ancient Greek mathematicians. The Pythagorean school (c 400 BC), studied both prime and perfect numbers for their mystical properties (see Activity 2.2). When Euclid's 'Elements' appeared in about 300 BC, several important results concerning primes had been proved. Euclid proved (by contradiction) that there are infinitely many primes; he also gave a proof of the 'Fundamental Theorem of Arithmetic', which states that every integer can be expressed as a product of primes in an essentially unique way (this is assumed in section 2.2).

The Greek, Eratosthenes, in about 200 BC, designed his algorithm for finding prime numbers (dealt with here in Activity 2.1), and his method is still a quick and efficient way of finding small prime numbers.

After the Greek mathematicians had made these advances there was little development in this area until Fermat, at the beginning of the 17th century. He established many results, including:

(i) any prime number of the form $p = 4n + 1$ (n integer) can be expressed uniquely as the sum of two squares, e.g. $5 = 1^2 + 2^2$, $13 = 2^2 + 3^2$.

(ii) if p is a prime number, $2^p - 2$ is divisible by p (this is known as Fermat's Little Theorem), e.g. $2^5 - 2 = 30$ is divisible by 5.

Prime numbers are still the subject of considerable study, and today very large prime numbers are used in the design of codes. For example, the number $n = p_1 \times p_2$, where p_1 and p_2 are both very large primes, is immensely difficult to factorise. In America, in fact, there have been attempts to copyright some of the very large prime numbers in response to their important commercial use both now and in the future.

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Misconceptions

Misconceptions with Decimals

- To round, say 14 489 to the nearest 1000, learners might first round to the nearest 10, then 100, then 1000, giving

  14 489 ⟶ 14 490 ⟶ 14 500 ⟶ 15 000 (as '5' rounds up)

  giving the *incorrect* answer that 14 489 to the nearest thousand is 15 000.
  The correct answer is that 14 489 to the nearest 1000 is 14 000.

- The number 1.329 is greater than 1.4 as it has more decimal places - reference to a number line will clearly show this to be false. Similarly, 1.09 is greater than 1.22, as the final digit 9 is greater than 2 - again reference to a number line shows this to be false.

- 0.1×0.1 is often thought to be 0.1; this can readily be seen to be false, as $0.1 \times 10 \times 0.1 = 0.1$

- 2.3×10 is *not* 2.30 or 20.3

- Not all divisions have exact answers, even simple ones such as $7 \div 3$, which equals 2, with remainder 1, or $2\frac{1}{3}$ or 2.3333...

- It is not always realised that $24 \div 8$ is the same number as the fraction $\frac{24}{8}$ $\left(= \frac{12}{4} = \frac{6}{2} = 3\right)$.

Misconceptions with Negative Numbers

- Many learners will have fundamental misconceptions, such as:

  (i) $-3 + 5 = 8$ (instead of 2)

  (ii) $-3 - 5 = 8$ (instead of -8)

  (iii) $-3 \times 5 = 15$ (instead of -15)

  (iv) $-3 \times (-5) = -15$ (instead of 15)

Misconceptions with Brackets

- Brackets often cause problems – they *do* matter and, in some cases, their use will produce a different answer,

  e.g. $7 - (4 + 1) = 7 - 5 = 2$

  whilst $7 - 4 + 1 = 3 + 1 = 4$

- $8 + 2 \times 3$ can easily be mistaken for $(8 + 2) \times 3 = 30$ (rather than $8 + 6 = 14$)

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Misconceptions with Fractions

- Note that equivalent fractions are always equal, e.g. $\frac{2}{4} = \frac{1}{2}$
- $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, not $\frac{2}{6}$ or $\frac{1}{6}$
- $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, not $\frac{1}{5}$
- $\frac{1}{3}$ of $\frac{1}{3} = \frac{1}{9}$, not 1
- The correct answer to $\frac{1}{4} \div \frac{1}{2}$ is $\frac{1}{2}$, not 2, and to $\frac{1}{3} \div \frac{1}{9}$ is 3, not $\frac{1}{3}$, etc.

Misconceptions with Percentages

- Learners must understand that fractions and percentages are in fact equivalent, i.e. 10% of a quantity is equivalent to $\frac{1}{10}$ of the quantity.
- Note that, although 10% is $\frac{1}{10}$, 5% is *not* $\frac{1}{5}$, but in fact, $\frac{5}{100} \equiv \frac{1}{20}$, etc.

Misconceptions with Ratio and Proportion

- Learners often do not realise that, for example, 2 : 8 is the same ratio as 1 : 4.
- It is a misconception that increasing a map scale increases the map distance (in fact, it decreases the map distance).
- Mistakes occur when using direct instead of proportional division; for example, if it takes 4 people 2 hours to address 200 envelopes, it takes 2 people 1 hour (incorrect) rather than the correct answer of 4 hours.

Misconceptions with Factors

- It is a misconception that 1 is a prime number: the definition of a prime number states that it has *two* factors, 1 and itself. The number 1 has only *one* factor so is *not* a prime number.