

11 Fractions and Percentages

11.1 Fractions, Decimals and Percentages

Percentages can be converted to fractions because 'percentage' simply means 'per hundred'. They can also be converted very easily to decimals, which can be useful when using a calculator. Fractions and decimals can also be converted back to percentages.



Worked Example 1

Convert each of the following percentages to fractions.

- (a) 50% (b) 40% (c) 8%



Solution

$$\begin{array}{lll} \text{(a)} \quad 50\% = \frac{50}{100} & \text{(b)} \quad 40\% = \frac{40}{100} & \text{(c)} \quad 8\% = \frac{8}{100} \\ & = \frac{1}{2} & = \frac{2}{25} \end{array}$$



Worked Example 2

Convert each of the following percentages to decimals.

- (a) 60% (b) 72% (c) 6%



Solution

$$\begin{array}{lll} \text{(a)} \quad 60\% = \frac{60}{100} & \text{(b)} \quad 72\% = \frac{72}{100} & \text{(c)} \quad 6\% = \frac{6}{100} \\ & = 0.6 & = 0.06 \end{array}$$



Worked Example 3

Convert each of the following decimals to percentages.

- (a) 0.04 (b) 0.65 (c) 0.9



Solution

$$\begin{array}{lll} \text{(a)} \quad 0.04 = \frac{4}{100} & \text{(b)} \quad 0.65 = \frac{65}{100} & \text{(c)} \quad 0.9 = \frac{9}{10} \\ & = 4\% & = \frac{90}{100} \\ & & = 90\% \end{array}$$



Information

'Per cent' comes from the Latin, 'per centum', which means 'for each hundred'.



Worked Example 4

Convert each of the following fractions to percentages.

(a) $\frac{3}{10}$ (b) $\frac{1}{4}$ (c) $\frac{1}{3}$



Solution

To convert fractions to percentages, multiply the fraction by 100%. This gives its value as a percentage.

$$\begin{array}{lll}
 \text{(a)} \quad \frac{3}{10} = \frac{3}{10} \times 100\% & \text{(b)} \quad \frac{1}{4} = \frac{1}{4} \times 100\% & \text{(c)} \quad \frac{1}{3} = \frac{1}{3} \times 100\% \\
 = 30\% & = 25\% & = 33\frac{1}{3}\%
 \end{array}$$



Exercises

1. Convert each of the following percentages to fractions, giving your answers in their simplest form.

- | | | | |
|---------|---------|---------|---------|
| (a) 10% | (b) 80% | (c) 90% | (d) 5% |
| (e) 25% | (f) 75% | (g) 35% | (h) 38% |
| (i) 4% | (j) 12% | (k) 82% | (l) 74% |

2. Convert each of the following percentages to decimals.

- | | | | |
|---------|---------|---------|---------|
| (a) 32% | (b) 50% | (c) 34% | (d) 20% |
| (e) 15% | (f) 81% | (g) 4% | (h) 3% |
| (i) 7% | (j) 18% | (k) 75% | (l) 73% |

3. Convert the following decimals to percentages.

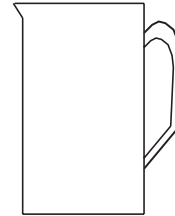
- | | | | |
|----------|----------|----------|----------|
| (a) 0.5 | (b) 0.74 | (c) 0.35 | (d) 0.08 |
| (e) 0.1 | (f) 0.52 | (g) 0.8 | (h) 0.07 |
| (i) 0.04 | (j) 0.18 | (k) 0.4 | (l) 0.3 |

4. Convert the following fractions to percentages.

- | | | | |
|--------------------|--------------------|--------------------|--------------------|
| (a) $\frac{1}{2}$ | (b) $\frac{7}{10}$ | (c) $\frac{1}{5}$ | (d) $\frac{3}{4}$ |
| (e) $\frac{1}{10}$ | (f) $\frac{9}{10}$ | (g) $\frac{4}{5}$ | (h) $\frac{4}{50}$ |
| (i) $\frac{8}{25}$ | (j) $\frac{7}{20}$ | (k) $\frac{7}{25}$ | (l) $\frac{2}{3}$ |

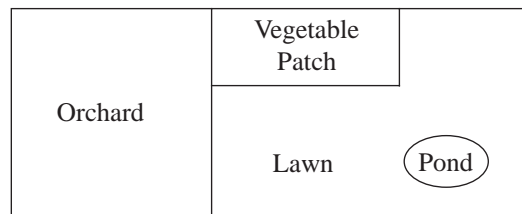
5. (a) Complete the equation $\frac{2}{3} = \frac{?}{15} = \frac{16}{?}$
- (b) Change $\frac{7}{20}$ to a percentage. (MEG)

6. (a) Water is poured into this jug.
Copy the diagram and show accurately the water level when the jug is three-quarters full.
- (b) What percentage of the jug is filled with water?



(SEG)

7. *Plan of a garden*

*Not to scale*

- (a) In the garden the vegetable patch has an area of 46.2 m^2 . The orchard has an area of 133.6 m^2 .
What is the total area of the vegetable patch and the orchard? Give your answer to the nearest square metre.
- (b) The garden has an area of 400 m^2 .
- (i) The lawn is 30% of the garden. Calculate the area of the lawn.
- (ii) A pond in the garden has an area of 80 m^2 . What percentage of the garden is taken up by the pond?

(SEG)

11.2 Fractions and Percentages of Quantities

Percentages are often used to describe changes in quantities or prices. For example,

'30% extra free' '10% discount' 'add $17\frac{1}{2}\%$ VAT'

This section deals with finding fractions or percentages of quantities.



Worked Example 1

Find 20% of £84.



Solution

This can be done by converting 20% to either a fraction or a decimal.

Converting to a fraction

Note that $20\% = \frac{20}{100} = \frac{1}{5}$

Therefore $20\% \text{ of } £84 = \frac{1}{5} \times £84$
 $= £16.80.$

Converting to a decimal

Note that $20\% = 0.2$

Therefore $20\% \text{ of } £84 = 0.2 \times £84$
 $= £16.80.$

**Worked Example 2**

A shopkeeper decides to increase some prices by 10%. By how much would she increase the price of:

- (a) a loaf of bread costing 90p (b) a packet of cereal costing £2.00?

**Solution**

First note that $10\% = \frac{1}{10}.$

(a) $10\% \text{ of } 90\text{p} = \frac{1}{10} \times 90\text{p}$
 $= 9\text{p}.$

So the cost of a loaf will be increased by 9p.

(b) $10\% \text{ of } £2 = \frac{1}{10} \times £2$
 $= £0.20 \text{ or } 20\text{p}.$

So the cost of a packet of cereal is increased by 20p.

**Worked Example 3**

A farmer decides to sell 25% of his 500 cows. How many cows does he sell?

**Solution**

First note that $25\% = \frac{1}{4}.$

$$25\% \text{ of } 500 = \frac{1}{4} \times 500$$

$$= 125.$$

So he sells 125 cows.



Worked Example 4

Natasha invests £200 in a building society account. At the end of the year she receives 5% interest. How much interest does she receive?



Solution

First convert 5% to a fraction. $5\% = \frac{5}{100} = \frac{1}{20}$

$$\begin{aligned} 5\% \text{ of } \pounds 200 &= \frac{1}{20} \times \pounds 200 \\ &= \pounds 10. \end{aligned}$$

So she receives £10 interest.



Exercises

1. Find

- | | | |
|------------------------------|-----------------|------------------|
| (a) 10% of 200 | (b) 50% of £5 | (c) 20% of £8 |
| (d) 25% of £100 | (e) 40% of £500 | (f) 90% of 200 |
| (g) $33\frac{1}{3}\%$ of £12 | (h) 75% of 800 | (i) 75% of 1000 |
| (j) 80% of 20 kg | (k) 70% of 5 kg | (l) 30% of 50 kg |
| (m) 5% of 100 m | (n) 20% of 50 m | (o) 25% of £30 |

2. Find

- | | | |
|--------------------------|--------------------------|---------------------------|
| (a) $\frac{2}{5}$ of 80 | (b) $\frac{3}{4}$ of 120 | (c) $\frac{1}{5}$ of 90 |
| (d) $\frac{1}{4}$ of 360 | (e) $\frac{4}{5}$ of 150 | (f) $\frac{3}{10}$ of 500 |

3. A firm decides to give 20% extra free in their packets of soap powder. How much extra soap powder would be given away free with packets which normally contain

- | | |
|--------------------|-----------------------|
| (a) 2 kg of powder | (b) 1.2 kg of powder? |
|--------------------|-----------------------|

4. A house costs £30 000. A buyer is given a 10% discount. How much money does the buyer save?

5. John has invested £500 in a building society. He gets 5% interest each year. How much interest does he get in a year?

6. Karen bought an antique vase for £120. Two years later its value had increased by 25%. What was the new value of the vase?

7. Ahmed wants to buy a new carpet for his house. The cost of the carpet is £240. One day the carpet shop has a special offer of a 25% discount. How much money does he save by using this offer?

8. When Wendy walks to school she covers a distance of 1800 m. One day she discovers a short cut which reduces this distance by 20%. How much shorter is the new route?
9. Chen earns £30 per week from his part-time job. He is given a 5% pay rise. How much extra does he earn each week?
10. Gareth weighed 90 kg. He went on a diet and tried to reduce his weight by 10%. How many kilograms did he try to lose?
11. Kim's mother decided to increase her pocket money by 40%. How much extra did Kim receive each week if previously she had been given £2.00 per week?
12. A new-born baby girl weighed 4 kg. In the first three months her weight increased by 60%. How much weight had the baby gained?
13. Work out
- (a) $\frac{7}{10}$ of £8 (b) 20% of £25 (c) $\frac{3}{8}$ of 6 metres.

(LON)

14. (a) Calculate 15% of £600.
 (b) List these fractions in order of size, starting with the smallest.

$$\frac{1}{3}, \frac{2}{9}, \frac{5}{6}, \frac{1}{6}$$

(MEG)

15. A cake weighs 850 grams. 20% of the cake is sugar. Calculate the weight of sugar in the cake.
16. An athletics stadium has 35 000 seats. 4% of the seats are fitted with headphones to help people hear the announcements. How many headphones are there in the stadium?

(NEAB)

17. Jane wants to buy this car.
 The deposit is $\frac{2}{5}$ of the price of the car.
 Jane's father gives her 30% of the price.
 Will this be enough for her deposit?
 You must explain your answer fully.



Investigation

The ancient Egyptians were the first to use fractions. However, they only used fractions with a numerator of one. Thus they wrote $\frac{3}{8}$ as $\frac{1}{4} + \frac{1}{8}$, etc.

What do you think the Egyptians would write for the fractions $\frac{3}{5}$, $\frac{9}{20}$, $\frac{2}{3}$ and $\frac{7}{12}$?

11.3 Quantities as Percentages

To answer questions such as,

Is it better to score 30 out of 40 or 40 out of 50?

it is helpful to express the scores as percentages.



Worked Example 1

Express '30 out of 40' and '40 out of 50' as percentages. Which is the better score?



Solution

'30 out of 40' can be written as $\frac{30}{40}$ and '40 out of 50' can be written as $\frac{40}{50}$.

Changing these fractions to percentages,

$$\begin{aligned} \frac{30}{40} &= \frac{30}{40} \times 100\% & \text{and} & \quad \frac{40}{50} = \frac{40}{50} \times 100\% \\ &= 75\% & & \quad = 80\% \end{aligned}$$

So '40 out of 50' is the better score, since 80% is greater than 75%.



Worked Example 2

A pupil scores 6 out of 10 in a test. Express this as a percentage.



Solution

'6 out of 10' can be written as $\frac{6}{10}$. Changing this fraction to a percentage,

$$\frac{6}{10} = \frac{6}{10} \times 100\% = 60\%.$$



Worked Example 3

Robyn and Rachel bought a set of CDs for £20. Robyn paid £11 and Rachel paid £9. What percentage of the total cost did each girl pay?



Solution

Robyn paid £11 out of £20, which is

$$\frac{11}{20} = \frac{11}{20} \times 100\% = 55\%.$$

Rachel paid £9 out of £20, which is

$$\frac{9}{20} = \frac{9}{20} \times 100\% = 45\%.$$



Worked Example 4

David earns £400 per week and saves £30 towards the cost of a new car.
What percentage of his earnings does he save?



Solution

He saves £30 out of £400, which is

$$\frac{30}{400} = \frac{30}{400} \times 100\% = 7.5\%.$$



Exercises

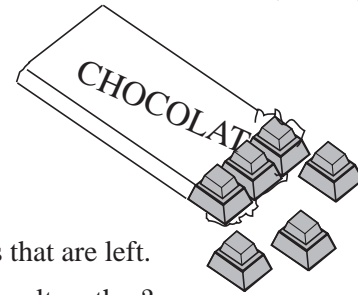
- Express each of the following as percentages.

| | | |
|--------------------|-------------------|--------------------|
| (a) 8 out of 50 | (b) 3 out of 25 | (c) 8 out of 20 |
| (d) 3 out of 10 | (e) 6 out of 50 | (f) 6 out of 40 |
| (g) 12 out of 80 | (h) 9 out of 30 | (i) 27 out of 30 |
| (j) 120 out of 300 | (k) 84 out of 200 | (l) 260 out of 400 |
| (m) 28 out of 70 | (n) 18 out of 60 | (o) 51 out of 60 |
- In a class of 25 children there are 10 girls. What percentage of the class are girls and what percentage are boys?
- The price of a bar of chocolate is 25p and includes 5p profit. Express the profit as a percentage of the price.
- The value of a house is £40 000 and the value of the contents is £3 200. Express the contents value as a percentage of the house value.
- In the crowd at a football match there were 28 000 *Manchester United* supporters and 22 000 *Tottenham* supporters. What percentage of the crowd supported each team?
- A school won a prize of £2000. The staff spent £1600 on a new computer and the rest on software. What percentage of the money was spent on software?
- A book contained 80 black and white pictures and 120 colour pictures. What percentage of the pictures were in colour?
- In a survey of 300 people it was found that 243 people watched *EastEnders* regularly. Express this as a percentage.
- James needs another 40 football stickers to complete his collection. There is a total of 500 stickers in the collection. What percentage of the collection does he have already?
- A 600 ml bottle of shampoo contains 200 ml of free shampoo. What percentage is free?

11. Adrian finds that in a delivery of 500 bricks there are 20 broken bricks. What percentage of the bricks are broken?
12. A glass of drink contains 50 ml of fruit juice and 200 ml of lemonade. What percentage of the drink is lemonade?
13. A recent survey shows that there are 20 000 different types of fish in the world. People catch only 9000 different types. What percentage of the different types of fish do people catch?

(NEAB)

14. Georgina buys a bar of chocolate.
The bar is divided into 18 equal pieces.



- (a) Georgina eats three pieces of chocolate.
What fraction of the bar has she eaten?
- (b) Later in the day Georgina eats $\frac{3}{5}$ of the pieces that are left.
How many pieces of chocolate have been eaten altogether?
- (c) What percentage of the bar has **not** been eaten?

(SEG)

11.4 More Complex Percentages

Not all percentages can be expressed as simple fractions and often figures such as 4.26% may need to be used. In these cases it is often better to work with decimals.



Worked Example 1

The cost of a hotel bill is £200. VAT at 17.5% has to be added to this bill. Find the VAT and the total bill.



Solution

Use $17.5\% = 0.175$.

Then

$$\begin{aligned} 17.5\% \text{ of } £200 &= 0.175 \times £200 \\ &= £35. \end{aligned}$$

So the total bill is

$$£200 + £35 = £235.$$



Worked Example 2

Imran has £486.27 in his building society account which earns interest of 8.21% per year. How much interest does he get and how much money does he have in his account after the first year?



Solution

Writing 8.21% as a decimal gives 0.0821.

$$\begin{aligned} 8.21\% \text{ of } \pounds 486.27 &= 0.0821 \times \pounds 486.27 \\ &= \pounds 39.92 \quad (\text{to the nearest penny}) \end{aligned}$$

So the account now contains

$$\pounds 486.27 + \pounds 39.92 = \pounds 526.19.$$



Worked Example 3

The cost of a large load of concrete blocks is £288 plus VAT at 17.5%. Find the total cost of the concrete blocks.



Solution

The problem can be solved in one stage by finding 117.5% of £288. This will give the original amount plus the VAT.

Note that 117.5% is 1.175 as a decimal.

So

$$\begin{aligned} 117.5\% \text{ of } \pounds 288 &= 1.175 \times \pounds 288 \\ &= \pounds 338.40. \end{aligned}$$

The total price is £338.40.



Worked Example 4

Jessica's salary of £12 000 is to be increased by 2.5%. Find her new salary.



Solution

Her new salary is 102.5% of her old salary.

$$\begin{aligned} 102.5\% \text{ of } \pounds 12\,000 &= 1.025 \times \pounds 12\,000 \\ &= \pounds 12\,300. \end{aligned}$$

Her new salary is £12 300.



Worked Example 5

A new car costs £9995, but a special offer gives an 8.5% discount. Find the discount price of the car.



Solution

With an 8.5% discount, 91.5% of the original price must be paid.

So

$$\begin{aligned} 91.5\% \text{ of } \pounds 9995 &= 0.915 \times \pounds 9995 \\ &= \pounds 9145.43 \quad (\text{to the nearest penny}) \end{aligned}$$

The discounted price is £9145.43.



Exercises

- Find each of the following, giving your answers to the nearest penny.
 - 32% of £50
 - 15% of £83
 - 12.6% of £40
 - 4.7% of £30
 - 6.9% of £52
 - 3.7% of £18.62
 - 0.8% of £4000
 - 92.3% of £211
 - 3.2% of £8.62
- Add 17.5% VAT to £415.
 - Add 3.2% interest to £1148.
 - Increase a salary of £15 000 by 1.6%.
 - Increase a price of £199 by 3.2%.
 - Decrease £420 by 7%.
 - Find the price of a £240 television offer with a 15% discount.
 - Find the price of an £11 999 car after a 22% discount.
- A portable CD player has a normal price of £150.
 - In a sale its normal price is reduced by 12%. Find the sale price.
 - After the sale, normal prices are increased by 2.5%. Find the new price of the CD player.
- An ice cream firm sells 20 000 ice-creams during one summer month. They expect sales to increase by 22% in the next month. How many ice-creams do they expect to sell?
- Peter earns £9000 per year in his new job. He does not pay tax on the first £3500 he earns and pays 25% tax on the rest. How much tax does he have to pay?
- Richard and Debbie cancel their holiday at short notice. The travel agents refund 65% of the £420 they had paid. How much money do Richard and Debbie lose?
- A chocolate manufacturer decides to introduce a range of *King Size* bars which are 35% larger than normal. A normal bar weighs 150 grams. What would a *King Size* bar weigh?
- A midi-hifi costs £186 plus VAT at $17\frac{1}{2}\%$. Its price is increased by 4%. How much would you have to pay to buy the midi-hifi at the new price?
- A company pays a Christmas bonus of £120 to each of its employees. This is taxed at 25%. One year they increase the bonus by 5%. How much does an employee take home?
- A new gas supplier offers a 20% discount on the normal price and a further 5% discount if customers pay directly from their banks. For one household the gas bill is normally £100. Find out how much they have to pay after both discounts.

11. A mountain bike costs £350 plus VAT.

VAT is charged at $17\frac{1}{2}\%$.

How much is the VAT?

(SEG)

- 12.

CASH
A discount of 15%
off the marked price
if you pay cash

SUPER 24" TV
£276

TERMS
A deposit of
 $\frac{1}{4}$ of the marked price
then 24 monthly payments
of £9.45 each

- (a) Mr. Smith buys the television set for cash. How much discount is he allowed?
- (b) Mr. Jones buys the set on terms.
- (i) How much must he pay as a deposit?
- (ii) Multiply 945 by 24 without using a calculator.
Show all your working.
- (iii) Work out the total price that Mr. Jones pays for his television set.

(MEG)

13. The usual price of a television set is £298 plus VAT at $17\frac{1}{2}\%$.

- (a) (i) Work out the exact value of $17\frac{1}{2}\%$ of £298.
(ii) What is the usual price of this television set?

GANNET STORE
BARGAIN OFFER!
You pay **NO VAT!**

BERRIES' STORE
SALE!
 $\frac{1}{6}$ OFF
USUAL PRICES

Gannet Store and Berries' Store are selling larger television sets at reduced prices. The usual price of these sets in both stores is £423 (£360 plus £63 VAT).

- (b) (i) Calculate the difference between the reduced prices in the two stores.
Show your working clearly.
- (ii) Which of the stores gives the bigger reduction?

(MEG)



Information

Did you know that a gallon in the UK is 20% bigger than a gallon in the USA?

11.5 Percentage Increase and Decrease

Percentage *increases* are calculated using

$$\text{Percentage increase} = \frac{\text{actual increase}}{\text{initial value}} \times 100\%$$

Similarly, percentage *decreases* are calculated using

$$\text{Percentage decrease} = \frac{\text{actual decrease}}{\text{initial value}} \times 100\%$$



Worked Example 1

The population of a village increased from 234 to 275 during one year. Find the percentage increase.



Solution

$$\text{Actual increase} = 275 - 234 = 41.$$

$$\begin{aligned} \text{Percentage increase} &= \frac{41}{234} \times 100\% \\ &= 17.52\% \quad (\text{to 2 decimal places}) \end{aligned}$$



Worked Example 2

When a beaker of sand is dried in a hot oven its mass reduces from 450 grams to 320 grams. Find the percentage reduction in its mass.



Solution

$$\begin{aligned} \text{Actual reduction} &= 450 \text{ grams} - 320 \text{ grams} \\ &= 130 \text{ grams.} \end{aligned}$$

$$\begin{aligned} \text{Percentage reduction} &= \frac{130}{450} \times 100\% \\ &= 28.9\% \end{aligned}$$



Worked Example 3

John buys calculators for £5 each and then sells them to other students for £6.90. Find his percentage profit.



Solution

$$\begin{aligned} \text{Actual profit} &= £6.90 - £5 \\ &= £1.90 \end{aligned}$$

$$\begin{aligned} \text{Percentage profit} &= \frac{1.90}{5} \times 100\% \\ &= 38\% \end{aligned}$$



Exercises

1. A baby weighed 5.6 kg and six weeks later her weight had increased to 6.8 kg. Find the percentage increase.
2. A factory produces video tapes at a cost of 88p and sells them for £1.10. Find the percentage profit.
3. A new car cost £11 500 and one year later it was sold for £9995. Find the percentage reduction in the value of the car.
4. An investor bought some shares at a price of £4.88 each. The price of the shares dropped to £3.96. Find the percentage loss.
5. A supermarket offers a £10 discount to all customers spending £40 or more. Karen spends £42.63 and John spends £78.82. Find the percentage saving for Karen and John.
6. After a special offer the price of baked beans was increased from 15p per tin to 21p per tin. Find the percentage increase in the price.
7. The size of a school increased so that it had 750 pupils instead of 680 and 38 teachers instead of 37. Find the percentage increases in the number of teachers and pupils. Comment on your answers.
8. In a science experiment the length of a spring increased by 4 cm to 20 cm. Find the percentage increase in the length of the spring.
9. The average cost of a local telephone call for one customer dropped by 8p to 27p. Find the percentage reduction in the average cost of a local call.
10. In a year, the value of a house increased from £46 000 to £48 000. Find the percentage increase in the value of the house and use this to estimate the value after another year.
11. A battery was tested and found to power a cassette player for 12 hours. An improved version of the battery powered the cassette player for an extra 30 minutes. Find the percentage increase in the life of the batteries.
12. The value of a car depreciates as shown in the table.

| <i>Vehicle</i> | <i>Value</i> |
|----------------|--------------|
| New | £12 000 |
| After 1 year | £10 000 |
| After 2 years | £ 8 800 |
| After 3 years | £ 8 000 |

During which year is the percentage decrease in the value of the car the greatest?

13.

Quality Garden Supplies
SUMMER SALE!

Save 20% on goods totalling
£30 or more.

- (a) Ken bought a ladder marked £35. How much did he save?
- (b) Tom needs a new spade. He can buy spade A which is marked £27.95 or spade B which is marked £32.45.
- (i) Calculate 20% of £32.45.
- (ii) How much cheaper would it be for Tom to buy spade B than to buy spade A?
- (c) Tom's wife suggests that he buys spade A, together with a plant costing £2.05 which she wants, so that he gets the 20% saving.
- If he buys the plant and spade A, express the saving as a percentage of the cost of spade A.

(MEG)

14.

Super Ace Games System

Normal Price £120

Sale Price $\frac{1}{3}$ off

- (a) Work out the sale price of the *Super Ace Games System*.

Mega Ace Games System

Normal Price £320

Sale Price £272

- (b) Find the percentage reduction on the *Mega Ace Games System* in the sale.

(LON)

15. Jimmy paid £120 for a CD player. He sold it for £105. What was his loss as a percentage of the price he paid?

(SEG)



Just For Fun

The growth rate of the human hair varies from person to person. On average, a human hair grows at a rate of 0.35 mm per day. If the length of a hair is 6 cm, how long will it take the hair to grow to a length of 26 cm?

11.6 Addition and Subtraction of Fractions



Note

The *numerator* is the **top** part of a fraction and the *denominator* is the **bottom** part of a fraction.

When adding or subtracting fractions they must have the same *denominator*.



Worked Example 1

$$\frac{4}{7} + \frac{5}{7} = ?$$



Solution

As both fractions have the same denominator (7), they can simply be added to give

$$\begin{aligned} \frac{4}{7} + \frac{5}{7} &= \frac{9}{7} \\ &= 1\frac{2}{7}. \end{aligned}$$



Worked Example 2

$$\frac{3}{4} + \frac{2}{5} = ?$$



Solution

As these fractions have different denominators, it is necessary to find the *lowest common denominator*, that is, the smallest number into which both denominators will divide exactly. In this case it is 20, since both 4 and 5 divide into 20 exactly.

$$\begin{aligned} \frac{3}{4} + \frac{2}{5} &= \frac{15}{20} + \frac{8}{20} \\ &= \frac{15 + 8}{20} \\ &= \frac{23}{20} \\ &= 1\frac{3}{20} \end{aligned}$$



Worked Example 3

$$\frac{2}{3} + \frac{7}{12} = ?$$



Solution

In this example, 12 is the lowest common denominator.

$$\begin{aligned} \frac{2}{3} + \frac{7}{12} &= \frac{8}{12} + \frac{7}{12} \\ &= \frac{8+7}{12} \\ &= \frac{15}{12} \\ &= 1\frac{3}{4} \\ &= 1\frac{1}{4} \end{aligned}$$



Worked Example 4

$$\frac{5}{8} - \frac{1}{3} = ?$$



Solution

Here 24 is the lowest common denominator.

$$\begin{aligned} \frac{5}{8} - \frac{1}{3} &= \frac{15}{24} - \frac{8}{24} \\ &= \frac{15-8}{24} \\ &= \frac{7}{24} \end{aligned}$$



Exercises

1. Give the answers to the following, simplifying them as far as possible.

(a) $\frac{1}{5} + \frac{1}{5}$

(b) $\frac{3}{8} + \frac{1}{8}$

(c) $\frac{5}{7} + \frac{1}{7}$

(d) $\frac{5}{7} - \frac{2}{7}$

(e) $\frac{8}{13} - \frac{5}{13}$

(f) $\frac{7}{9} - \frac{4}{9}$

(g) $\frac{7}{9} + \frac{8}{9}$

(h) $\frac{3}{5} + \frac{4}{5}$

(i) $\frac{6}{7} + \frac{5}{7}$

(j) $\frac{7}{10} - \frac{3}{10}$

(k) $\frac{8}{9} - \frac{5}{9}$

(l) $\frac{4}{15} - \frac{1}{15}$

2. Complete each of the following.

(a) $\frac{2}{5} + \frac{3}{7} = \frac{?}{35} + \frac{15}{35}$
 $= \frac{?}{35}$

(b) $\frac{1}{5} + \frac{1}{6} = \frac{?}{30} + \frac{?}{30}$
 $= \frac{?}{30}$

$$(c) \quad \frac{1}{2} + \frac{1}{4} = \frac{?}{4} + \frac{1}{4}$$

$$= \frac{?}{4}$$

$$(d) \quad \frac{3}{16} + \frac{5}{8} = \frac{3}{16} + \frac{?}{16}$$

$$= \frac{?}{16}$$

$$(e) \quad \frac{4}{7} + \frac{2}{3} = \frac{?}{21} + \frac{?}{21}$$

$$= \frac{?}{21}$$

$$(f) \quad \frac{3}{5} + \frac{7}{12} = \frac{?}{60} + \frac{?}{60}$$

$$= \frac{?}{60}$$

3. Find the answers to the following, simplifying them if possible.

$$(a) \quad \frac{1}{6} + \frac{3}{8}$$

$$(b) \quad \frac{5}{7} + \frac{2}{5}$$

$$(c) \quad \frac{1}{8} + \frac{3}{32}$$

$$(d) \quad \frac{1}{10} + \frac{1}{3}$$

$$(e) \quad \frac{3}{7} + \frac{5}{8}$$

$$(f) \quad \frac{1}{2} + \frac{2}{3}$$

$$(g) \quad \frac{1}{7} + \frac{1}{10}$$

$$(h) \quad \frac{5}{8} + \frac{4}{3}$$

$$(i) \quad \frac{6}{7} + \frac{2}{3}$$

$$(j) \quad \frac{4}{7} - \frac{1}{2}$$

$$(k) \quad \frac{6}{11} - \frac{1}{4}$$

$$(l) \quad \frac{2}{3} - \frac{1}{6}$$

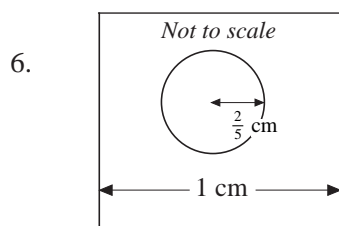
$$(m) \quad \frac{3}{4} - \frac{2}{3}$$

$$(n) \quad \frac{5}{8} - \frac{5}{12}$$

$$(o) \quad \frac{11}{12} - \frac{3}{8}$$

4. A garden has an area of $\frac{2}{5}$ hectare. The owner buys an extra $\frac{1}{3}$ hectare of land to increase the size of the garden. What is the new size of the garden?

5. A large company makes a profit of £ $\frac{3}{4}$ million in one year and £ $\frac{2}{3}$ million the next year. Find the total profits for the two-year period.



A hole of radius $\frac{2}{5}$ cm is drilled in the middle of a metal sheet of width 1 cm.

How far is it from the edge of the sheet to the hole?

7. A council decides to turn $\frac{1}{3}$ of a park into a dog-free zone. It later bans dogs from the play area which occupies $\frac{1}{10}$ of the park and which was originally outside the dog-free zone. What fraction of the park is now open to dogs?

8. Mike has filled $\frac{3}{5}$ of the space on the hard disc in his computer with software.

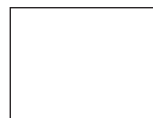
He wants to keep $\frac{1}{4}$ of the disc free from software. What fraction of the disc is left for extra software?

9. In a school $\frac{1}{3}$ of the children eat school dinners, $\frac{1}{2}$ bring packed lunches and the rest go home. What fraction of the children go home for lunch?
10. A shopper buys $1\frac{1}{4}$ kg of *Golden Delicious* apples and $1\frac{1}{3}$ kg of *Cox's* apples. Find the total weight of the apples bought.

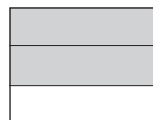
11.7 Multiplication and Division of Fractions

Multiplication

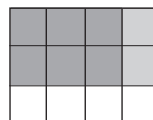
Consider finding $\frac{3}{4}$ of $\frac{2}{3}$ by starting with this rectangle.



First select $\frac{2}{3}$ of the rectangle, as shown by the shaded area.



Then select $\frac{3}{4}$ of the shaded area.



This represents $\frac{3}{4}$ of $\frac{2}{3}$ of the original rectangle, that is, $\frac{6}{12}$ or $\frac{1}{2}$ of the original rectangle.

Now $\frac{3}{4}$ of $\frac{2}{3}$ is the same as $\frac{3}{4} \times \frac{2}{3}$, so

$$\frac{3}{4} \times \frac{2}{3} = \frac{6}{12} = \frac{1}{2}.$$

When multiplying two fractions, the *numerators* (top parts) should be multiplied together to give the numerator of the result. Similarly, the two denominators should be multiplied together.

In general terms,

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$



Worked Example 1

$$\frac{3}{4} \times \frac{5}{7} = ?$$



Solution

$$\begin{aligned} \frac{3}{4} \times \frac{5}{7} &= \frac{3 \times 5}{4 \times 7} \\ &= \frac{15}{28} \end{aligned}$$



Worked Example 2

$$\frac{3}{5} \times \frac{7}{12} = ?$$



Solution

$$\begin{aligned} \frac{3}{5} \times \frac{7}{12} &= \frac{1 \times 7}{5 \times 4} \\ &= \frac{7}{20} \end{aligned}$$



Worked Example 3

$$1\frac{1}{2} \times 3\frac{4}{5} = ?$$



Solution

$$\begin{aligned} 1\frac{1}{2} \times 3\frac{4}{5} &= \frac{3}{2} \times \frac{19}{5} \\ &= \frac{57}{10} \\ &= 5\frac{7}{10} \end{aligned}$$

Division

To understand how to *divide* with fractions, first consider how multiplication and division are related.

Take as an example,

$$3 \times 4 = 12.$$

Then it is also true that

$$12 \div 4 = 3.$$

So ' $\times 4$ ' and ' $\div 4$ ' are *inverse* operations.

Note that

$$12 \times \frac{1}{4} = 3,$$

so $\div 4$ is the same as $\times \frac{1}{4}$.

Similarly, because $\div \frac{1}{2}$ is the same as $\times 2$,

$$6 \div \frac{1}{2} = 12 \quad (\text{check: } 12 \times \frac{1}{2} = 6)$$

and, alternatively, $6 \times 2 = 12$.

So $\div \frac{1}{2}$ is the same as $\times 2$.

You can generalise these examples to give

$$\div a \quad \text{is the same as} \quad \times \frac{1}{a}$$

$$\div \frac{1}{b} \quad \text{is the same as} \quad \times b$$

and combining the two results gives

$$\div \frac{a}{b} \quad \text{is the same as} \quad \times \frac{b}{a}.$$

For example,

$$\begin{aligned} 6 \div \frac{3}{4} &= 6 \times \frac{4}{3} \\ &= 8. \end{aligned}$$

(This result can be seen more easily from the diagram opposite.)

Similarly,

$$\begin{aligned} \frac{6}{20} \div \frac{2}{5} &= \frac{6}{20} \times \frac{5}{2} \\ &= \frac{3}{4} \end{aligned}$$

| | | |
|---|---------------|---------------|
| 1 | $\frac{3}{4}$ | |
| 2 | $\frac{3}{4}$ | $\frac{3}{4}$ |
| 3 | $\frac{3}{4}$ | |
| 4 | $\frac{3}{4}$ | |
| 5 | $\frac{3}{4}$ | $\frac{3}{4}$ |
| 6 | $\frac{3}{4}$ | |

So to divide by a fraction, the fraction should be *inverted*, that is, turned upside down, and then multiplied.

In general terms,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$



Worked Example 4

$$\frac{3}{4} \div \frac{7}{8} = ?$$



Solution

$$\begin{aligned} \frac{3}{4} \div \frac{7}{8} &= \frac{3}{4} \times \frac{8}{7} \\ &= \frac{3 \times 2}{1 \times 7} \\ &= \frac{6}{7} \end{aligned}$$



Exercises

1. Find each of the following, cancelling when possible.

(a) $\frac{3}{4} \times \frac{5}{7}$

(b) $\frac{1}{5} \times \frac{7}{8}$

(c) $\frac{4}{5} \times \frac{1}{12}$

(d) $\frac{3}{7} \times \frac{9}{10}$

(e) $\frac{4}{7} \times \frac{5}{8}$

(f) $\frac{6}{7} \times \frac{3}{4}$

(g) $\frac{2}{7} \times \frac{3}{8}$

(h) $\frac{1}{6} \times \frac{4}{7}$

(i) $\frac{3}{5} \times \frac{10}{9}$

(j) $1\frac{1}{2} \times 1\frac{1}{3}$

(k) $4\frac{1}{6} \times 2\frac{1}{2}$

(l) $1\frac{3}{4} \times 2\frac{1}{7}$

(m) $3\frac{3}{7} \times 4\frac{1}{5}$

(n) $5\frac{1}{2} \times 1\frac{3}{4}$

(o) $8\frac{1}{2} \times 3\frac{4}{7}$

(p) $2\frac{3}{4} \times 4\frac{1}{7}$

(q) $5\frac{3}{8} \times 1\frac{5}{6}$

(r) $1\frac{2}{7} \times 1\frac{3}{8}$

2. Find

(a) $\frac{3}{4} \div \frac{1}{2}$

(b) $\frac{6}{7} \div \frac{3}{4}$

(c) $\frac{1}{5} \div \frac{1}{7}$

(d) $\frac{3}{8} \div \frac{4}{5}$

(e) $\frac{3}{7} \div \frac{9}{10}$

(f) $\frac{7}{4} \div \frac{2}{5}$

(g) $1\frac{1}{4} \div \frac{3}{4}$

(h) $5\frac{1}{2} \div \frac{1}{4}$

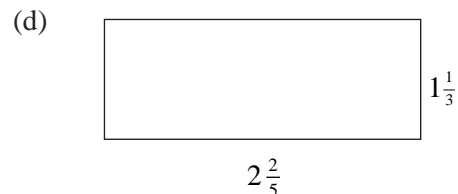
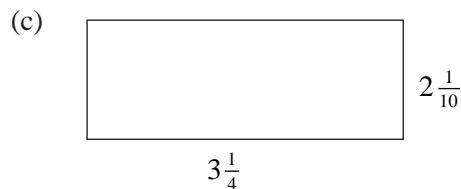
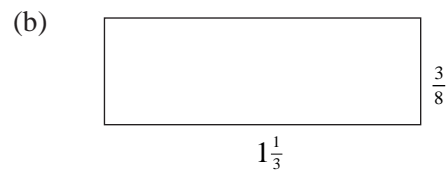
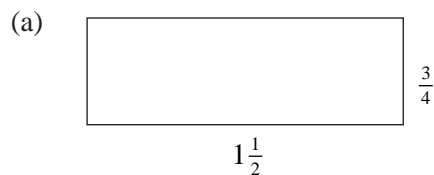
(i) $1\frac{1}{7} \div 2\frac{3}{8}$

(j) $4\frac{1}{2} \div 1\frac{1}{5}$

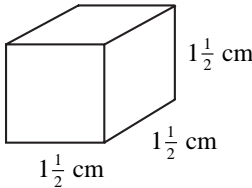

(k) $1\frac{3}{4} \div 1\frac{5}{8}$

(l) $3\frac{1}{7} \div 1\frac{7}{8}$

3. Find the area of each rectangle below.



4. In a garden, $\frac{1}{2}$ of it is used for growing vegetables and $\frac{1}{4}$ of this vegetable area for potatoes. What fraction of the garden is used for growing potatoes?

5. In a school, $\frac{4}{7}$ of the children are boys and $\frac{1}{10}$ of these are colour-blind.
What fraction of the school are colour-blind boys?
6. A cube is made with sides of length $1\frac{1}{2}$ cm.
Find the volume and surface area of the cube.
- 
7. A petrol can holds $5\frac{1}{2}$ litres when full. How much petrol is in the can if it is $\frac{3}{4}$ full?
8. A car travels at 50 m.p.h. for $\frac{3}{4}$ hour. How far does the car travel?
9. Find the length of the unmarked side of this rectangle if its area is $1\frac{1}{2}$ m².
- 
10. A recipe requires $\frac{1}{4}$ kg of sugar for a cake. How many cakes could be made with $1\frac{3}{4}$ kg of sugar?
11. Alison cycles 3 miles in $\frac{2}{3}$ hour. What is her speed?
12. It takes a factory $\frac{3}{4}$ hour to assemble a finished product. How many items could be assembled in an 8 hour day?

11.8 Compound Interest and Depreciation

When money is invested the interest is often *compounded*, which means that interest is given on the interest.



Worked Example 1

A person invests £200 in a building society account which pays 4% interest each year. Find the value of the investment after 3 years.



Solution

Interest of 4% will be added at the end of each year by multiplying by 1.04.

- So, value of account after 1 year: $£200 \times 1.04 = £208$
 value of account after 2 years: $£208 \times 1.04 = £216.32$
 value of account after 3 years: $£216.32 \times 1.04 = £224.97$.

Note that the amount of interest added increases each year.

The final value could have been found in one calculation:

$$£200 \times 1.04^3 = £224.97 .$$



Worked Example 2

When Gemma was born, her grandmother invested £200 in a building society for her. Find the value of this investment after 18 years if the interest rate is 6% per year.



Solution

$$\begin{aligned} \text{Final value} &= £200 \times 1.06^{18} \\ &= £570.87. \end{aligned}$$

Problems with *depreciation* can be tackled in a similar way.



Worked Example 3

A car was bought for £14 000. Its value decreases by 8% each year. Find the value of the car after:

- (a) 1 year (b) 5 years (c) 10 years.



Solution

Decreasing the value by 8% leaves 92% of the original value.

$$\begin{aligned} \text{(a) Value after one year} &= £14\,000 \times 0.92 \\ &= £12\,880 \end{aligned}$$

$$\begin{aligned} \text{(b) Value after 5 years} &= £14\,000 \times 0.92^5 \\ &= £9227.14 \end{aligned}$$

$$\begin{aligned} \text{(c) Value after 10 years} &= £14\,000 \times 0.92^{10} \\ &= £6081.44 \end{aligned}$$



Note

You can see from these worked examples that the total amount in an account after n years, A_n , with interest of $r\%$ is given by

$$A_n = \left(1 + \frac{r}{100}\right)^n A_0$$

where A_0 is the initial sum invested.



Exercises

- Jane invests £1200 in a bank account which earns interest at the rate of 6% per annum. Find the value of her investment after:
(a) 1 year (b) 2 years (c) 5 years.
- A sum of £5000 is to be invested for 10 years. What is the final value of the investment if the annual interest rate is:
(a) 5% (b) 4.8% (c) 7.2%?
- Which of the following investments would earn most interest?
A £300 for 5 years at 2% interest per annum,
B £500 for 1 year at 3% interest per annum,
C £200 for 3 years at 8% interest per annum
- The value of a computer depreciates at a rate of 25% per annum. A new computer costs £1600. What will the value of the computer be after:
(a) 2 years (b) 6 years (c) 10 years?
- A car costs £9000 and depreciates at a rate of 20% per annum. Find the value of the car after 3 years.
- John invests £500 in a building society with interest of 8.4% per annum. Karen invests £200 at the same rate.
(a) How many years does it take for the value of Karen's investment to become greater than £300?
(b) How many years does it take for the value of John's investment to become greater than
(i) £700 (ii) £900?
- If the rate of inflation were to remain constant at 3%, find what the price of a jar of coffee, currently priced at £1.58, would be in 4 years' time.
- The population of a third world country is 42 million and growing at 2.5% per annum.
(a) What size will the population be in 3 years' time?
(b) In how many years' time will the population exceed 50 million?
- The value of a car depreciates at 15% per annum. A man keeps a car for 4 years and then sells it.
(a) If the car initially cost £6000, find:
(i) its value after 4 years,
(ii) the selling price as a percentage of the original value.

- (b) Repeat (a) for a car which cost £12 000.
- (c) Comment on your answers.
10. A couple borrow £1000 to furnish their new home. They have to pay interest of 18% on this amount.
- (a) Find the amount of interest which would be charged at the end of the first year.
- (b) If they repay £300 at the end of each year, how much do they owe at the end of the third year of the loan?

11.9 Reverse Percentage Problems

Sometimes it is necessary to *reverse* percentage problems. For example if the price of a television includes VAT, you might need to know how much of the price is the VAT.



Worked Example 1

The price of a computer is £1410, including VAT at $17\frac{1}{2}\%$. Find the actual cost of the computer and the amount of VAT which has to be paid.



Solution

To add 17.5% VAT to a price it should be *multiplied* by 1.175. So to remove the VAT it should be *divided* by 1.175.

$$\begin{aligned}\text{Original Price} &= \frac{\pounds 1410}{1.175} \\ &= \pounds 1200.\end{aligned}$$

$$\begin{aligned}\text{VAT} &= \pounds 1410 - \pounds 1200 \\ &= \pounds 210.\end{aligned}$$



Worked Example 2

A customer is offered a 20% discount when buying a new bed. The discounted price is £158.40. Find the full price of the bed.



Solution

To find the discounted price of the bed, the full price should be *multiplied* by 0.8. So to find the full price, the discounted price should be *divided* by 0.8.

$$\begin{aligned}\text{Full price} &= \frac{\pounds 158.40}{0.8} \\ &= \pounds 198.\end{aligned}$$



Worked Example 3

Sharon invests some money in a building society at 6% interest per annum. After two years the value of her investment is £280.90. Find the amount she invested.



Solution

To find the final value, the amount invested would be *multiplied* by 1.06^2 .

To find the amount invested, *divide* the final value by 1.06^2 .

$$\begin{aligned} \text{Amount invested} &= \frac{\pounds 280.90}{1.06^2} \\ &= \pounds 250. \end{aligned}$$



Exercises

1. A foreign tourist can reclaim the VAT he has paid on the following items, the prices of which include VAT.

| | |
|--------------------|---------|
| Video Camera | £149.60 |
| Portable CD Player | £110.45 |
| Watch | £42.77 |
| FAX Machine | £406.08 |

- (a) Find the total cost of the items without VAT at 17.5%.
- (b) How much VAT can the tourist reclaim?
2. The price of a television is £225.60 including 17.5% VAT. What would be the price with no VAT?
3. A gas bill of £43.45 includes VAT at 8%. Find the amount of VAT paid.
4. The end of year profits of a large company increased this year by 12% to £90 944. Find the profits made last year.
5. A special bottle of washing up liquid contains 715 ml of liquid. The bottle is marked '30% extra free'. How much liquid is there in a normal bottle?
6. In a sale the following items are offered at discount prices as listed.

| <i>Item</i> | <i>Sale Price</i> | <i>Discount</i> |
|----------------|-------------------|-----------------|
| Television | £288.00 | 10% |
| Video Recorder | £373.12 | 12% |
| Computer | £1124.80 | 24% |
| Calculator | £13.78 | 5% |

What were the prices of these items before the sale?

7. After one year, the value of a car has fallen by 15% to £8330. What was the value of the car at the beginning of the year?

8. A sum is invested in a building society at 4% interest per annum and after 3 years the value of the investment is £562.43. How much was originally invested?
9. Jenny's pocket money is increased by 25% each year on her birthday. When she is 16 years old, her pocket money is £12.86 per week. How much did she get per week when she was:
- (a) 15 years old (b) 13 years old (c) 10 years old?
10. Jai buys a car, keeps it for 4 years and then sells it for £2100. If the value of the car has depreciated by 12% per year, how much did Jai originally pay for the car?



Information

Did you know that in 1996 a Japanese mathematician (using a computer!) took just 5 days to compute the value of π to over 6 billion digits.



Just For Fun

The sum of $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ of the enrolment of School A is exactly the enrolment of School B.

The sum of $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$ and $\frac{1}{8}$ of the enrolment of School A is exactly the enrolment of School C.

What are the enrolments of these three schools, assuming that no school has more than 1 000 pupils?



Information

The Chinese represented negative numbers by indicating them in red and the Hindus denoted them by putting a circle or a dot over the numbers. The Chinese had knowledge of negative numbers as early as 200 BC and the Hindus as early as the 7th century.

In Europe, as late as the 16th century, some scholars still regarded negative numbers as absurd. In 1545, Cardano (1501–1570), an Italian scholar, called positive numbers 'true' and negative numbers 'fictitious' numbers.