14 Loci and Transformations
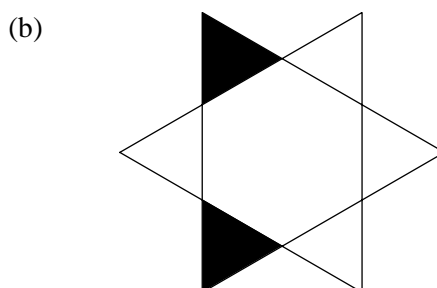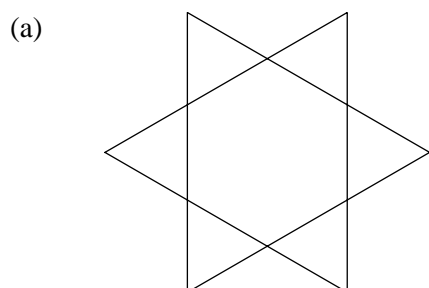
14.1 Drawing and Symmetry

This section revises the ideas of symmetry first introduced in Unit 3 and gives you practice in drawing simple shapes.
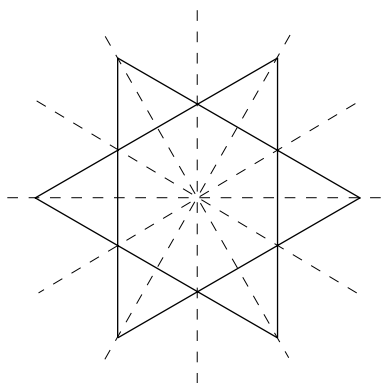
Worked Example 1

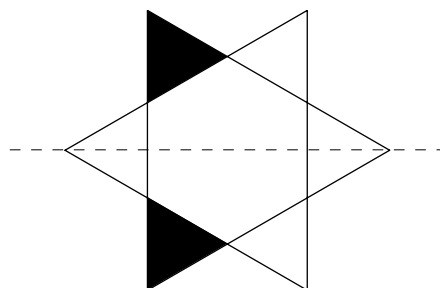Describe the symmetries of each shape below.

(a)

(b)

Solution

(a) This shape has 6 lines of symmetry, as shown in the diagram.

(b) This shape has one line of symmetry as shown below.

It has rotational symmetry of order 6 as it can be rotated about its centre to 6 different positions.

It has rotational symmetry of order 1, since it can rotate a full 360° back to its original position.

Worked Example 2

Draw accurately a rectangle with sides of length 8 cm and 5 cm.
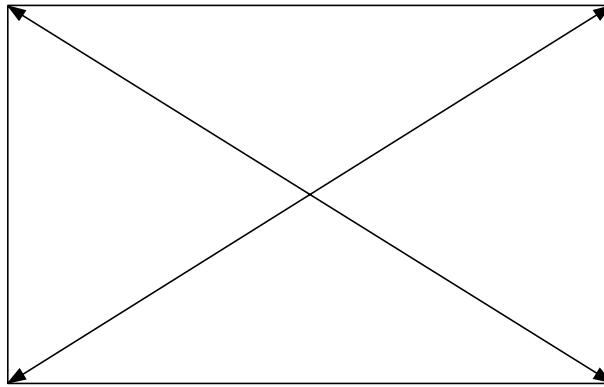
Solution

First draw a line 8 cm long.

Then draw lines 5 cm long at each end, making sure they are at right angles to the base line.

Finally, join these two lines to complete the rectangle.

Measure the diagonals and check that they are both the same length.

Exercises

1. Draw accurately rectangles with the following sizes.
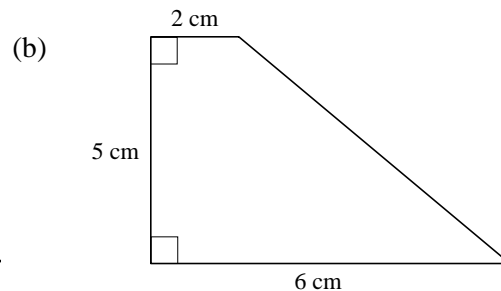
   (a) 3 cm by 8 cm   (b) 10 cm by 3 cm

   (c) 6 cm by 7 cm   (d) 6 cm by 4 cm

   For each rectangle check that both diagonals are the same length.
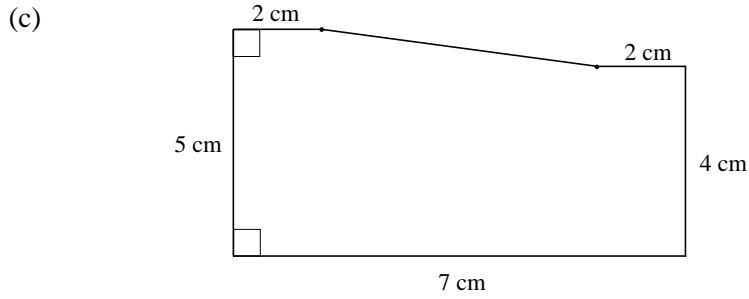
2. Make accurate drawings of each of the shapes shown below and answer the question below each shape.
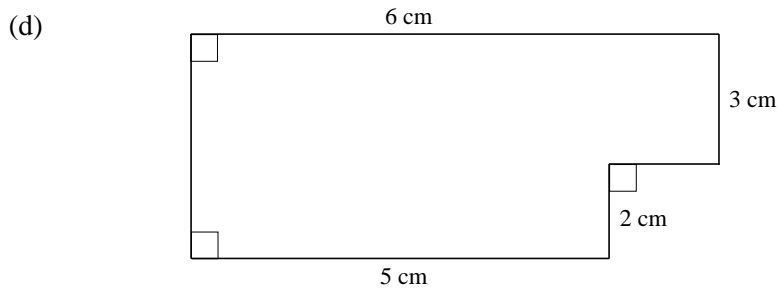
   (a)

   5 cm

   4 cm

   8 cm

   What is the length
   of the sloping side?

   (b)

   2 cm

   5 cm

   6 cm
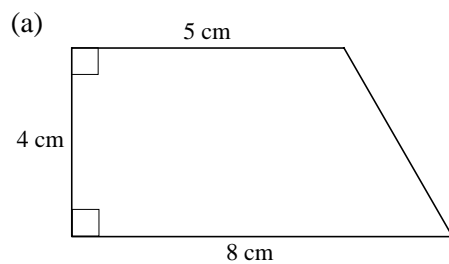
   What is the length of the
   longest diagonal?

(c)

2 cm

2 cm

5 cm

4 cm

7 cm

What is the length of the sloping side?

(d)

6 cm

3 cm

2 cm

5 cm

What is the length of the longest straight line which can be drawn inside the shape?

3.   Each shape below includes a semi-circle. Make an accurate drawing of each shape and state the radius of the semi-circle.
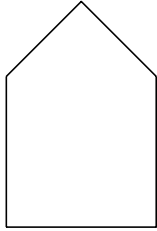
(a)

5 cm   5 cm

4 cm

(b)

3 cm

4 cm

(c)

2 cm

4 cm   4 cm

8 cm

(d)

10 cm

4 cm

8 cm
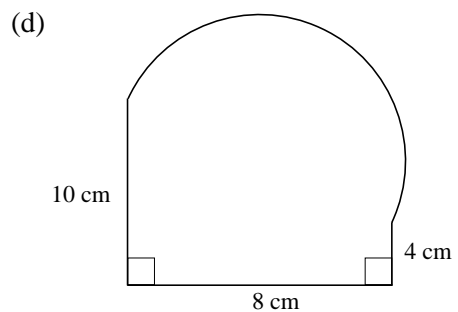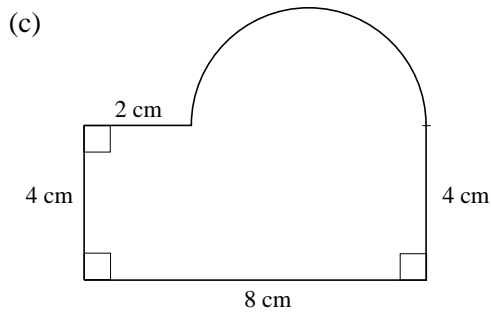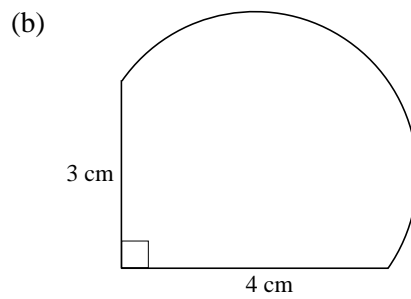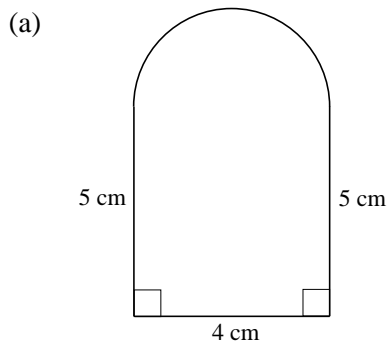
Investigation

Are human beings symmetrical? Do you know of any animal that is not symmetrical?

4. For each shape below:

   (i) state the order of rotational symmetry,

   (ii) copy the shape and draw any line of symmetry.

   (a)

   (b)
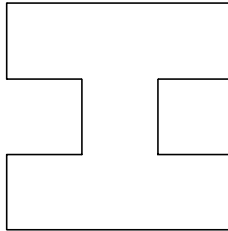
   (c)
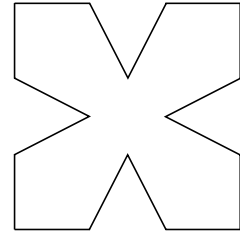
   (d)
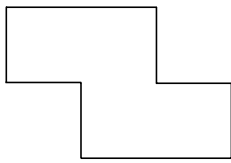
   (e)

   (f)

5. (a) Copy and shade part of the shape below so that it has 3 lines of symmetry.

   (b) What is the order of rotational symmetry of the shape?

6. State the number of lines of symmetry and the order of rotational symmetry for each shape below.

   (a)

   (b)
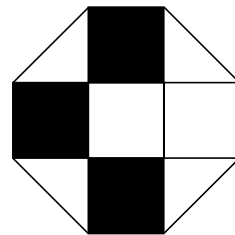
   (c)
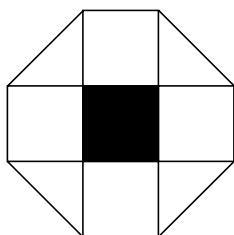
   (d)
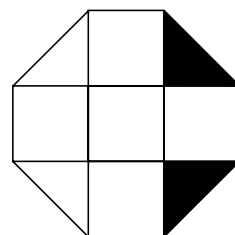
7.   Which of the shapes below have:

(a)   rotational symmetry of order 1   (b)   no lines of symmetry

(c)   more than two lines of symmetry   (d)   rotational symmetry of order 2

(e)   rotational symmetry of an order greater than 2?

A   B   C

D   E   F
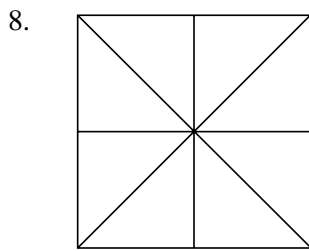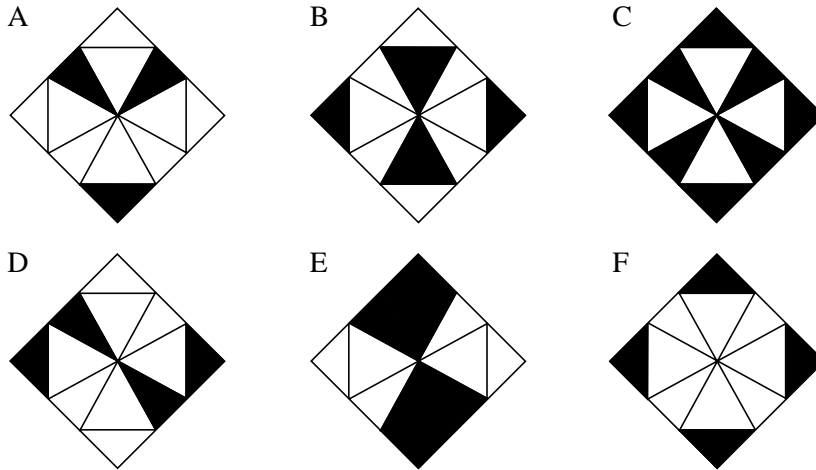
8.   Make 4 copies of the shape opposite. Shade triangles in the shape to produce shapes with:

(a)   2 lines of symmetry,

(b)   one line of symmetry,

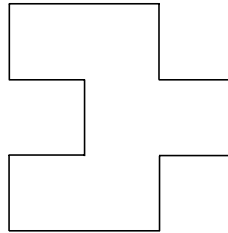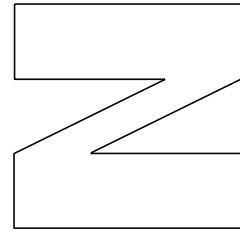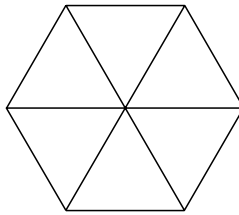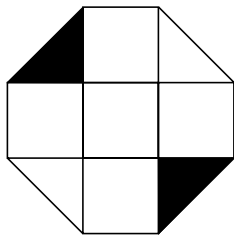(c)   rotational symmetry of order 2,

(d)   rotational symmetry of order 4.

14.2 Scale Drawings

Scale drawings are often used to produce plans for houses or new kitchens. For example, a scale of 1: 40 means that 1 cm on the plan is equivalent to 40 cm in reality.

Worked Example 1

The diagram shows a scale drawing of the end wall of a house. The scale used is 1:100.

Find:

(a)   the height of the top of the chimney,

(b)   the height of the door,

(c)   the width of the house.
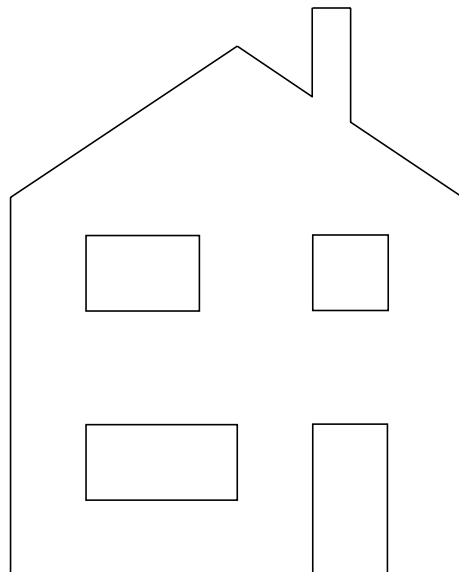
Solution

As the scale is 1:100, every 1 cm on the drawing represents 100 cm or 1 m in reality.

(a)   The height of the top of the chimney is 7.5 cm on the drawing. This corresponds to
$7.5 \times 100 = 750$ cm or 7.5 m in reality.

(b)   The height of the door is 2 cm on the drawing and so is 2 m in reality.

(c)   The width of the house is 6 cm on the drawing and so is 6 m in reality.

Worked Example 2

The diagram shows a rough sketch of the layout of a garden.
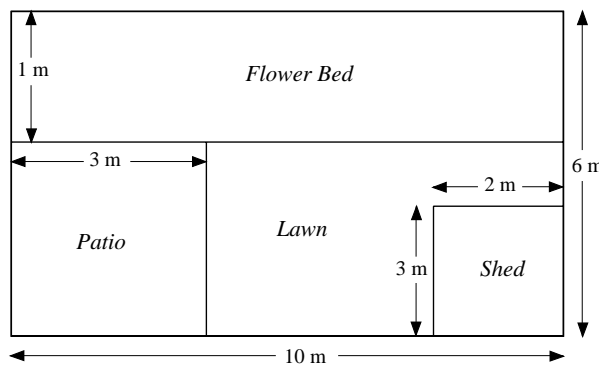
Produce a scale drawing with a scale of 1: 200.

Solution

A scale of 1: 200 means that 1 cm on the plan represents 200 cm or 2 m in reality.
The table below lists the sizes of each part of the garden and the size on the scale drawing.

Area	Real Size	Size on Drawing
Garden	10 m $\times$ 6 m	5 cm $\times$ 3 cm
Patio	5 m $\times$ 3 m	2.5 cm $\times$ 1.5 cm
Lawn	7 m $\times$ 5 m	3.5 cm $\times$ 2.5 cm
Flower Bed	10 m $\times$ 1 m	5 cm $\times$ 0.5 cm
Shed	3 m $\times$ 2 m	1.5 cm $\times$ 1 cm

A scale drawing can now be
produced and is shown opposite
drawn on squared paper.
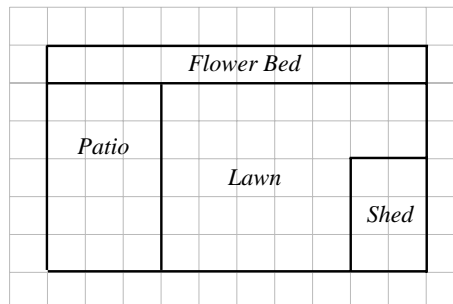
Exercises

1.   The scale drawing below of a flat has been drawn on a scale of 1: 200.
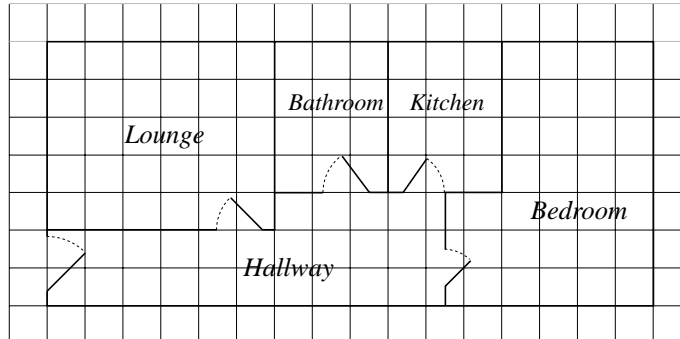
(a)   Find the actual sizes of the lounge and bathroom.

(b)   Find the area of the bedroom floor.

(c)   Find the length of the hallway.

2.
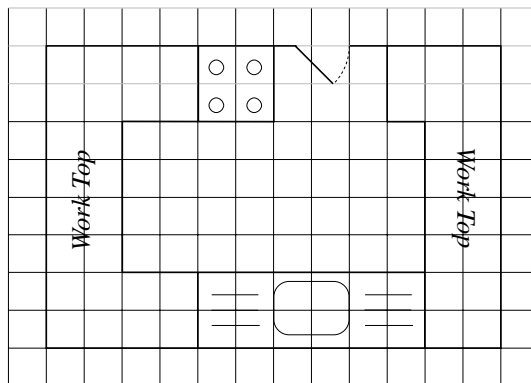
The scale drawing shown a plan of a kitchen on a scale of 1: 60.

(a)   What are the length and width of the kitchen?

(b)   What is the size of the cooker?

(c)   What is the size of the sink?

(d)   Find the area of the worktops in the kitchen.

3.   A rough sketch is made of a set of offices. It is shown below.
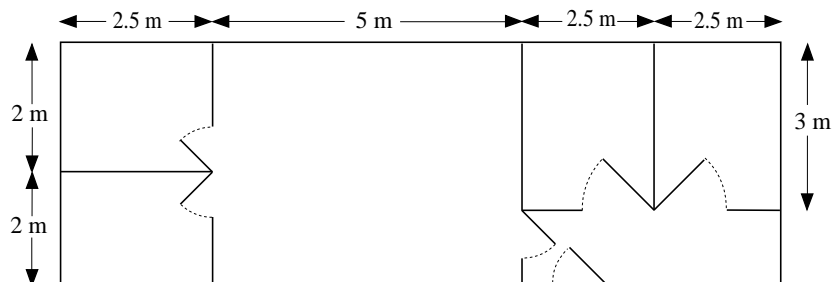
Use the information given to produce a scale drawing with a scale of 1: 200.

85

4.   Hannah produces a scale drawing of her ideal bedroom on a scale of 1: 50.
      The plan is shown below.

      (a)   What is the size of the room?   (b)   How long is her bed?

      (c)   What is the area of the top of her desk?

      (d)   What is the floor area of the room?

5.   The diagram shows a rough sketch of the
      end wall of a house.
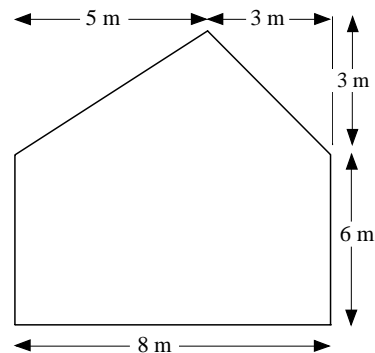
      (a)   Produce a scale drawing using a
             scale of 1:100.

      (b)   Use the drawing to find the sloping
             lengths of both sides of the roof.

6.   The diagram shows a scale
      drawing of a garden.

      It is drawn with a scale of 1: 80.
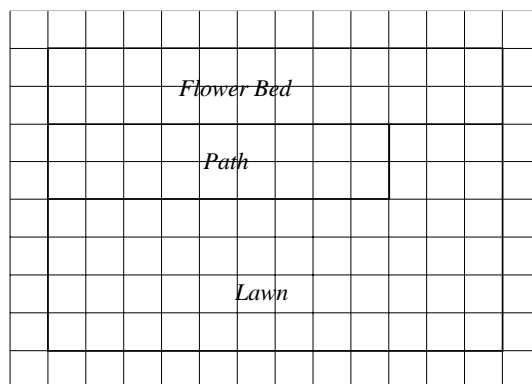
      (a)   How wide is the garden?

      (b)   How long is the path?

      (c)   A shed with a base of size
             2 m by 1.5 m is to be added
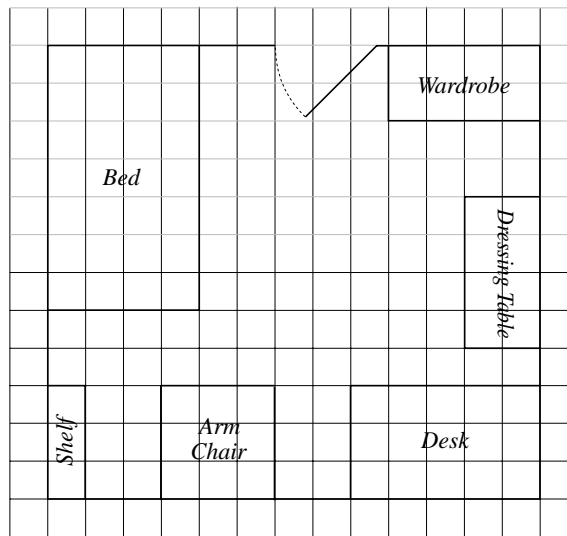   to the plan.
             Find the size of the rectangle
             that should be drawn on the plan.

      (d)   The garden also contains a pond of radius 0.6 m. What would be the radius
             of the circle which should be added to the plan?

7. A classroom is rectangular with width 4 m and length 5 m.

   What would be the size of the rectangle used to represent the classroom on plans with a scale of:

   (a)   1: 50   (b)   1: 25   (c)   1: 100?

8. The diagram shows a scale plan of a small industrial estate drawn on a scale of 1: 500.
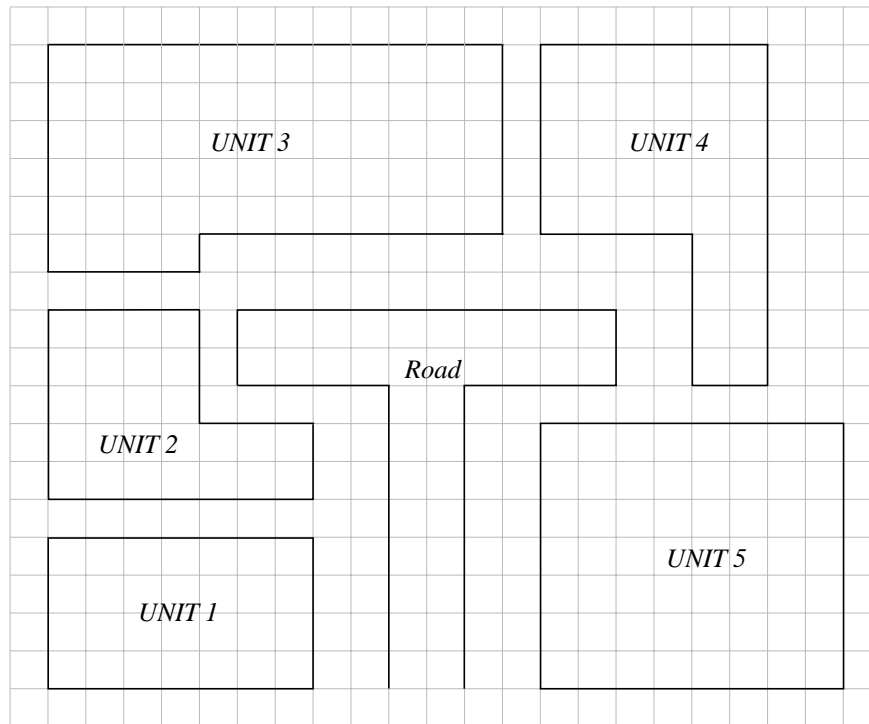
   (a)   Find the floor area of each unit.

   (b)   Re-draw the plan with a scale of 1:1000.

9. The plan of a house has been drawn using a scale of 1: 20.

   (a)   (i)   On the plan, the length of the lounge is 25 cm. What is the actual length of the lounge in metres?

           (ii)   The actual lounge is 3.2 m wide. How wide is the lounge on the plan?

   (b)   The actual kitchen is 2.6 m wide. Estimate, in feet, the width of the actual kitchen.

   (SEG)

10. A classroom is drawn on a plan using a scale of 1: 50.

    (a)   On the plan, how many centimetres represent one metre?

   (b)   The width of the classroom is 6.7 m. How many centimetres represent this width on the plan?

    (SEG)

11.   Byron and Shelley are two dogs.

    (a)   Byron's lead is 1 m long. One end of the lead can slide along a railing,
           which is fixed to the wall of the house.

           Shelley's lead is 1.5 m long. One end of this lead is attached to a post, A, at
           the corner of his kennel.

           The scale diagram below represents the fenced garden, PQRS, where the
           dogs live.

Scale
2 cm
represents
1 m
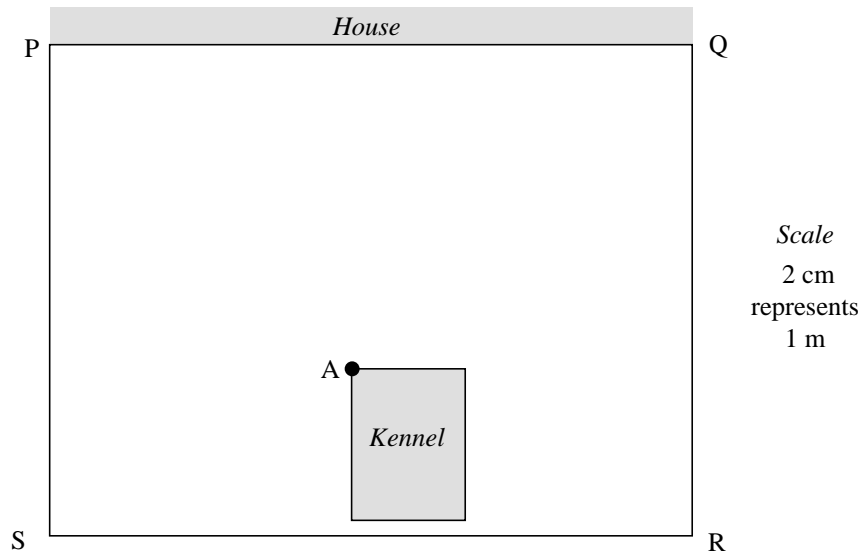
           Copy the diagram and show on your drawing all the possible positions of
           each dog if the leads remain tight.

    (b)

Not to scale
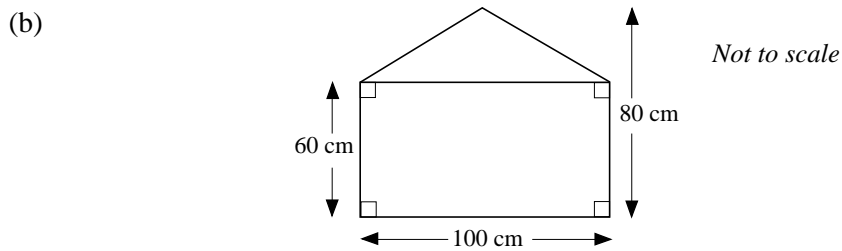
           The diagram represents the cross-section of Shelley's kennel. Calculate the
   area of this cross-section, giving your answer in cm^2.

(MEG)

Investigation

Look at an atlas and find out the scales used in maps of the UK, Europe and the world.
Are the same scales used for all the different maps? If not, why not?

12. The diagram is an isometric drawing of a kitchen with cupboards at floor level.
The kitchen is a cuboid 300 cm wide, 450 cm long and 250 cm high.
The work top above the cupboards is 100 cm above the floor and 50 cm wide.
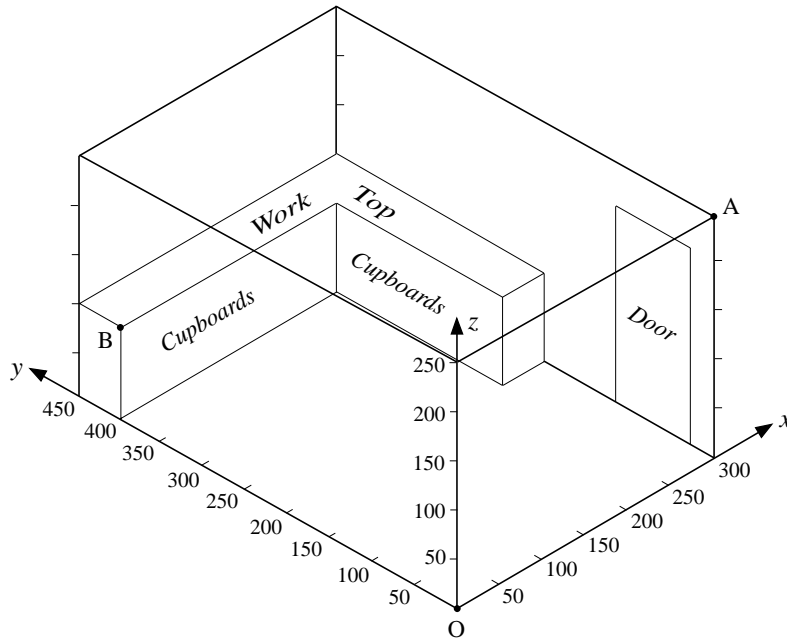
(a) Another cupboard is to be fixed above the work top on the shorter wall.
It is a cuboid 300 cm long, 50 cm high and 25 cm from back to front.
Its top is to be 50 cm below the ceiling.

Draw this cupboard on a copy of the diagram.

(b) Axes, O*x*, O*y*, O*z* are taken as shown in the diagram. Write down the
coordinates of

(i) the point A (ii) the point B.

(c) This is a scale drawing of the floor of the kitchen.

The part of the floor within 50 cm of the cupboards must be kept clear.

On a copy of the diagram, show clearly and accurately this part of the floor.
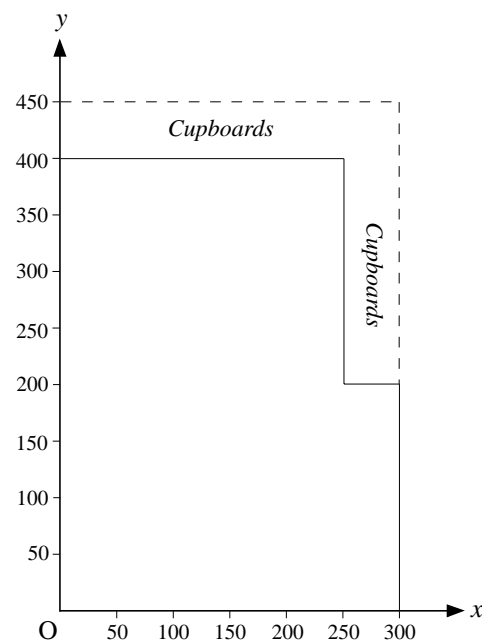
(MEG)

14.3 Constructing Triangles and Other Shapes

A protractor and a compass can be used to produce accurate drawings of triangles and other shapes.
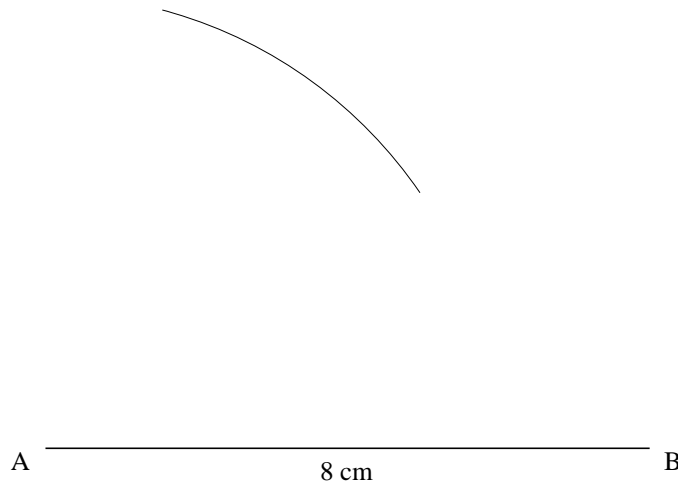
Worked Example 1

Draw a triangle with sides of length 8 cm, 6 cm and 6 cm.
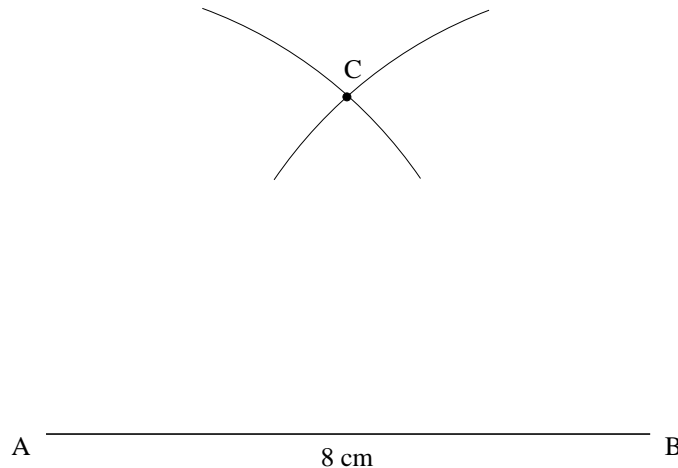
Solution

First draw a line of length 8 cm.

Then set the distance between the point and pencil of your compass to 6 cm and draw an arc with centre A as shown below.

A ————————————————————— B
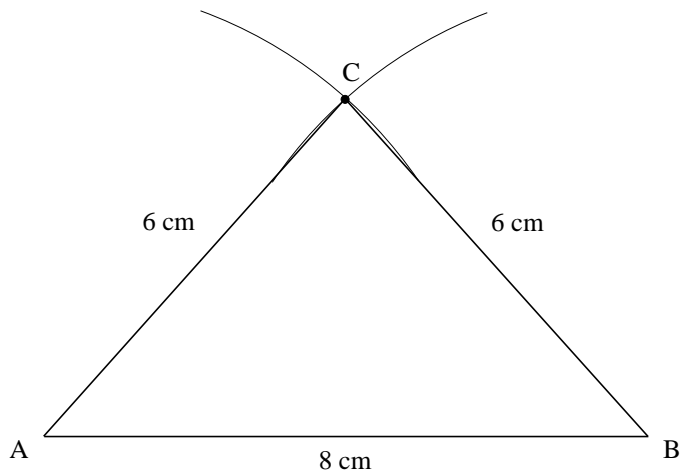   8 cm

The arc is a distance of 6 cm from A.

With your compass set so that the distance between the point and the pencil is still 6 cm, draw an arc centred at B, as shown below.
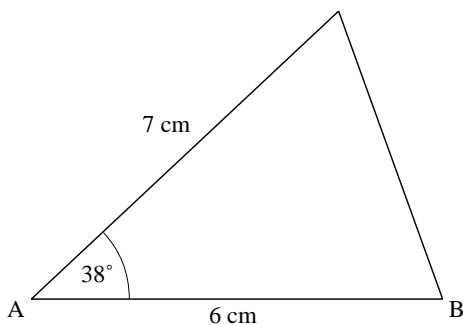
C

A ————————————————————— B
                    8 cm

The point, C, where the two arcs intersect is the third corner of the triangle.

The triangle can now be completed.

Worked Example 2

The diagram shows a rough sketch of a triangle.
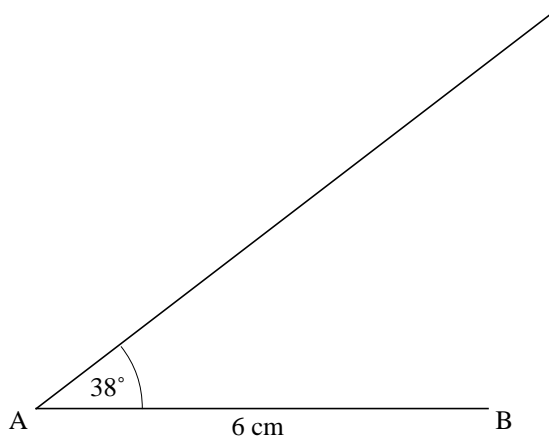

Make an accurate drawing of the triangle and find the length of the third side.
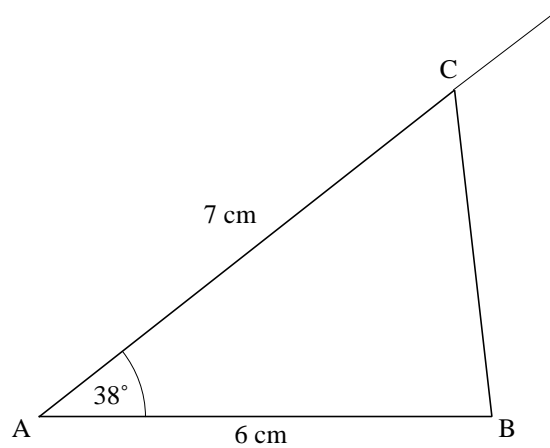
Solution

First draw a line of length 6 cm and measure an angle of 38°.

Then measure 7 cm along the line and the triangle can be completed.

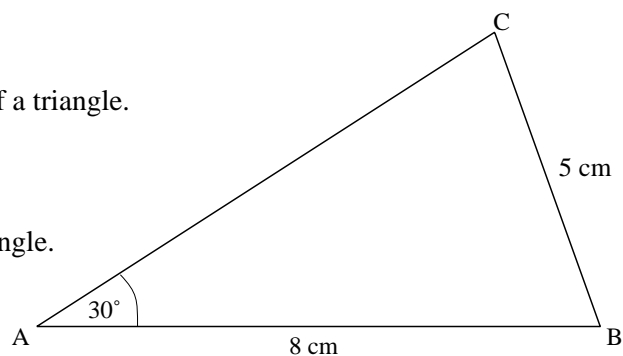The third side of the triangle can then be measured as 4.3 cm.
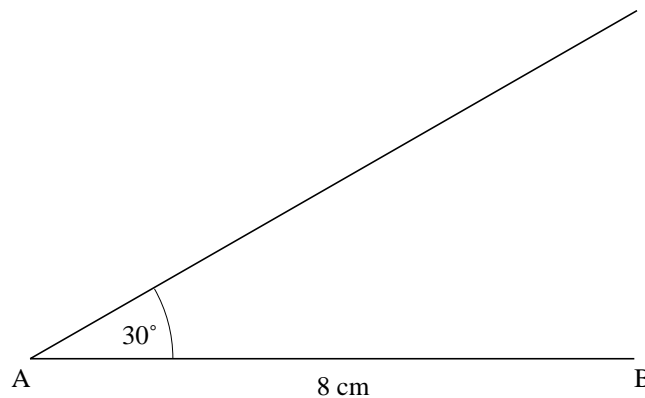
Worked Example 3

The diagram shows a rough sketch of a triangle.

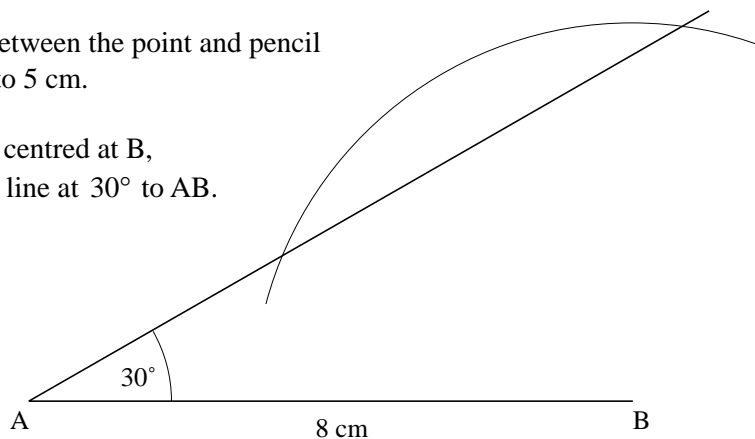Make an accurate drawing of the triangle.

Solution

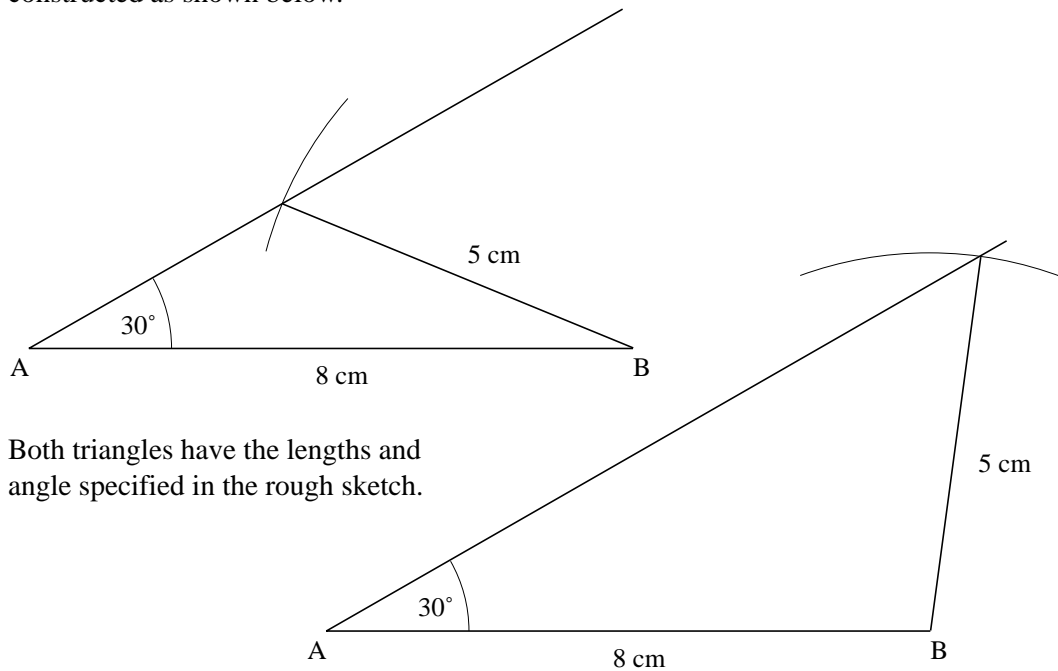First draw the side of length 8 cm and measure the angle of $30°$, as shown below.

Set the distance between the point and pencil
of your compass to 5 cm.

Then draw an arc centred at B,
which crosses the line at $30°$ to AB.

As the arc crosses the line in two places, there are two possible triangles that can be constructed as shown below.

Both triangles have the lengths and angle specified in the rough sketch.

Note

An arc must be taken when constructing triangles to ensure that all possibilities are considered.

Exercises

1.   Draw triangles with sides of the following lengths.

    (a)   10 cm, 6 cm, 7 cm   (b)   5 cm, 3 cm, 6 cm

    (c)   4 cm, 7 cm, 6 cm

2.   Draw accurately the triangles shown in the rough sketches below and answer the question given below each sketch.

    (a)

        How long is the side BC?

    (b)
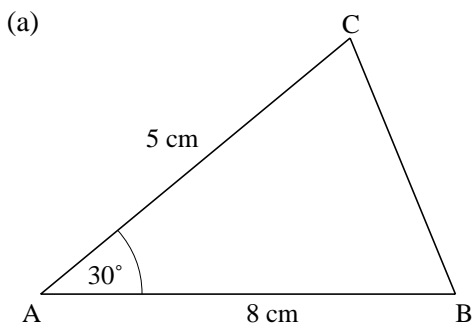
        How long are the sides AC and BC?

(c)

C

120°

5 cm   7 cm

A   B

How long is the side AB?

(d)

C

6 cm   7 cm

A   8 cm   B
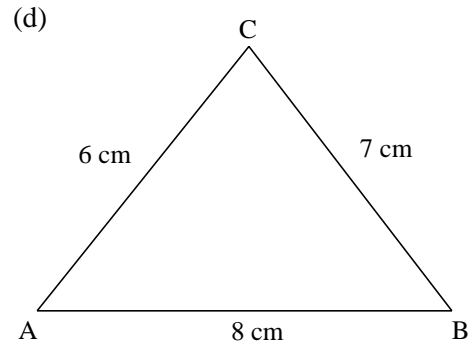
What is the size of the angle ABC?

(e)

C

80°

70°

A   7 cm   B
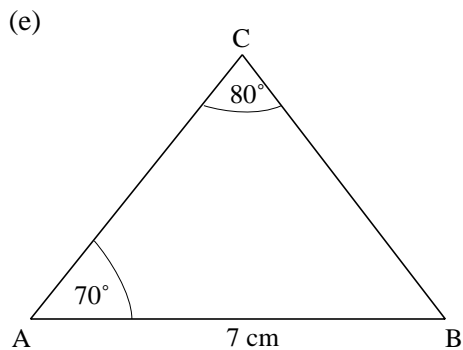
How long is the side AC?

(f)

C

8 cm

40°

A   6 cm   B
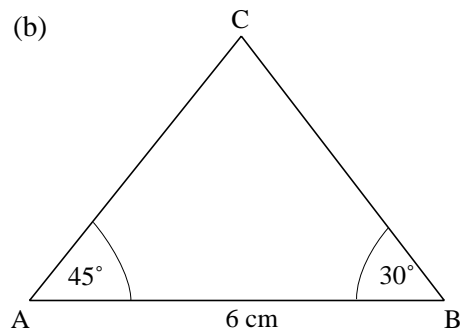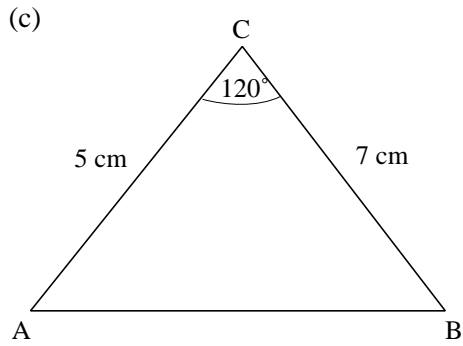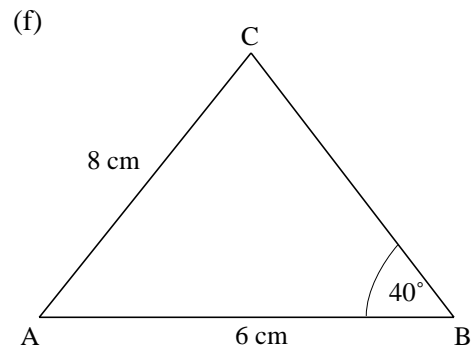
How long is the side BC?

3.   An isosceles triangle has a base of length 6 cm and base angles of 50°. Find the lengths of the other sides of the triangle.

4.   An isosceles triangle has 2 sides of length 8 cm and one side of length 4 cm. Find the sizes of all the angles in the triangle.

5.   Draw an equilateral triangle with sides of length 5 cm.

6.   For each rough sketch shown below, draw two possible triangles.

(a)

5 cm

45°

6 cm

(b)

4.5 cm

60°

5 cm

(c)

5 cm

30°

7 cm

(d)

3.5 cm

50°

4 cm
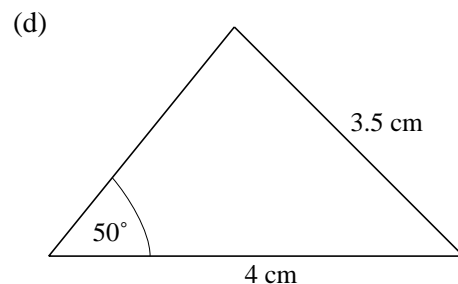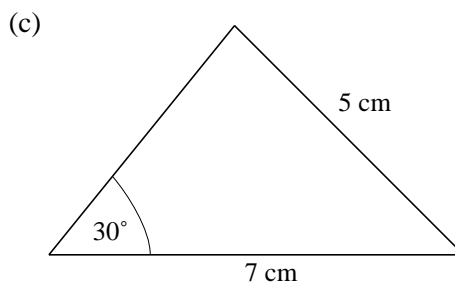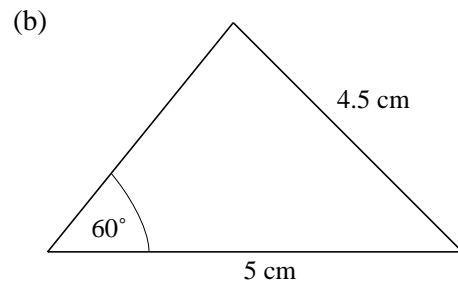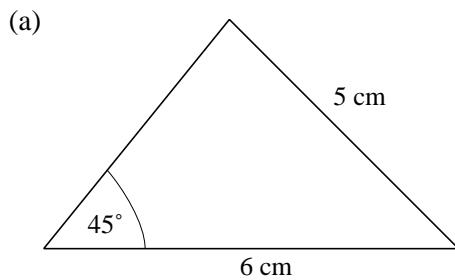
7. (a) Draw accurately the parallelogram shown below.

5 cm

60°

4 cm
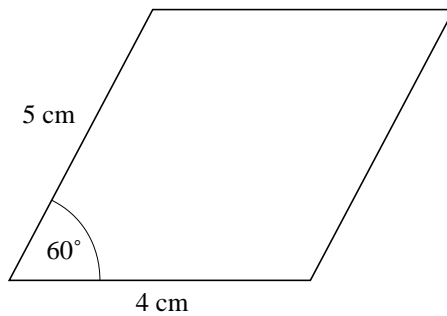
(b) Measure the two diagonals of the parallelogram.

8. (a) Draw the kite shown in the rough sketch opposite.
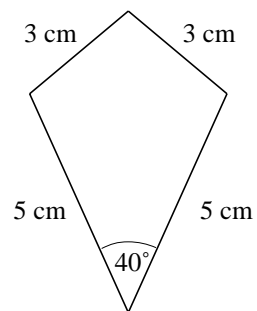
(b) Check that the diagonals of the kite are at right angles.

3 cm   3 cm

5 cm   5 cm

40°

9. A pile of sand has the shape shown below. Using an accurate diagram, find its height.

4 m   4 m

6 m
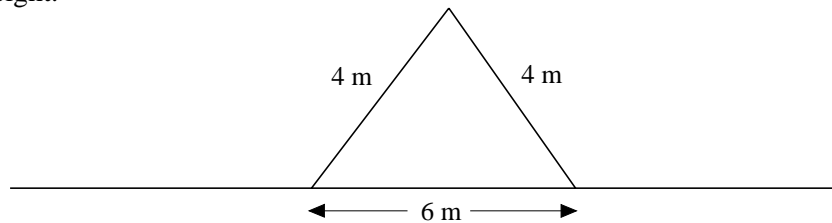
10. Draw accurately the shape shown opposite.

Find the size of the angle marked θ.

θ

7 cm   7 cm

4 cm   4 cm
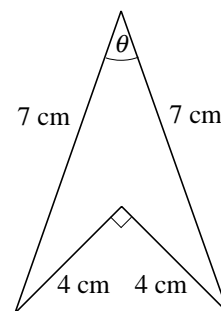
11. The sketch shows the design for a church window.

CB and EA are perpendicular to BA.
CD is part of a circle, centre E.
DE is part of a circle, centre C.

Using a ruler and compasses draw the design accurately.

Not to scale

D

C   E

8 cm   8 cm

B   A
    6 cm
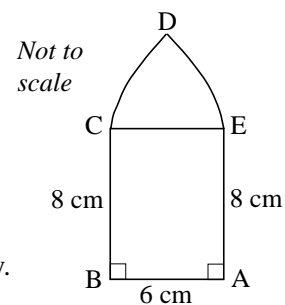
(SEG)

12. John is required to construct a pyramid with a square base, as shown opposite.
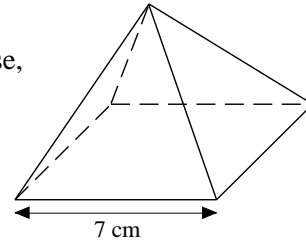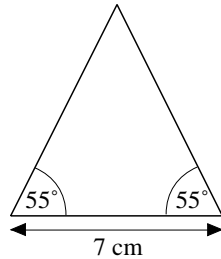
(a)   Each sloping face is a triangle with base angles of 55°.

Construct one of these triangles accurately and to full size.

(b) Construct the square base of the pyramid accurately and to full size.

(MEG)

13. A rectangle has sides of 8 cm and 5 cm.

(a) Calculate the perimeter of the rectangle.

(b) Construct the rectangle accurately.

(MEG)

14. Construct a rhombus ABCD with the line AB as base and with $B\hat{A}D = 50°$.

(MEG)

14.4   Enlargements

An *enlargement* is a transformation which enlarges (or reduces) the size of an image. Each enlargement is described in terms of a *centre of enlargement* and a *scale factor*.
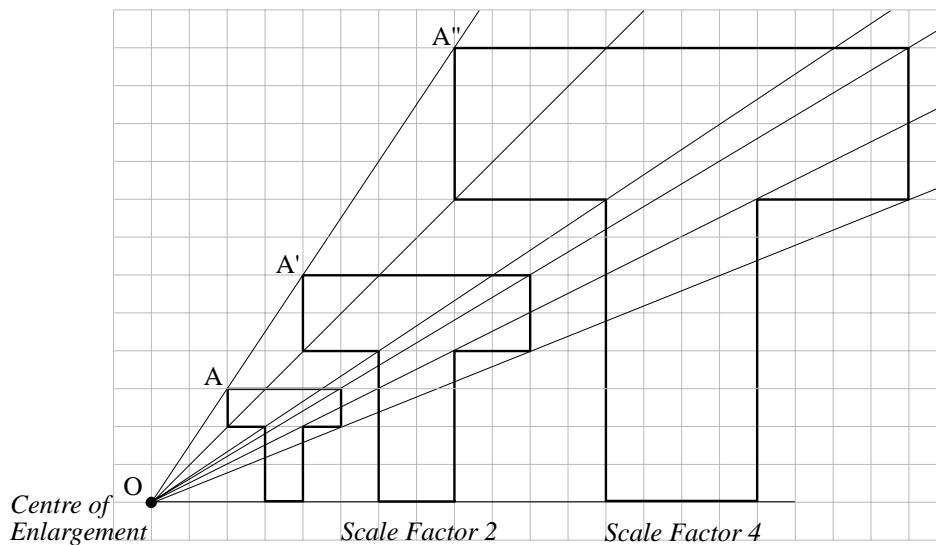
The example shows how the original, A, was enlarged with scale factors 2 and 4. A line from the *centre of enlargement* passes through the corresponding corners of each image.

Note

The distances, OA$'$ and OA$''$, are related to OA:

$$OA' = 2 \times OA$$

$$OA'' = 4 \times OA$$

The same is true of all the other distances between O and corresponding points on the images.

Worked Example 1

Enlarge the triangle shown using the centre of enlargement marked and scale factor 3.
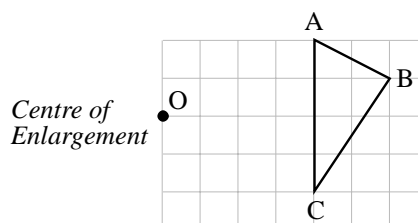
Solution

The first step is to draw lines from the centre of enlargement through each corner of the triangle as shown below.
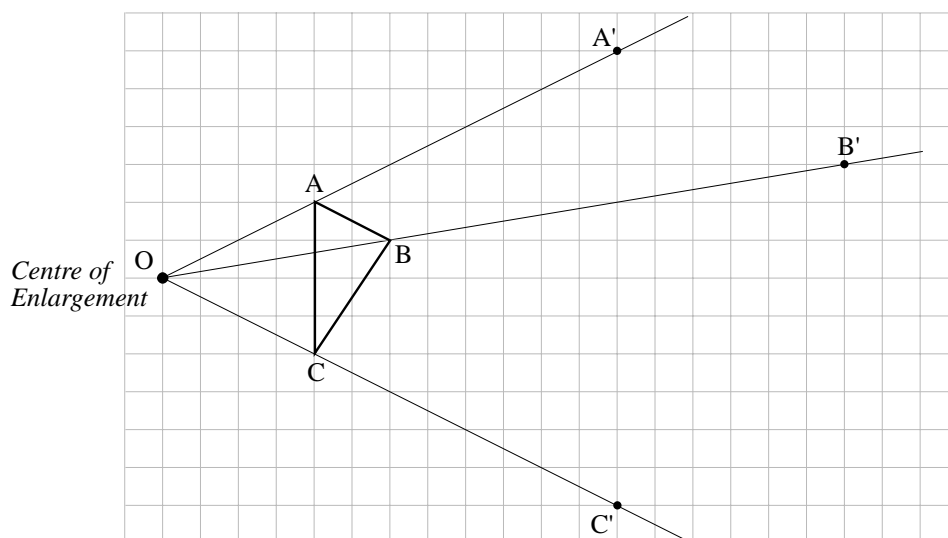
As the scale factor is 3, then

$$OA' = 3 \times OA$$

$$OB' = 3 \times OB$$

$$OC' = 3 \times OC$$

The points A$'$, B$'$ and C$'$ have also been marked on the diagram. Once these points have been found they can be used to draw the enlarged triangle.
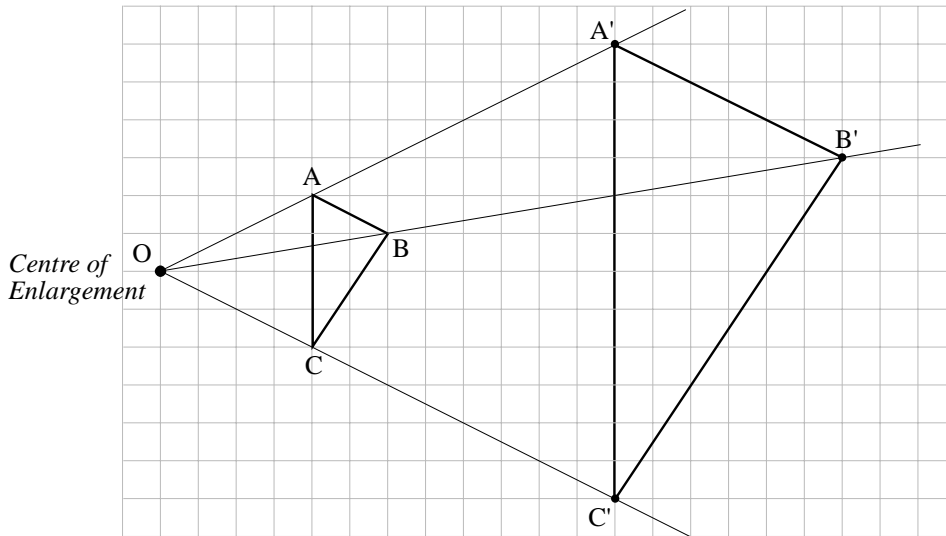
Worked Example 2

Enlarge the pentagon with scale factor 2 using the centre of enlargement marked on the diagram.
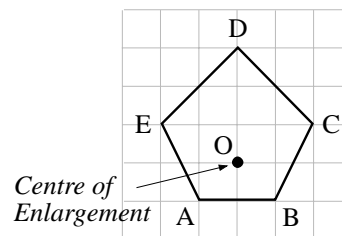
Solution

The first step is to draw lines from the centre of enlargement which pass through the five corners of the pentagon.

As the scale factor is 2 the distances from the centre of enlargement to the corners of the image will be

$$OA' = 2 \times OA$$
$$OB' = 2 \times OB$$
$$OC' = 2 \times OC$$
$$OD' = 2 \times OD$$
$$OE' = 2 \times OE .$$
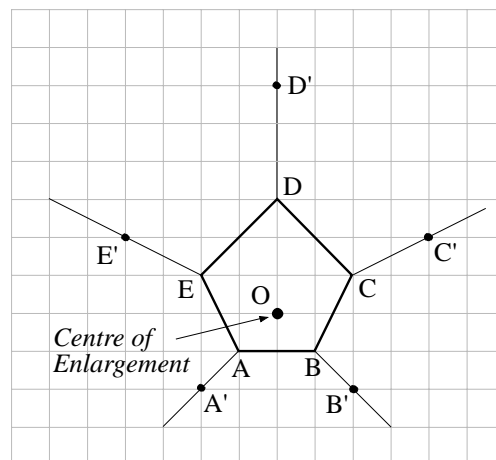
These points can then be marked and joined to give the enlargement.
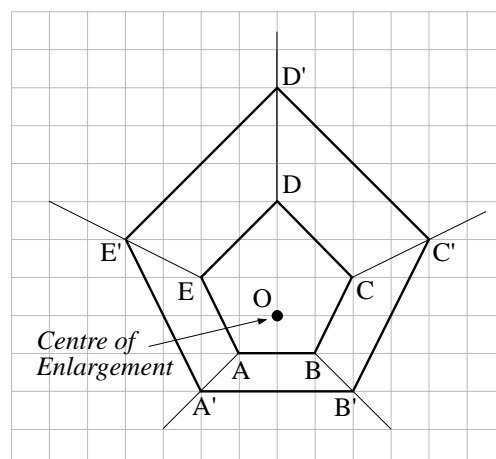
Worked Example 3

The diagram shows the square ABCD which has been enlarged to give the squares $A'B'C'D'$ and $A''B''C''D''$.

(a) Find the scale factor for each enlargement.

(b) Find the centre of enlargement.
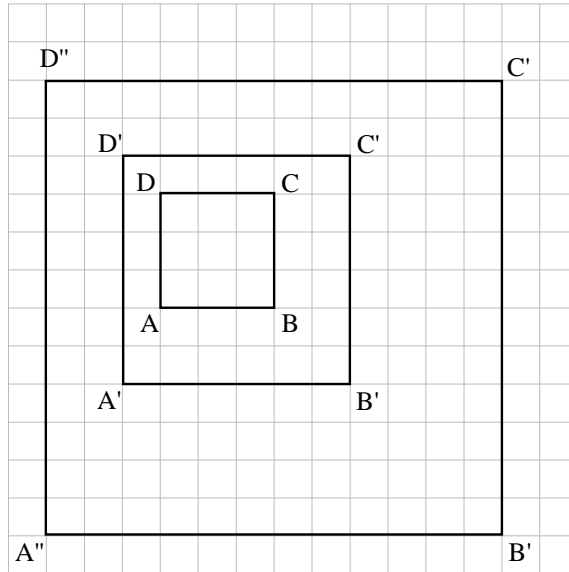
Solution

(a) The sides of the square ABCD are each 1.5 cm. The sides of the square $A'B'C'D'$ are 3 cm. As these are twice as long as the original, the scale factor for this enlargement is 2.

The sides of the square $A''B''C''D''$ are 6 cm, which is 4 times longer than the original square. So the scale factor for this enlargement is 4.

(b) To find the centre of enlargement draw lines through A, A' and A'', then repeat for B, B' and B'', C, C' and C'' and D, D' and D''.
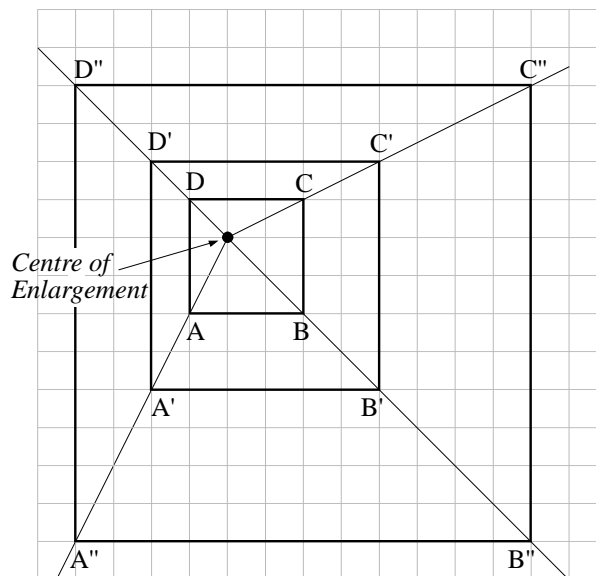
These lines cross at the centre of enlargement as shown in the diagram.

Investigation

Points X and Y lie on a straight line AB. Given that AX : XB = 1 : 2 and AY : YX = 2 : 3, write down the ratio AY : XB.

Exercises

1. Enlarge the triangle shown with scale factor 3 and the centre of enlargement shown.

Centre of Enlargement
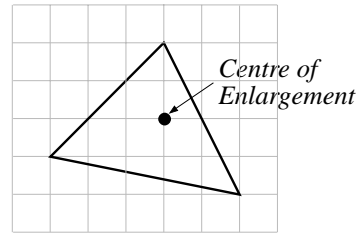
2. Copy the diagrams below on to squared paper. Enlarge each shape with scale factor 2 using the point marked as the centre of enlargement.

(a)

Centre of Enlargement

(b)

Centre of Enlargement

(c)

Centre of Enlargement

(d)

Centre of Enlargement

(e)

Centre of Enlargement

(f)

Centre of Enlargement
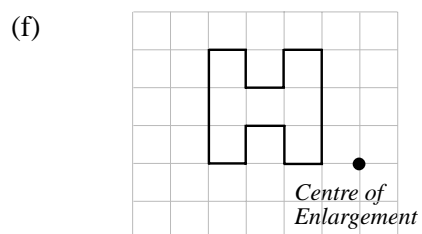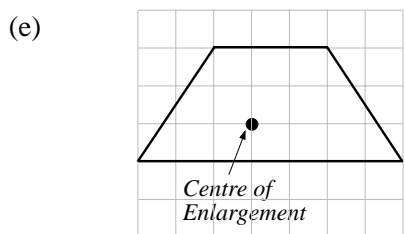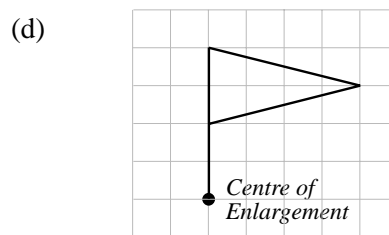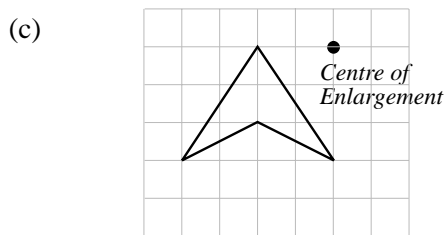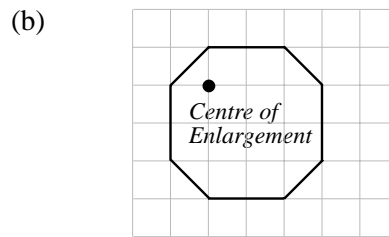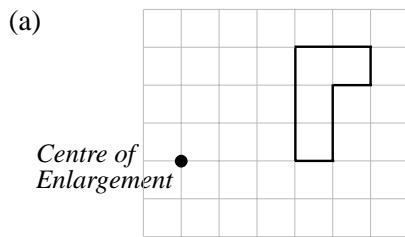
3. (a) A boy writes his initials as shown below. Use the marked centre of enlargement to enlarge his initials with scale factor 3.

Centre of Enlargement
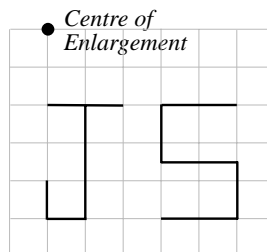
(b) Repeat part (a) using your own initials.

4.   In each diagram below, the smaller shape has been enlarged to obtain the larger shape. For each example state the scale factor.

(a)
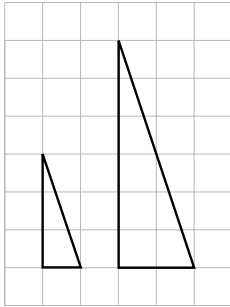
(b)
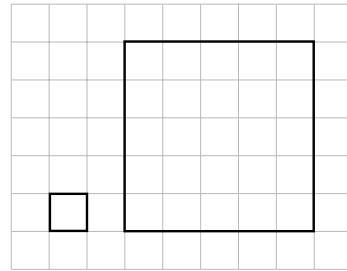
(c)
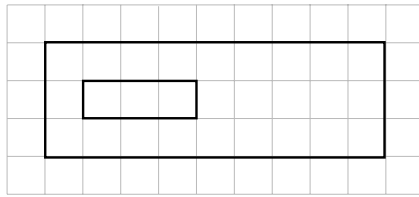
(d)
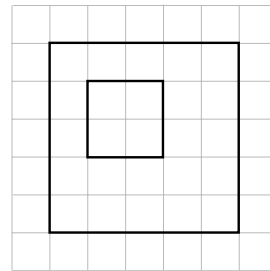
(e)
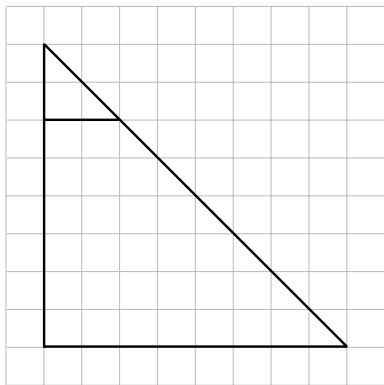
(f)
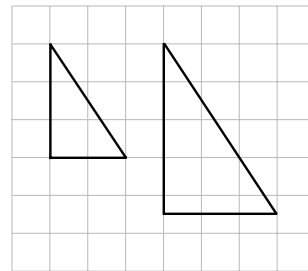
5.   Copy each diagram on to squared paper.   Then find the centre of enlargement and the scale factor when the smaller shape is enlarged to give the bigger shape.

(a)   (b)

(c)

(d)

(e)

(f)
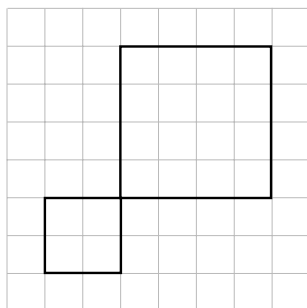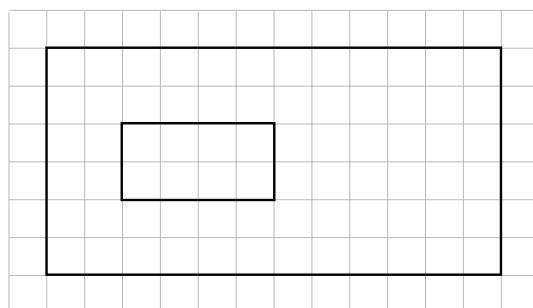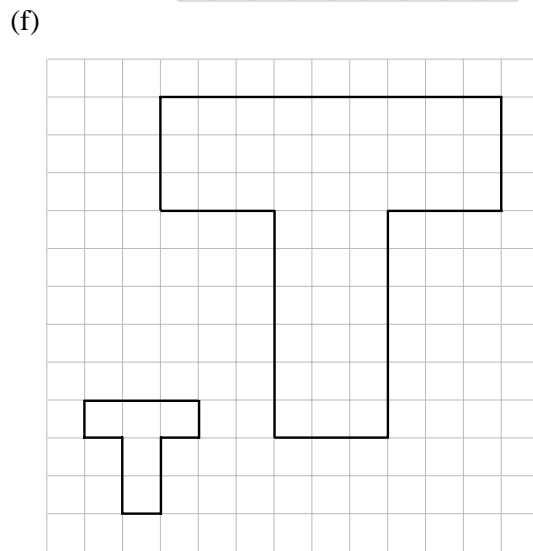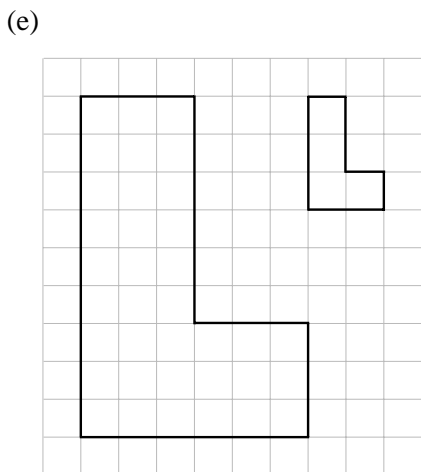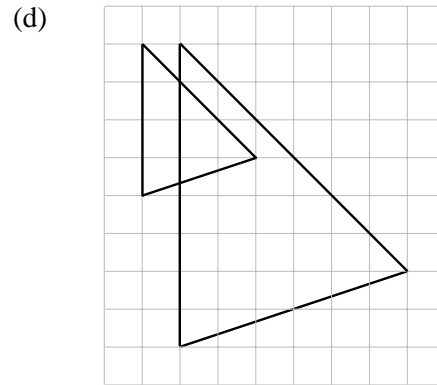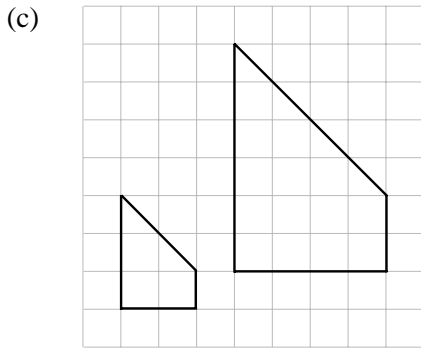
6. Copy each diagram below and enlarge it with scale factor 3.

    (a)

    Centre of Enlargement

    (b)

    Centre of Enlargement
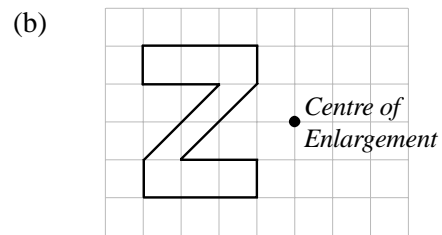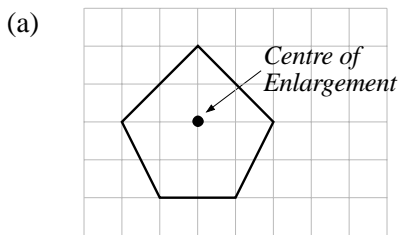
7. (a) Draw a set of axes with *x* and *y* values from 0 to 15.

    (b) Plot the points $(2, 2)$, $(6, 2)$, $(6, 6)$ and $(2, 6)$, then join them to form a square.

    (c) Enlarge the square with scale factor 3, using the point with coordinates $(3, 3)$ as the centre of enlargement.

8. (a) Draw a circle with centre at $(4, 4)$ and radius 2.

    (b) Enlarge the circle with scale factor 2 and centre of enlargement $(4, 4)$.

    (c) Enlarge the circle with scale factor 3 and centre $(5, 5)$.

9.   A triangle with vertices at the points with coordinates (2, 1) (7, 1) and (7, 6) is
        enlarged to give the triangle with coordinates at the points (6, 3), (21, 3) and (21, 18).

   (a)   Draw both triangles.
   (b)   What is the scale factor of the enlargement?
   (c)   What are the coordinates of the centre of enlargement?

10. (a)   On a set of axes draw a triangle.

   (b)   Enlarge the triangle with scale factor 2 using the point (0, 0) as the centre of
         enlargement.

   (c)   Write down the coordinates of both triangles. How do they compare?

   (d)   What would you expect to be the coordinates of your triangle if it were to be
         enlarged with scale factor 3 using (0, 0) as the centre of enlargement?
         Check your answer by drawing the triangle.

   (e)   Enlarge your original triangle with a different centre. Is there a simple
         relationship between the coordinates of the original and the enlargement,
         when the centre of enlargement is used?

11.   Using the point O as the centre of enlargement,
      enlarge the rectangle with scale factor 3.
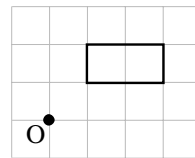
(MEG)

12.
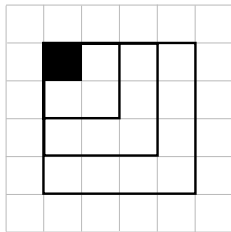
      The shaded square has sides of length 1 cm.
      It is enlarged a number of times as shown.

   (a)   Copy and complete the table below

.

Length of Side of Square	1 cm	2 cm	3 cm	4 cm
Perimeter of Square	4 cm	8 cm	12 cm	
Area of Square	1 cm^2	4 cm^2		16 cm^2

The shaded square continues to be enlarged.

   (b)   Copy and complete the following table.

Length of Side of Square	
Perimeter of Square	
Area of Square	64 cm^2

(SEG)

14.5 Reflections

Reflections are obtained when you draw the image that would be obtained in a mirror.
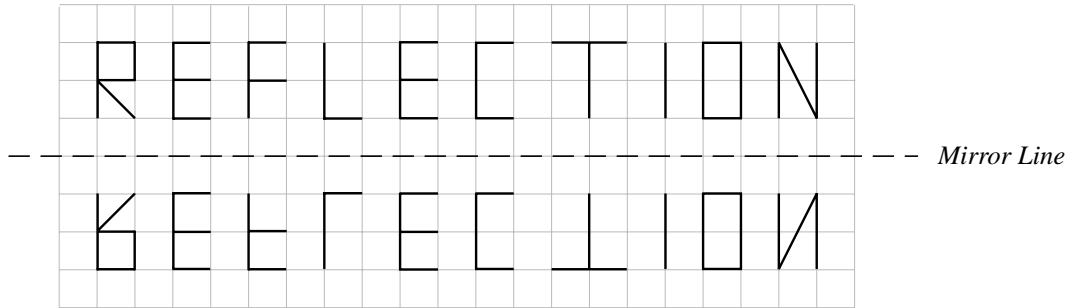
Mirror Line

Every point on a reflected image is always the same distance from the *mirror line* as the original. This is shown below.
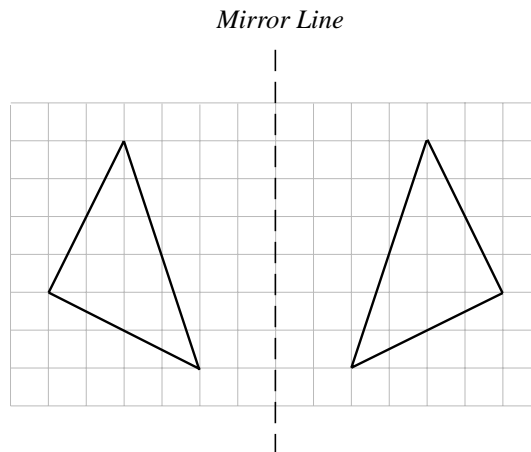
Mirror Line

Note

Distances are always measured at right angles to the *mirror line*.

Worked Example 1

Draw the reflection of the shape in the mirror line shown.

Mirror Line
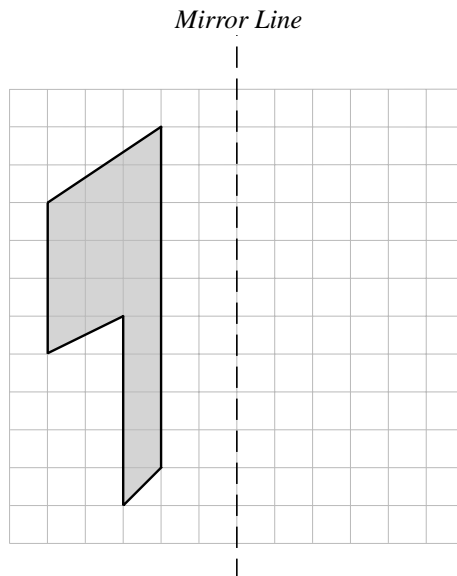
Mirror Line

Solution

The lines added to the diagram show how to find the position of each point after it has been reflected.

Remember that the image of each point is the same distance from the mirror line as the original.
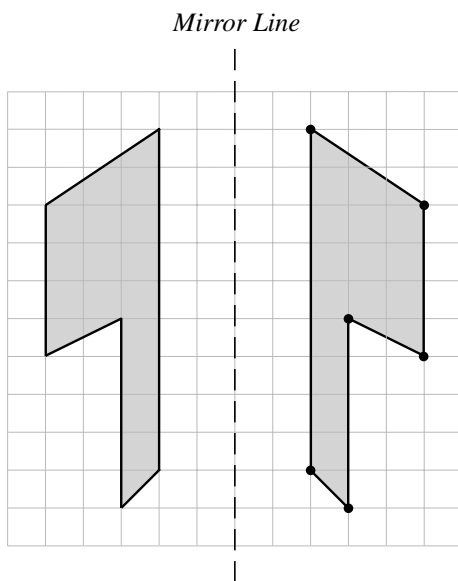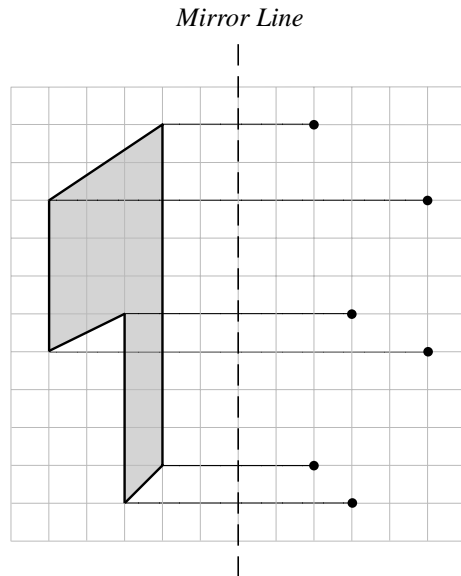
Mirror Line

The points can then be joined to give the reflected image.

If the construction lines have been drawn in pencil they can be rubbed out.

Worked Example 2

Reflect this shape in the mirror line shown in the diagram.
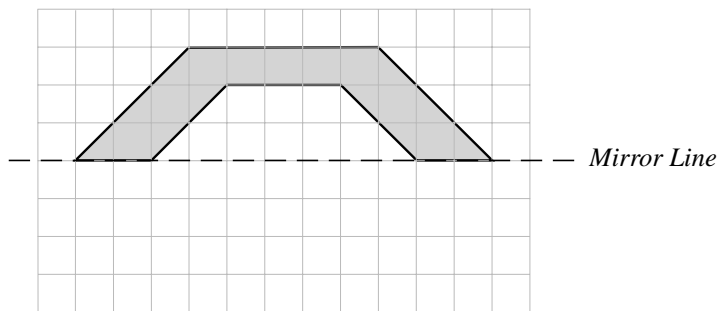
Mirror Line

Solution

The lines are drawn at right angles to the mirror line. The points which form the image must be the same distance from the mirror lines as the original points. The points which were on the mirror line remain there.

Mirror Line
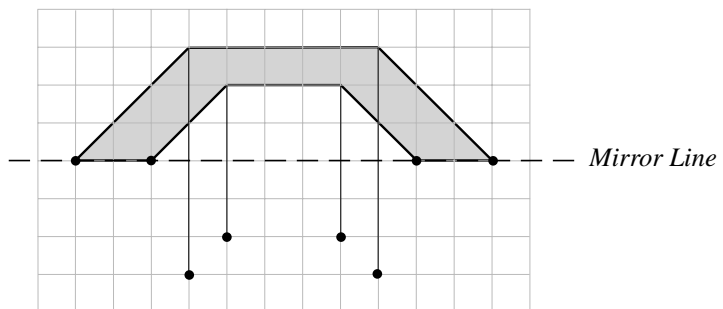
The points can then be
joined to give the
reflected image.
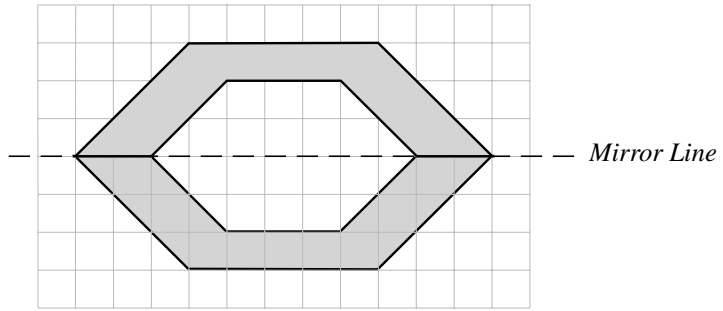
Mirror Line

Exercises

1. Copy the diagram below and draw the reflection of each object.

(a)

Mirror Line

(b)

Mirror Line

(c)

Mirror Line

(d)

Mirror Line

(e)

Mirror Line

(f)
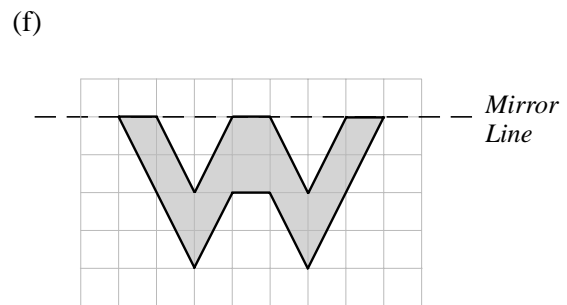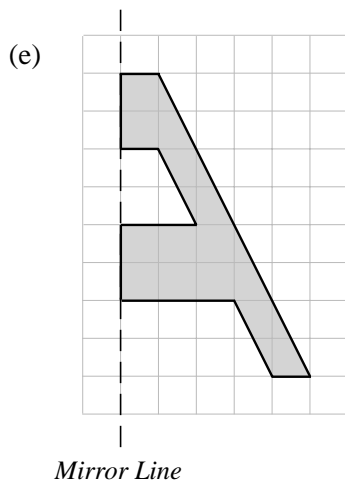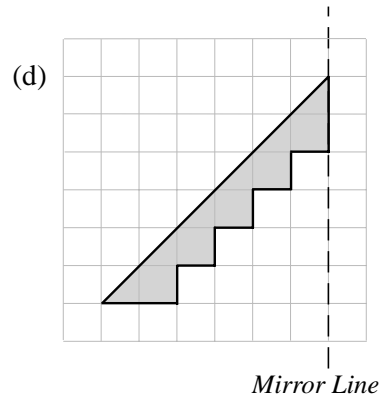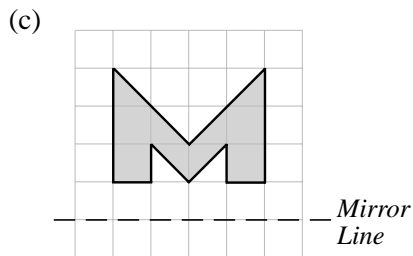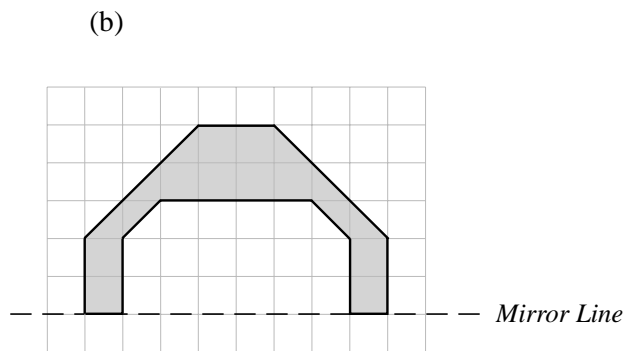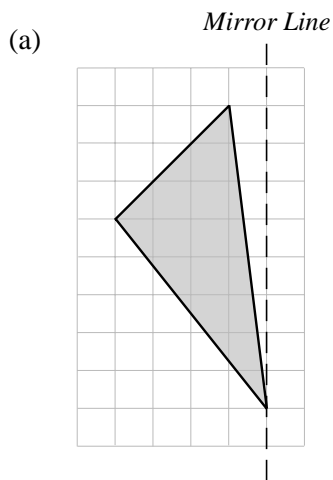
Mirror Line

2.   Copy each diagram and draw the reflection of each shape in the mirror line shown.

(a)

(b)

(c)

(d)

(e)

(f)
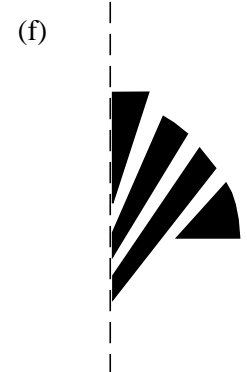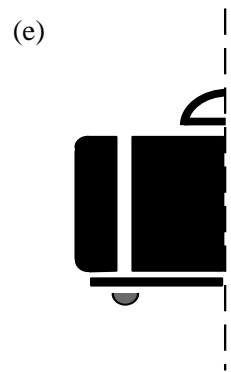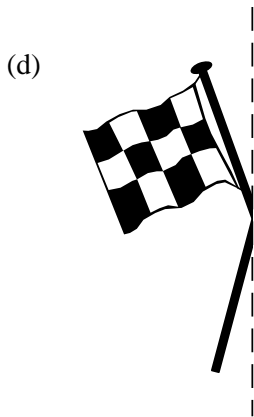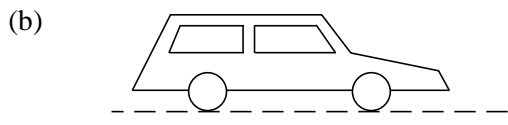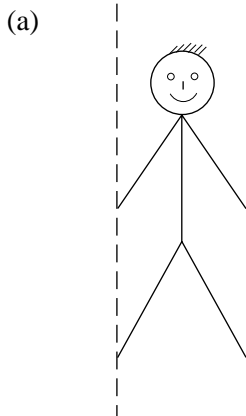
3.   (a)   Draw a set of axes with *x* and *y* values from -5 to 5.

(b)   Plot the points with coordinates

(1, 1), (1, 5), (4, 5), (4, 3), (2, 3), (2, 1).

Join the points in that order to form a shape.

(c)   Reflect the image in the *y*-axis. Write down the coordinates of the corners of this shape.

(d)   Reflect the image obtained in (c) in the *x*-axis. List the coordinates of the corners.

(e)   Reflect the image obtained in (d) in the *y*-axis. Describe how this shape could have been obtained directly from the original shape.

4.   A student reflected his two initials, the first in the *y*-axis and the second in the *x*-axis, to obtain the image opposite.

Copy the diagram and show the original position of the initials.
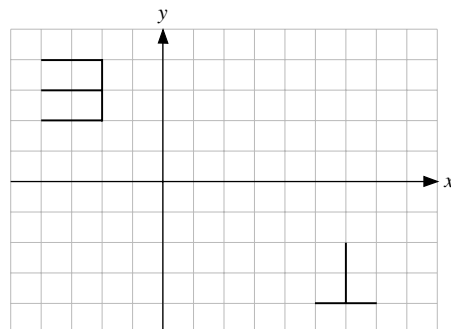
5.   Copy the diagram below.
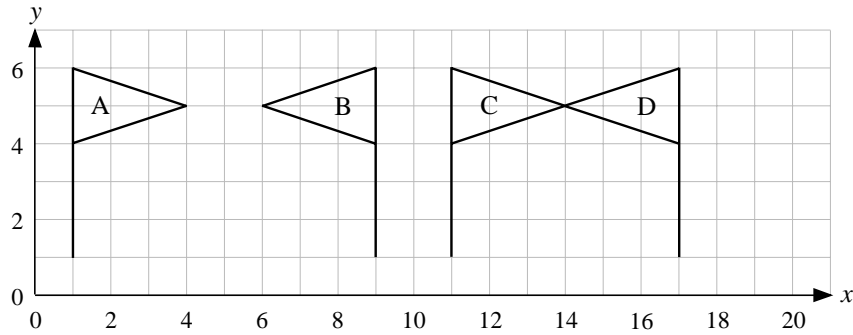
Draw in the mirror line for each reflection described.

(a)   A → B   (b)   B → C   (c)   C → D   (d)   A → D

6.   (a)   Copy the axes and shape shown.

      (b)   (i)   Draw the reflection of the shape
                    in the *y*-axis.

             (ii)   Compare the coordinates of
                    each shape.

             (iii) Describe what happens to the
                    coordinates of a point when it
                    is reflected in the *y*-axis.
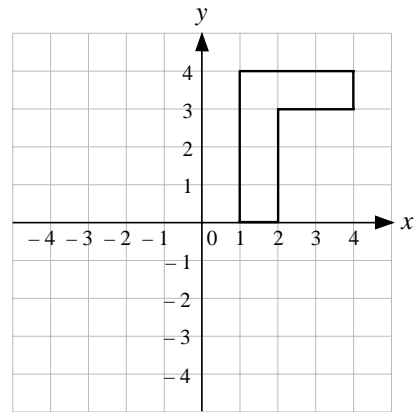
      (c)   Repeat (b) using the *x*-axis.

7.
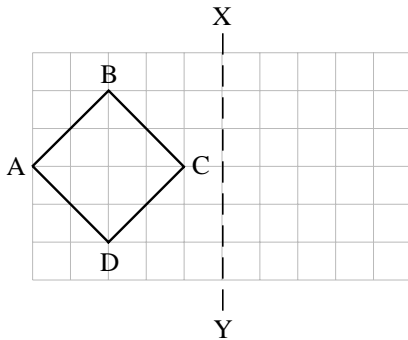
      (a)   Copy the diagram and draw the
             reflection of ABCD in the mirror line
             XY.

      (b)   ABCD has rotational symmetry.
             Mark with a cross its centre of
             rotation.

                                              (SEG)

8.   (a)   Find the area of the shaded shape.

      (b)   Copy the diagram and draw the
             reflection of the shaded shape
             in the mirror line.
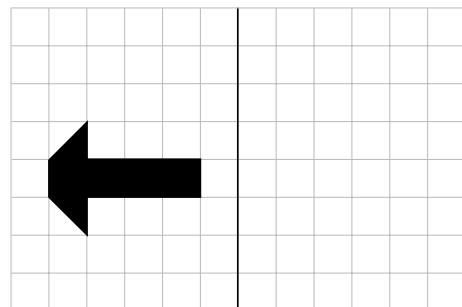
Mirror Line

                                              (NEAB)

14.6 Construction of Loci

When a person moves so that they always satisfy a certain condition, their possible path is called a *locus*. For example, consider the path of a person who walks around a building, always keeping the same distance away from the building.

Building

— *Path of the person*
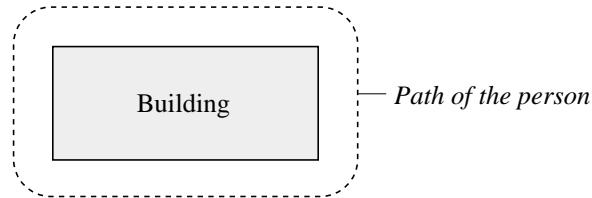
The dotted line in the diagram shows the path taken – i.e. the *locus*.

Worked Example 1

Draw the locus of a point which is always a constant distance from another point.

Solution

The fixed point is marked A.
The locus is a circle around the fixed point.

A
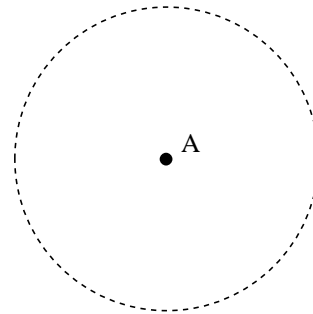
Worked Example 2

Draw the locus of a point which is always the same distance from A as it is from B.

A   B

Solution

The locus of the line AB will be the same distance from both points.

A ————————————— B
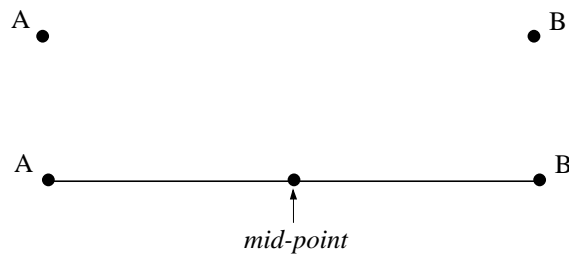
mid-point

However, any point on a line perpendicular to AB and passing through the mid-point of AB will also be the same distance from A and B.

Locus

A ———————————— B

The diagram shows how to construct this line.

This line is called the *perpendicular bisector* of AB.
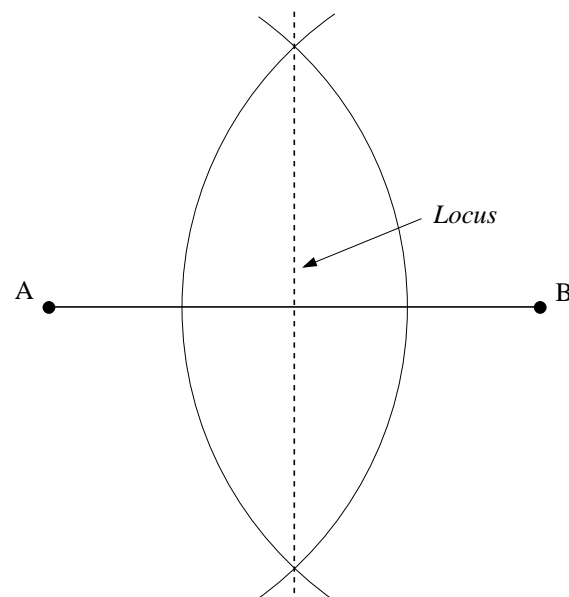
Worked Example 2

Draw the locus of a point that is the same distance from the lines AB and AC shown in the diagram below.
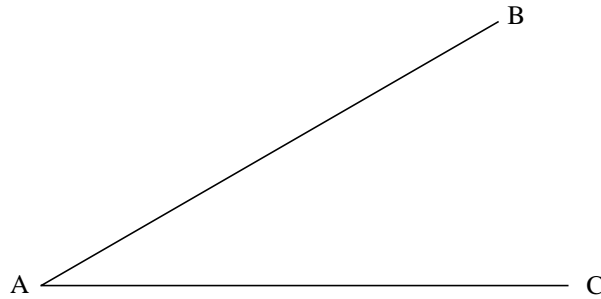
Solution

The locus will be a line which divides the BÂC into two equal angles. The diagram below shows how to construct this locus.

This line is called the *bisector* of BÂC.
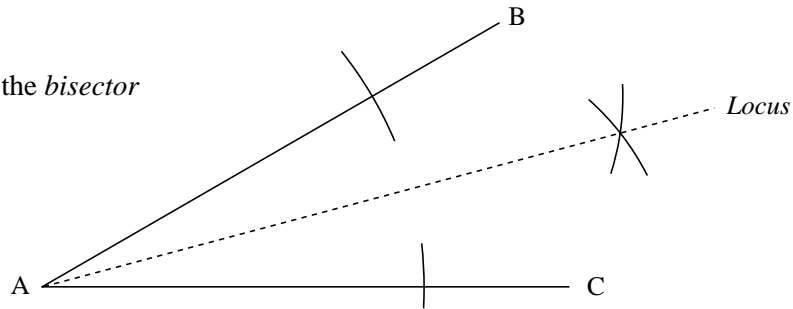
Exercises

1.   Draw the locus of a point which is always a distance of 4 cm from a fixed point A.

2.   The line AB is 4 cm long. Draw the locus of a point which is always 2 cm from the line AB.

3.   Copy the diagram and draw in the locus of a point which is the same distance from the line AB as it is from CD.
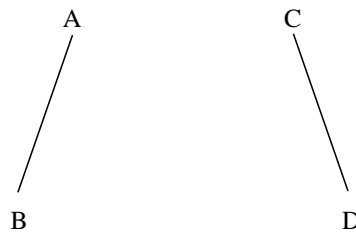
4.   (a)   Draw an equilateral triangle with sides of length 5 cm.

      (b)   Draw the locus of a point that is always 1 cm from the sides of the triangle.

5.   Copy the triangle opposite.

      Draw the locus of a point which is the same distance from AB as it is from AC.
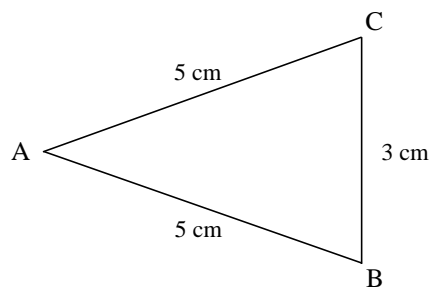
6. Draw the locus of a point that is the same distance from both lines shown in the diagram below.
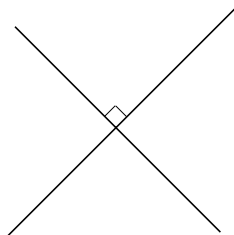
7. (a) Draw 2 parallel lines.

    (b) Draw the locus of a point which is the same distance from both lines.

8. The diagram below shows the boundary fence of a high security army base.
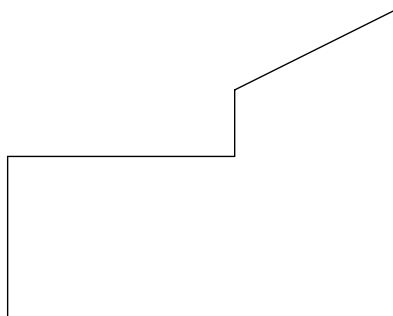
    (a) Make a copy of this diagram.

    (b) A security patrol walks round the *outside* of the base, keeping a constant distance from the fence. Draw the locus of the patrol.

    (c) A second patrol walks *inside* the fence, keeping a constant distance from the fence. Draw the locus of this patrol.

9. The points A and B are 3 cm apart. Draw the locus of a point that is twice as far from A as it is from B.

10. A ladder leans against a wall, so that it is almost vertical. It slides until it is flat on the ground. Draw the locus of the mid-point of the ladder.

11. The points A and B are 4 cm apart.

    (a) Draw the possible positions of the point, P, if

        (i)   AP $=$ 4 cm   and   BP $=$ 1 cm
        (ii) AP $=$ 3 cm   and   BP $=$ 2 cm
        (iii) AP $=$ 2 cm   and   BP $=$ 3 cm
        (iv) AP $=$ 1 cm   and   BP $=$ 4 cm

    (b) Draw the locus of the point P, if AP $+$ BP $=$ 5 cm.

    (c) Draw the locus of the point P, if AP $+$ BP $=$ 6 cm.

14.7 Enlargements which Reduce

When the scale factor of an enlargement is a fraction, the size of the enlargement is reduced. The image of the original is then between the *centre of enlargement* and the original.

Worked Example 1

The diagram shows three triangles.

ABC was enlarged with different scale factors to give $A'B'C'$ and $A''B''C''$.

(a)   Find the centre of enlargement.

(b)   Find the scale factor for each enlargement.
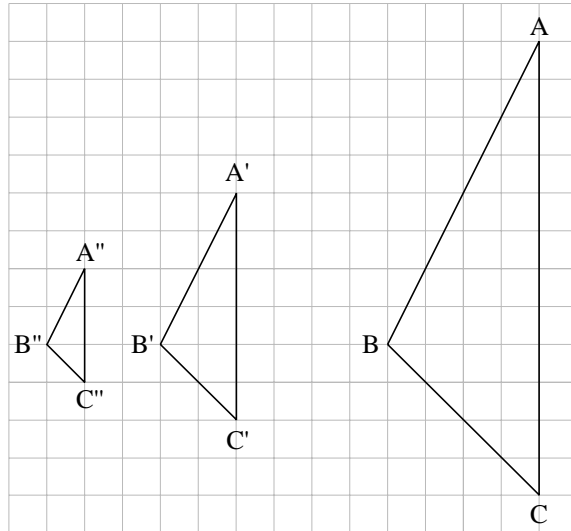
Solution

(a)   To find the centre of enlargement, lines should be drawn through the corresponding points on each figure.
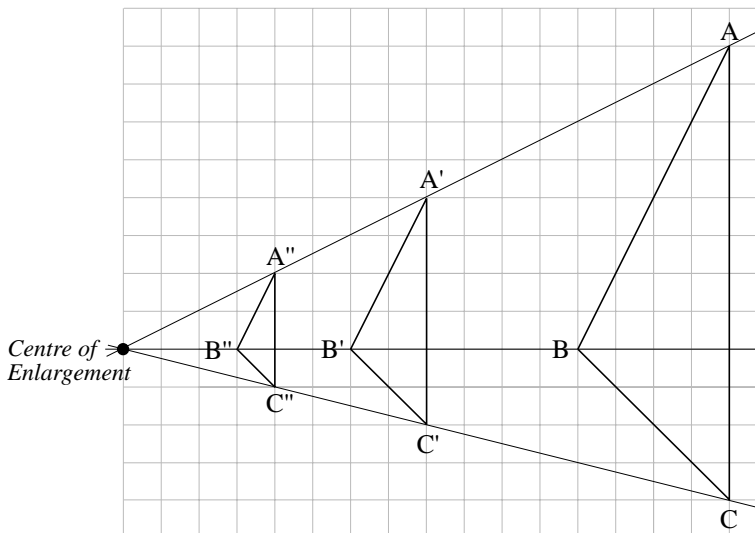
(b)   To find the scale factors, compare the lengths of sides in the different triangles. First consider triangles ABC and $A'B'C'$:

$$AC = 6 \text{ cm} \quad \text{and} \quad A'C' = 3 \text{ cm},$$

so
$$A'C' = \frac{1}{2} \times AC,$$

which means that the scale factor is $\frac{1}{2}$.

For triangles ABC and A″B″C″,

$$AC = 6 \text{ cm} \quad \text{and} \quad A''C'' = 1.5 \text{ cm},$$

so
$$A''C'' = \frac{1}{4} \times AC$$

which means that the scale factor is $\frac{1}{4}$.

Worked Example 2

Enlarge the triangle below with scale factor $\frac{1}{3}$ and centre of enlargement as shown.
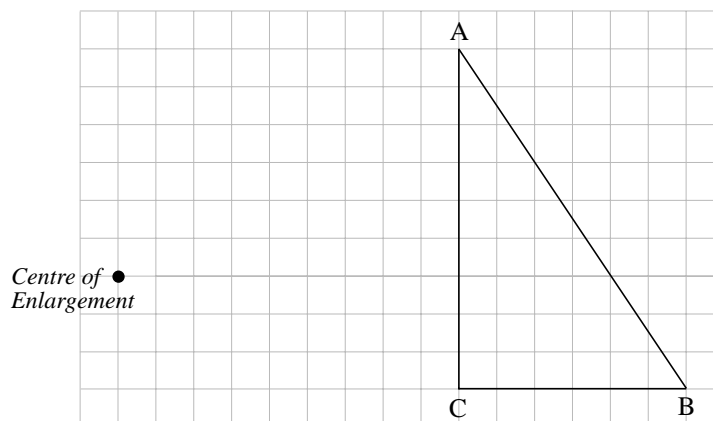
Solution

The first stage is to draw lines from each corner of the triangle through the centre of enlargement.
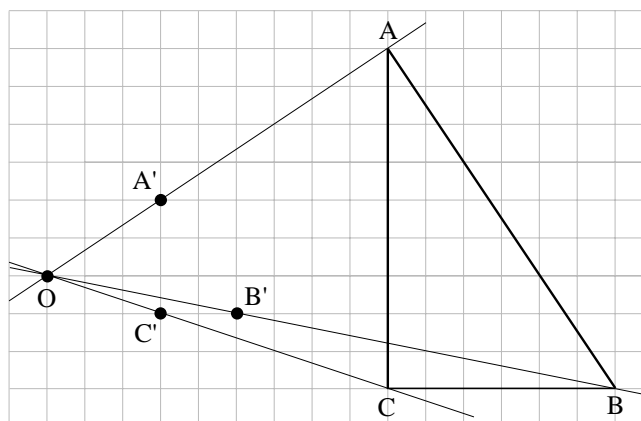
Then the corners of the image should be fixed so that

$$OA' = \frac{1}{3} \times OA$$

$$OB' = \frac{1}{3} \times OB$$

$$OC' = \frac{1}{3} \times OC.$$

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These points can then be joined to give the image.
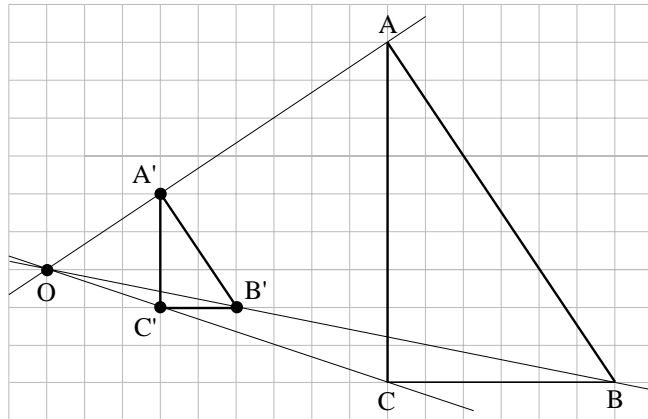
Exercises

1. For each pair of objects, state the scale factor of an enlargement which produces the smaller image from the larger one.

    (a)

    (b)

    (c)

    (d)

    (e)

    (f)
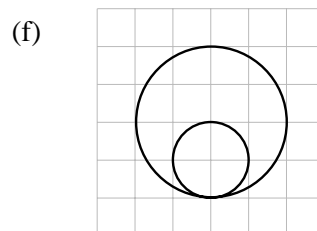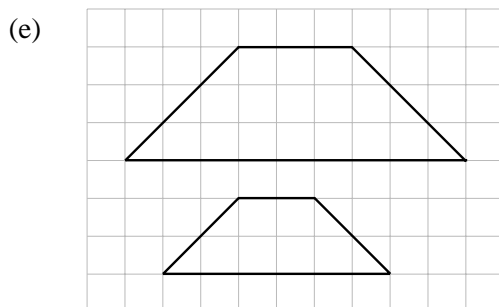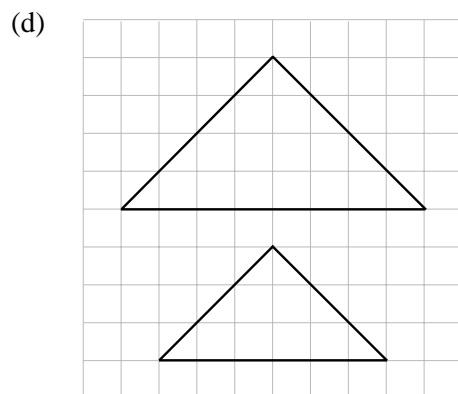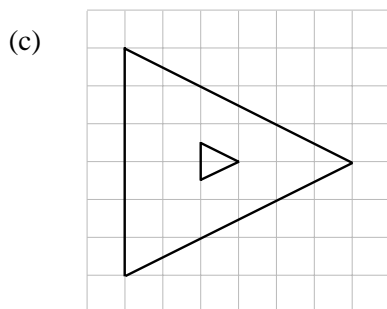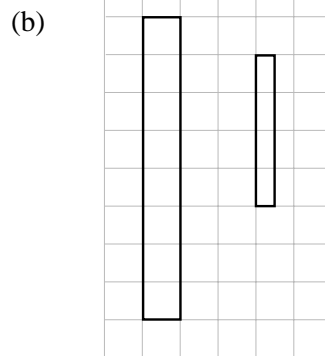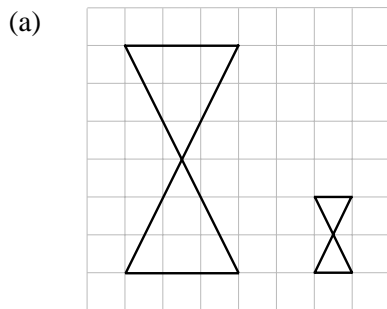
2.	In each example below, the smaller shape has been obtained from the larger shape by an enlargement. For each example, state the scale factor and the coordinates of the centre of enlargement.
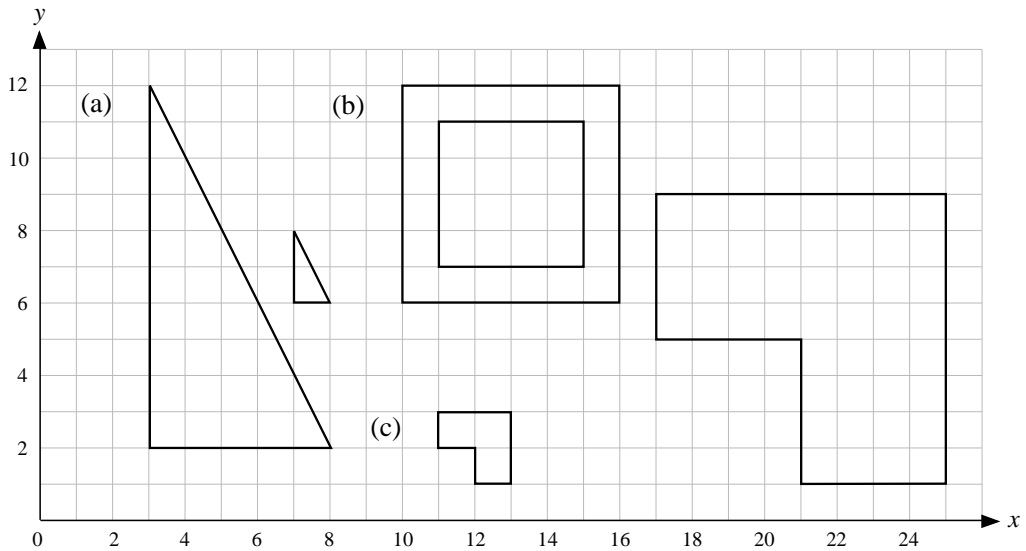
3.	Copy each shape and enlarge using the centre of enlargement shown and the scale factor specified.

(a)

Centre of Enlargement

Scale factor $\dfrac{1}{2}$

(b)

Centre of Enlargement

Scale factor $\dfrac{2}{3}$

(c)
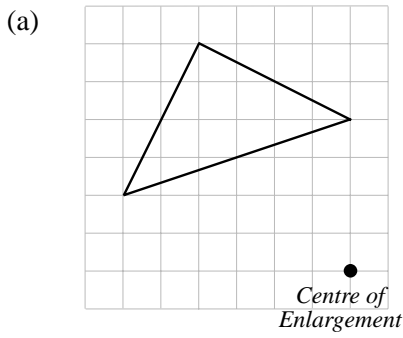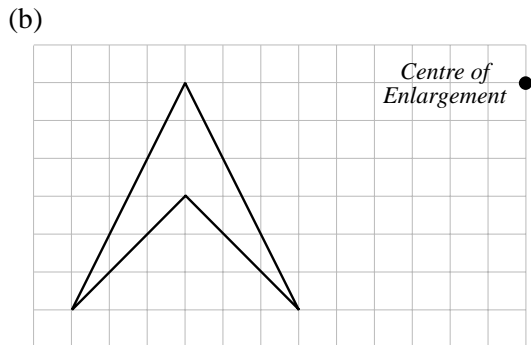
Centre of Enlargement
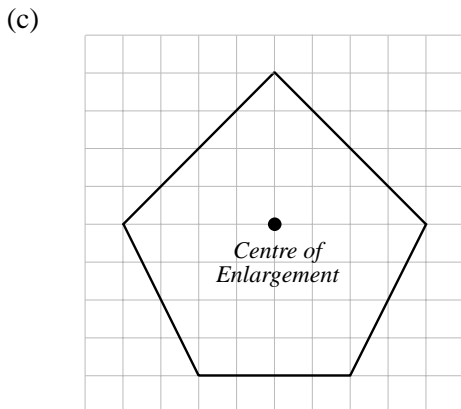
Scale factor $\dfrac{1}{4}$ and $\dfrac{3}{4}$

(d)

Centre of Enlargement
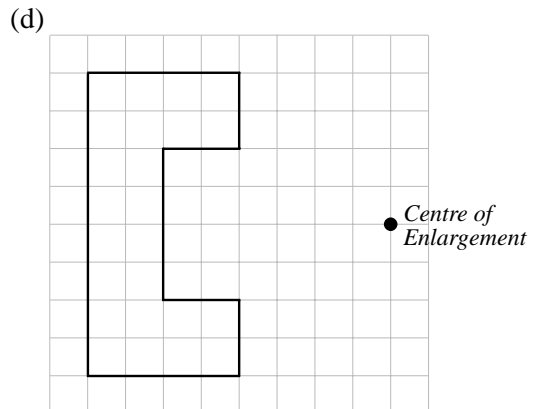
Scale factor $\dfrac{3}{4}$ and $\dfrac{1}{8}$

4. For each of (a) to (d) below:

    (i)   draw the triangle which has corners at the points given;

    (ii)   enlarge each triangle using the scale factor and centre of enlargement given;

    (iii)   write down the coordinates of the corners of the image.

    (a)   (1, 2), (3, 6), (7, 4)   *Scale factor* $\dfrac{1}{2}$ *Centre of enlargement* (3, 0)

    (b)   (3, 4), (6, 7), (12, 4)   *Scale factor* $\dfrac{1}{3}$ *Centre of enlargement* (0. 1)

    (c)   (1, 1), (7, 1), (7, 7)   *Scale factor* $\dfrac{5}{6}$ *Centre of enlargement* (1, 1)

    (d)   (2, 0), (4, 4), (6, 2)   *Scale factor* $\dfrac{1}{4}$ *Centre of enlargement* (8, 6)

5. (a)   Plot the points:

             (2, 3), (2, 6), (5, 9), (8, 9), (11, 6), (11, 3), (8, 0) and (5, 0).

          Join the points to form an octagon.

    (b)   When the octagon is enlarged, the corners move to the points with
          coordinates:

             (4, 3), (4, 4), (5, 5), (6, 5), (7, 4), (7, 3), (6, 2) and (5, 2).

          Draw this octagon.

    (c)   Find the centre of enlargement and the scale factor.

    (d)   Use the same centre of enlargement to enlarge the original octagon with
          scale factor $\dfrac{2}{3}$.

6. The larger triangle shown below is reduced in size by using a photocopier to give
    the smaller triangle.
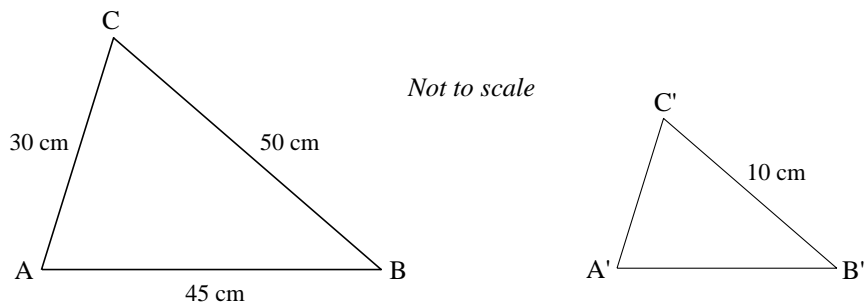
    (a)   What is the scale factor of the enlargement which took place?

    (b)   What are the lengths of A′C′ and B′C′?

7. Each diagram below shows a shape and its image after an enlargement.
   In each case, state the scale factor and the lengths of each side of the image.

(a)

15 cm

18 cm

12 cm

10 cm

9 cm

x

y

z

(b)

50 cm

40 cm

20 cm

30 cm

x

y

(c)

63 cm

81 cm

54 cm

36 cm

9 cm

x

y

z

(d)

64 cm

40 cm

48 cm

24 cm

x

y
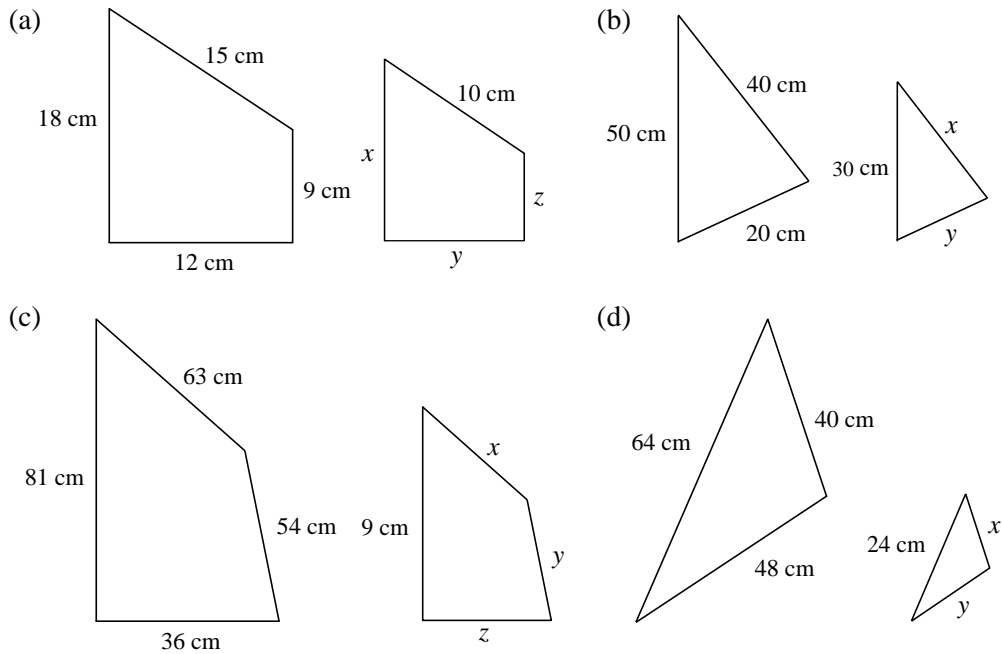
8. The parallelogram ABCD has vertices $(6, 3)$, $(9, 3)$, $(12, 9)$ and $(9, 9)$ respectively.

y (cm)
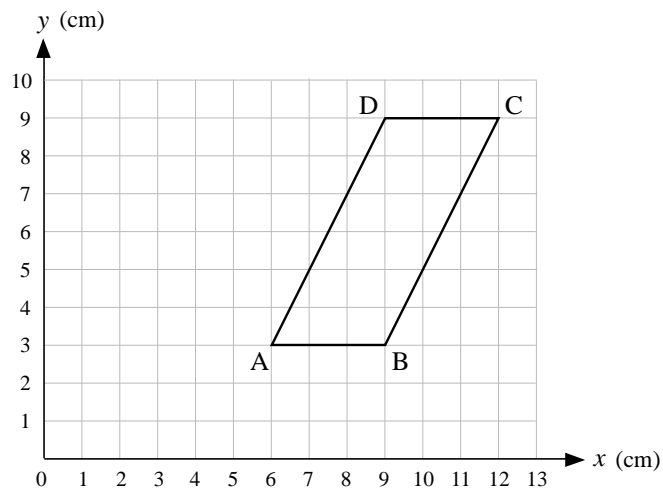
An enlargement scale factor $\dfrac{1}{3}$ and centre $(0, 0)$ transforms parallelogram ABCD onto parallelogram $A'B'C'D'$.

(a) (i) Draw the parallelogram $A'B'C'D'$.

   (ii) Calculate the area of parallelogram $A'B'C'D'$.

(b) The side AB has length 3 cm. The original shape ABCD is now enlarged with a scale factor of $\dfrac{2}{5}$ to give $A''B''C''D''$.

   Calculate the length of the side $A''B''$.

(SEG)

9.   (a)   Draw a rectangle and write down the coordinates of each corner.

         (b)   Enlarge the rectangle with scale factor $\frac{1}{2}$ using (0, 6) as the centre of enlargement. Write down the coordinates of each corner of the rectangle you obtain.

         (c)   How are the coordinates of the image related to the coordinates of the original?

         (d)   What would you expect the coordinates of the smaller rectangle to be if the original was enlarged with scale factor $\frac{1}{4}$? Check your answer by drawing this enlargement.

         (e)   Find the area of each rectangle. How does the scale factor of an enlargement affect the area of a shape?

14.8   Further Reflections

This section considers reflections in lines which are not simply horizontal or vertical. It also makes more use of equations to describe mirror lines.

Worked Example 1

Draw the reflection of the shape in the mirror line shown.
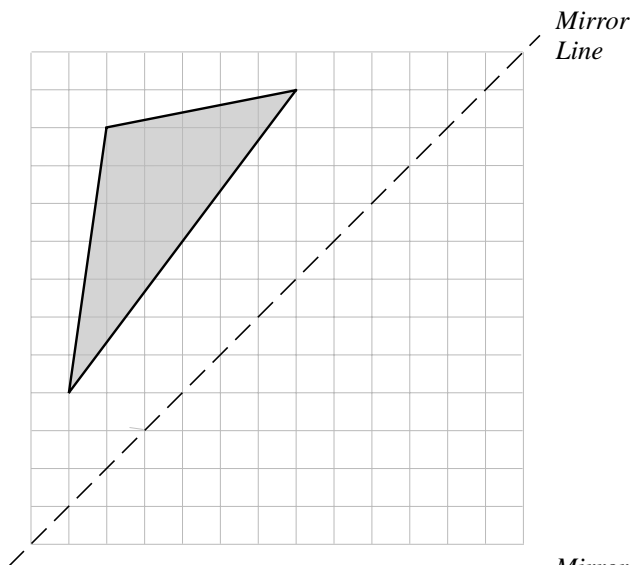
Mirror Line

Solution

The image of each point will be the same distance from the mirror line as the original points.

These distances should be measured so that they are perpendicular to the mirror line.
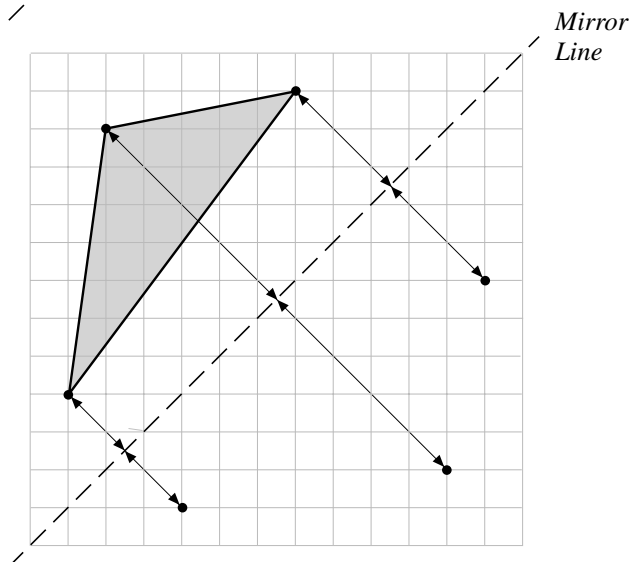
Mirror Line

These points can then be joined
to give the reflected image.
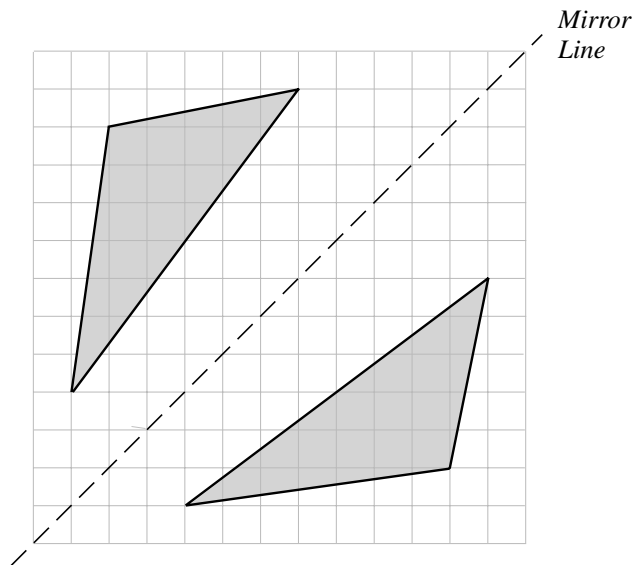
Mirror
Line

Worked Example 2

The diagram shows a number of triangles which can be obtained from each other by
reflections in suitable lines.
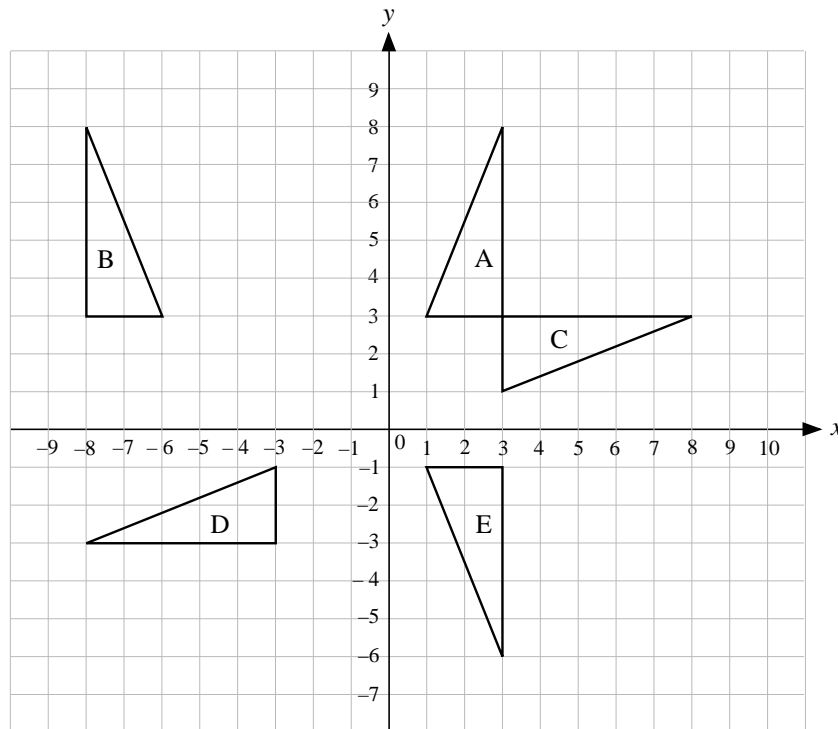
Find the equation of the mirror line which reflects:

(a)   A to B   (b)   A to C

(c)   A to D   (d)   A to E.

Solution

(a)   The diagram shows the
      mirror line.

      This vertical line passes
      through -2.5 on the x-axis
      and has equation

$$x = -2.5.$$
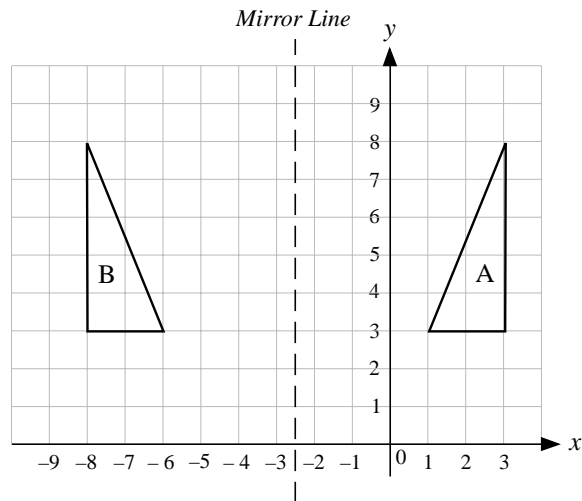
(b)   The diagram shows the mirror line.

      This line passes through points with
      coordinates $(1, 1)$, $(2, 2)$, $(3, 3)$, etc.
      and has equation

$$y = x.$$
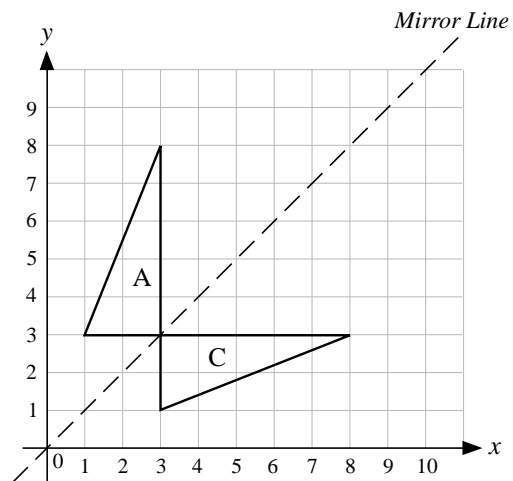
(c)   The diagram shows the sloping
      mirror line which is used to
      reflect A on to D.

      It passes through the points with
      coordinates $(-1, 1)$, $(-2, 2)$,
      $(-3, 3)$, etc. and has equation
$$y = -x.$$
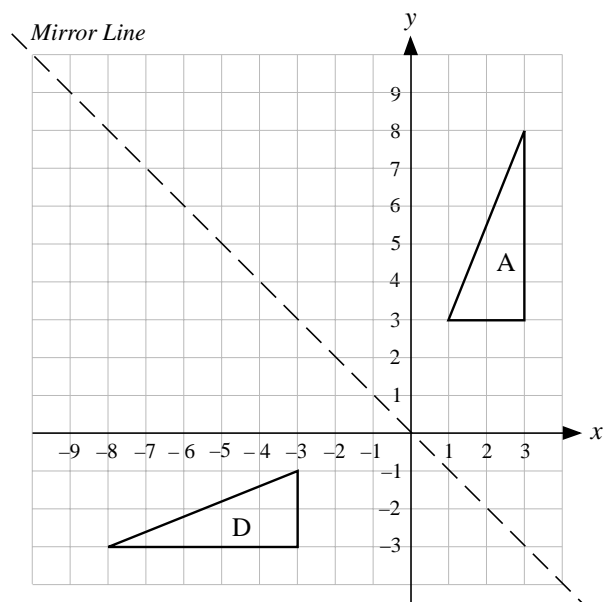
(d) The diagram shows the horizontal mirror line which is needed for this reflection.

It passes through 1 on the *y*-axis and so has equation

$$y = 1.$$
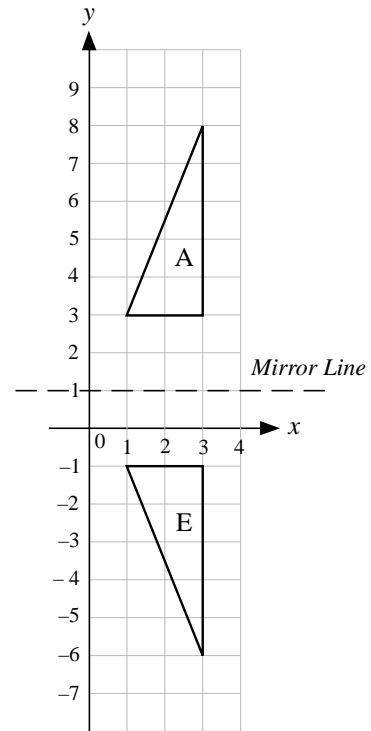
Exercises

1. Copy the diagrams below and draw the reflection of each shape in the mirror line shown.

    (a)   (b)

    (c)   (d)

(e)   (f)   *Mirror Line*

Mirror Line
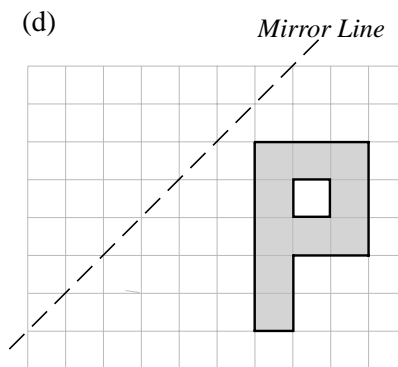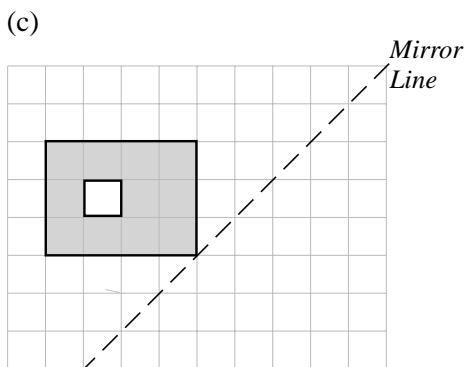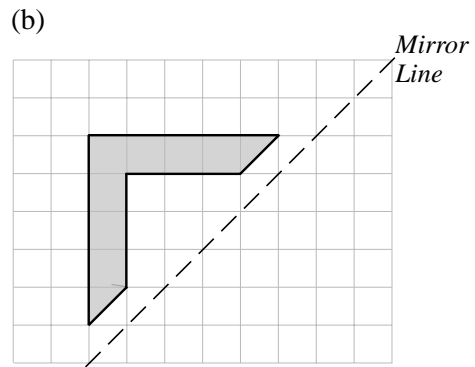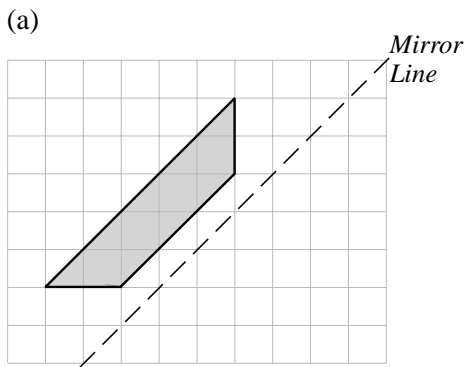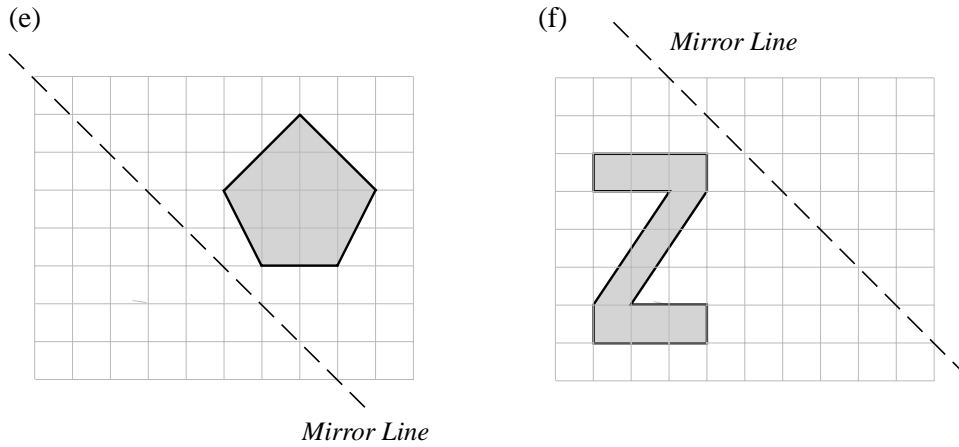
2.   The diagram shows the positions of the shapes A, B, C, D and E.
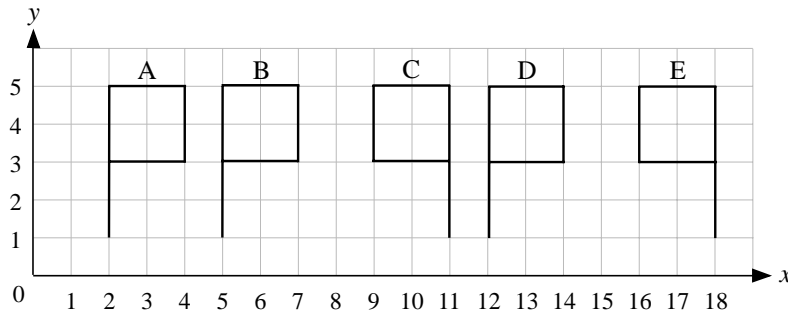
The table below gives the equations of lines of reflection needed to obtain one
shape from another. The symbol ✗ means that it is impossible to obtain one
shape from the other by a reflection.

	E	D	C	B
A				✗
B				
C	✗			
D	$x=15$			

Copy and complete the table.

3.   (a)   Draw a set of axes with *x*-values from 0 to 15 and *y*-values from 0 to 5.

     (b)   Join in order the points with the following coordinates:

                    (1, 1), (1, 5), (4, 5), (4, 3) and (1, 3).

     (c)   Draw the line $x = 5$ and reflect the original shape in this line.

     (d)   Draw the line $x = 10$ and reflect the image obtained in (c) in this line.

4.   Copy the set of axes below and the shape labelled A.
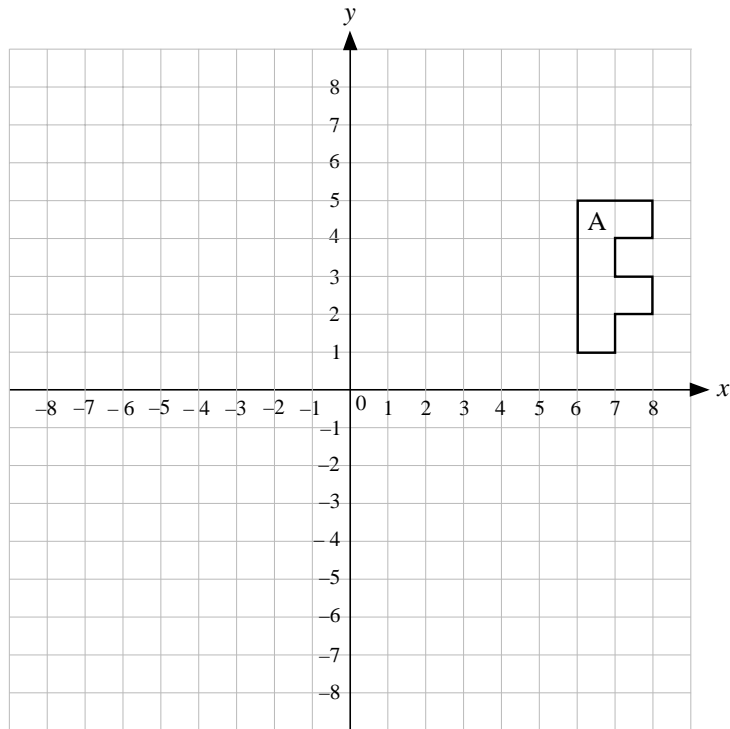
(a)   Reflect A in the line $x = 2$ to obtain B.

(b)   Reflect B in the line $y = -2$ to obtain C.

(c)   Reflect A in the line $y = x$ to obtain D.

(d)   Reflect B in the line $y = -x$ to obtain F.

5.   The diagram shows a number of shapes which have been reflected in various lines.
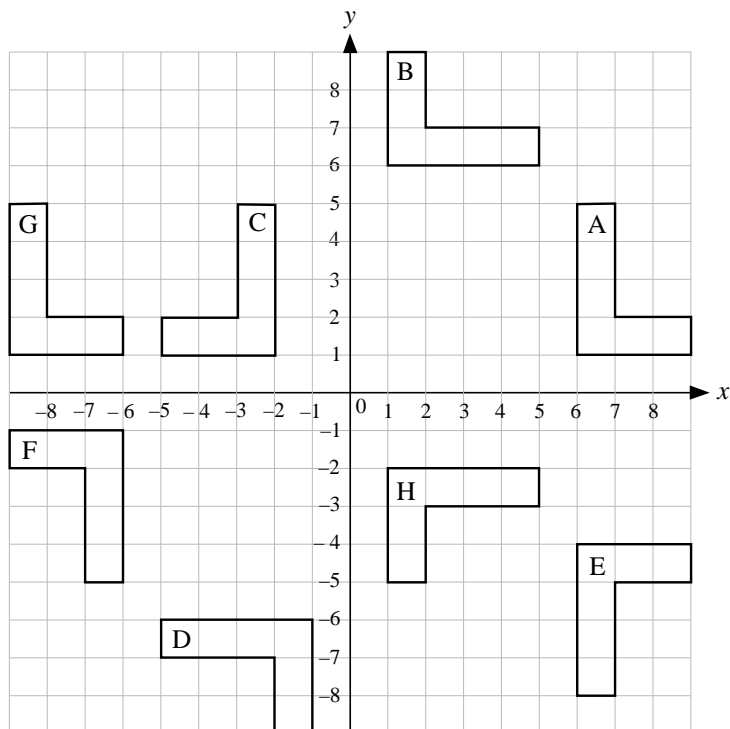
Find the equation for the reflection which maps

(a)   A to B   (b)   A to C   (c)   A to D

(d)   B to F   (e)   C to G   (f)   C to H

(g)   B to H.

6.   (a)   Draw the triangle which has corners at (1, 1), (3, 1) and (1, 4) on a set of
            axes with x and y values from -3 to 8.

      (b)   Reflect the triangle in the lines:

            (i) $y = x + 3$ (ii) $y = 2 - x$

7.   (a)   Draw a set of axes with x and y values from 0 to 6.

      (b)   Join the points

                  (3, 1), (6, 1), (6, 2), (5, 2), (5, 3), (4, 3), (4, 2), (3, 2) and (3, 1).

      (c)   Draw the line $y = x$ and the reflection of this shape in this line.

      (d)   Complete a copy of the table below.

Coordinates of the original points	Coordinates of the reflected points
(3, 1)	(?, ?)
(6, 1)	(?, ?)
(6, 2)	(?, ?)
(5, 2)	(?, ?)
(5, 3)	(?, ?)
(4, 3)	(?, ?)
(4, 2)	(?, ?)
(3, 2)	(?, ?)

      (e)   Describe what happens to the coordinates of a point when it is reflected in
            the line $y = x$.

      (f)   The coordinates of the corners of a rectangle are

                  (3, 2), (5, 2), (3, 3) and (5, 3).

            If the rectangle was reflected in the line $y = x$, what would be the
            coordinates of the corners of the image?

      (g)   Draw both rectangles and check your answers.

Information

*The Babylonians (around 2000 BC) were the first to divide a circle into 360 parts called
'degrees'. This was the consequence of the Babylonians' estimation of 360 days in a year.
The Babylonians used a **sexagesimal** system (base 60). Thus angles at a point equal 360˚,
one degree equals 60 minutes and one minute equals 60 seconds.*

14.9 Rotations

Rotations are obtained when a shape is rotated about a fixed point, called the *centre of rotation*, through a specified angle. The diagram shows a number of rotations.
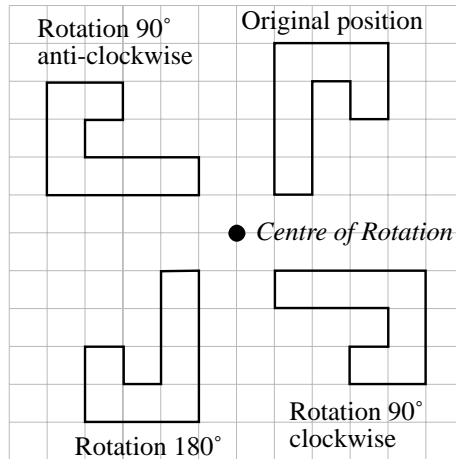
It is often helpful to use tracing paper to find the position of a shape after a rotation.

Worked Example 1

Rotate the triangle shown in the diagram through 90° clockwise about the point with coordinates (0, 0).
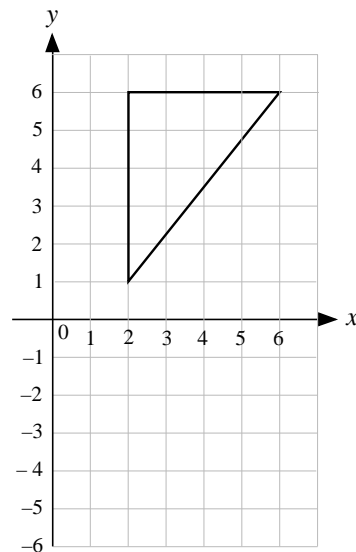
Solution
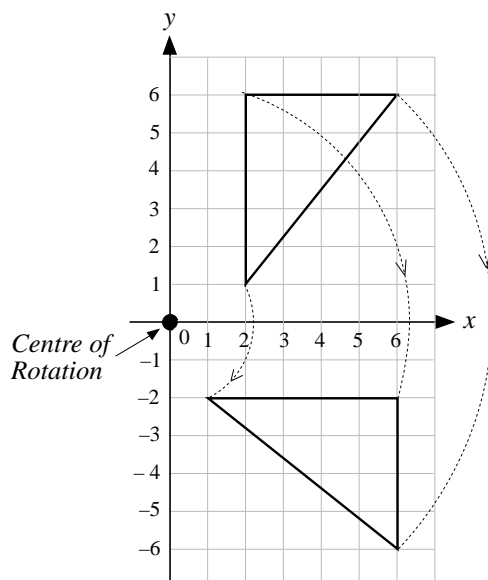
The diagram opposite shows how each corner can be rotated through 90° to give the position of the new triangle.

Worked Example 2

The diagram shows the position of a shape A and the shapes, B, C, D, E and F which are obtained from A by rotation.
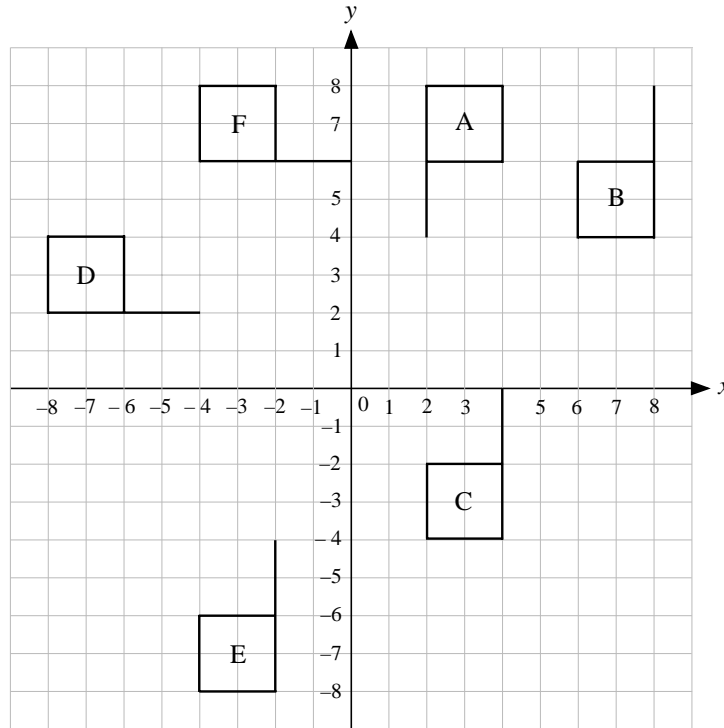
Describe the rotation which moves A onto each other shape.

Solution

The diagram shows the centres of rotation and how one corner of the shape A was rotated.
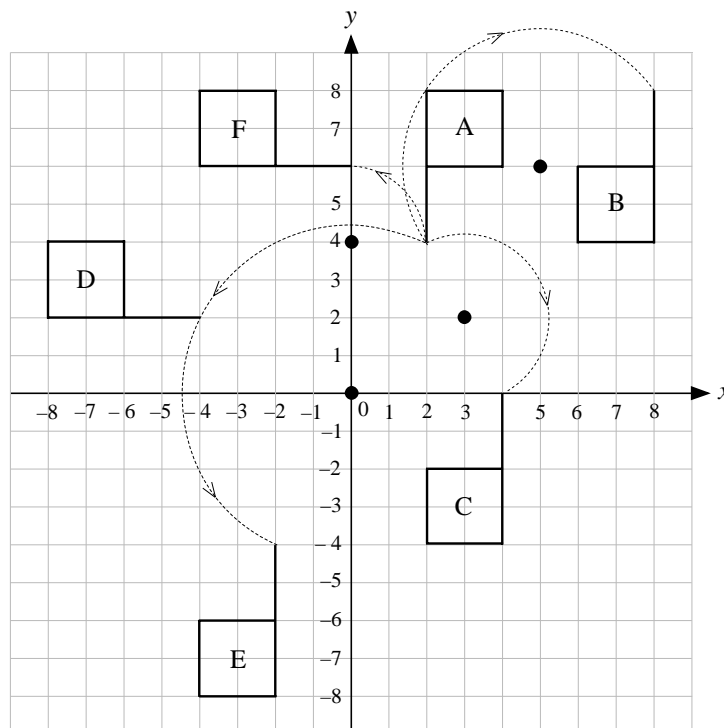
Each rotation is now described.

- A to B: Rotation of 180° about the point (5, 6).
- A to C: Rotation of 180° about the point (3, 2).
- A to D: Rotation of   90° anti-clockwise about the point (0, 0).
- A to E: Rotation of 180° about the point (0, 0).
- A to F: Rotation of   90° anti-clockwise about the point (0, 4).

Exercises

1.   Copy the axes and triangle
      shown opposite.

      (a)   Rotate A through 90°
             clockwise around (0, 0)
             to obtain B.

      (b)   Rotate A through 90°
             anticlockwise around
             (0, 0) to obtain C.

      (c)   Rotate A through 180°
             around (0, 0) to obtain D.
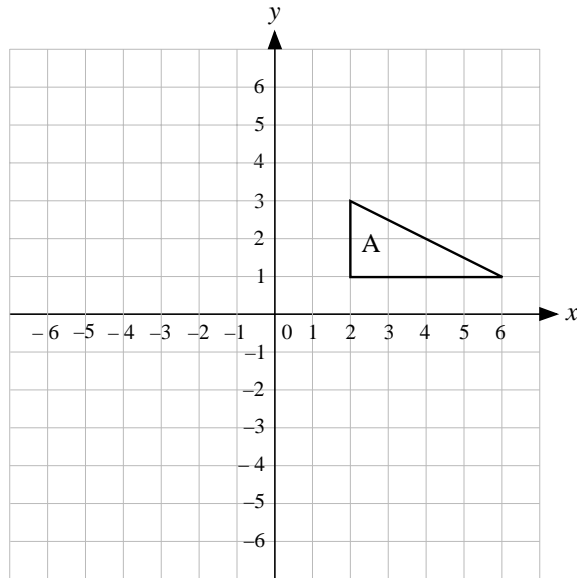
2.   Repeat *Question 1* for the triangle with coordinates (3, 1), (6, 2) and (0, 4).

3.   Copy the axes and triangle shown below.
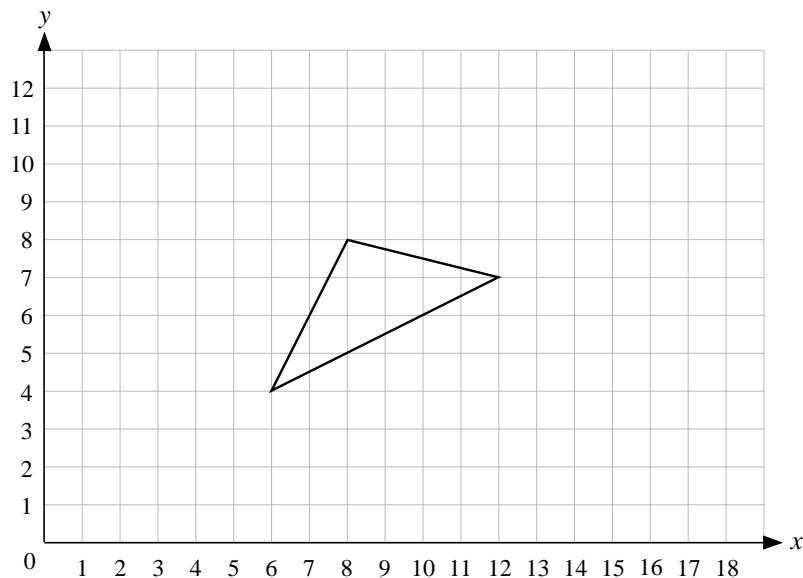
      Rotate the triangle through 180° using each of its corners as the centre of rotation.

4.   Copy the axes and shape shown below.
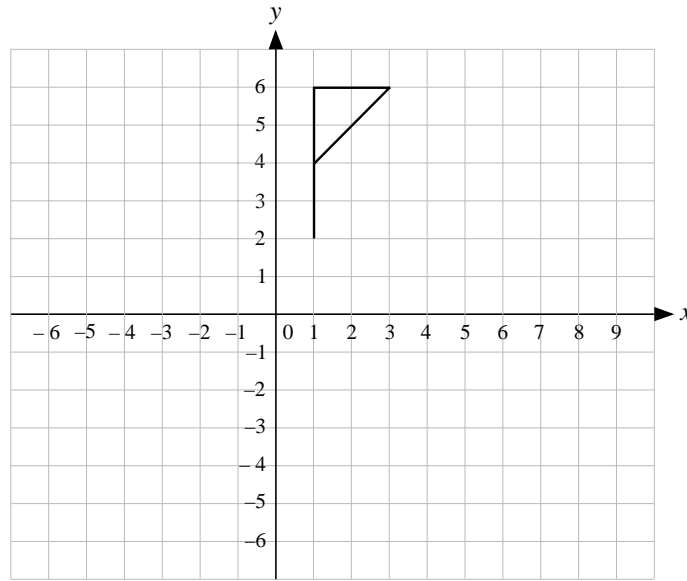
(a)   Rotate the original shape through 90° clockwise around the point (1, 2).

(b)   Rotate the original shape through 180° around the point (3, 4).

(c)   Rotate the original shape through 90° clockwise around the point (1, −2).

(d)   Rotate the original shape through 90° anti-clockwise around the point (0, 1).

5.   Repeat *Question 4* for the triangle with corners at (2, 2), (1, 3) and (3, 5).

6.   The diagram shows the position of a shape labelled A and other shapes which were obtained by rotating A.
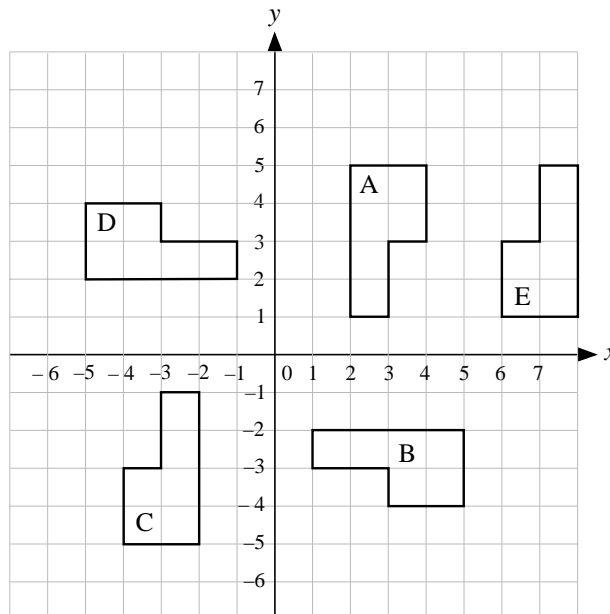
(a)   Describe how each shape can be obtained from A by a rotation.

(b)   Which shapes can be obtained by rotating the shape E?

7. The shape A has been rotated to give each of the other shapes shown. For each shape, find the centre of rotation.
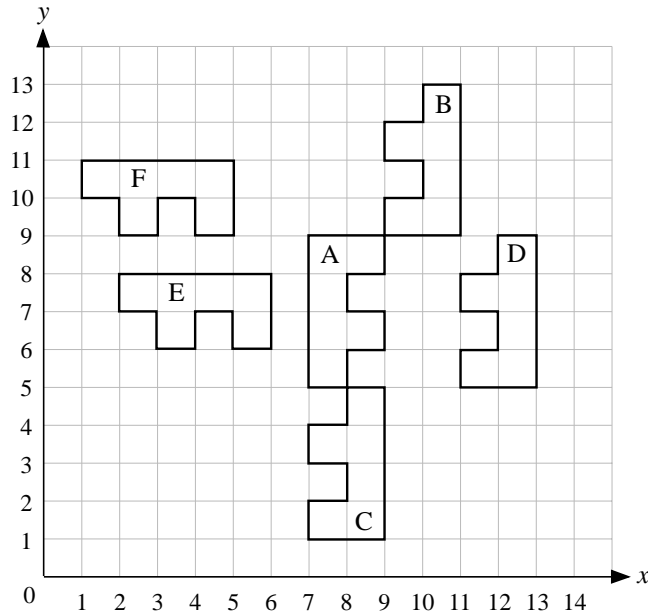
8. (a) Describe how each shape shown below can be obtained from A by a rotation.
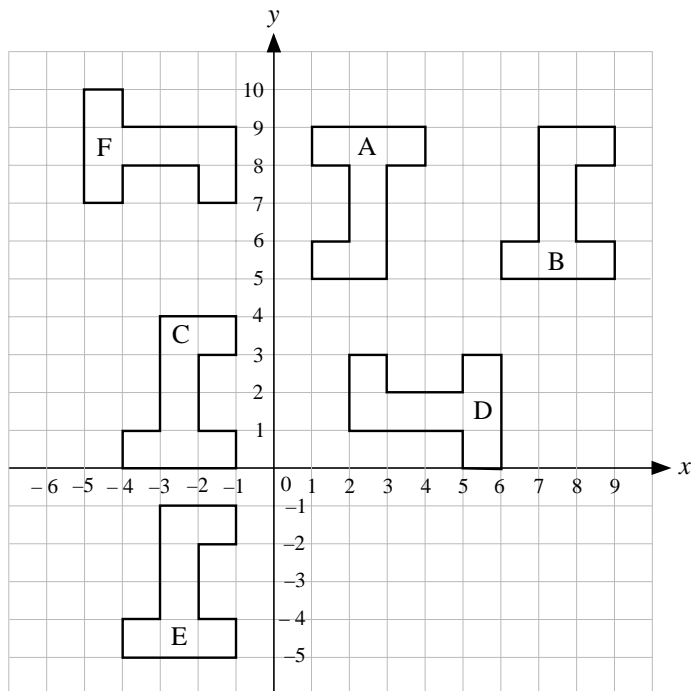
(b) Which shapes cannot be obtained from C by a rotation?

9.   On a set of axes with x and y values from -2 to 12, draw the triangle with corners at the points $(0, 0)$, $(4, 5)$ and $(1, 4)$.

(a)   Rotate the triangle through $90°$ clockwise about the point $(5, 6)$.

(b)   Rotate the second triangle through $90°$ clockwise about the point $(4, 3)$.

(c)   Describe how to obtain the third triangle from the original triangle by a single rotation.

10.   The shape B can be obtained from A by two rotations. Describe these rotations.
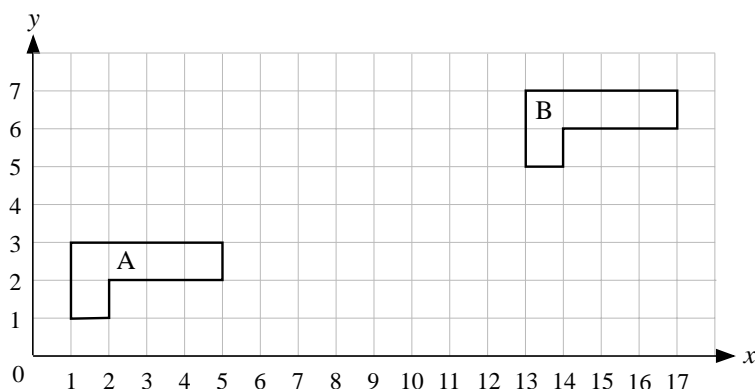
14.10 Translations

A *translation* moves all the points of an object in the same direction and the same distance. The diagram shows a translation.

Here every point has been moved 8 units to the right and 3 units up.

This translation is described by the vector

$$\begin{pmatrix} 8 \\ 3 \end{pmatrix}.$$
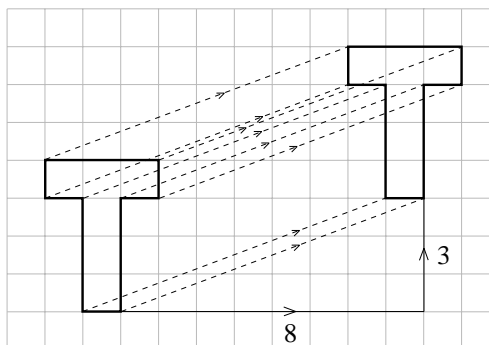
Worked Example 1

Describe the translation which moves the shaded shape to each of the other shapes shown.
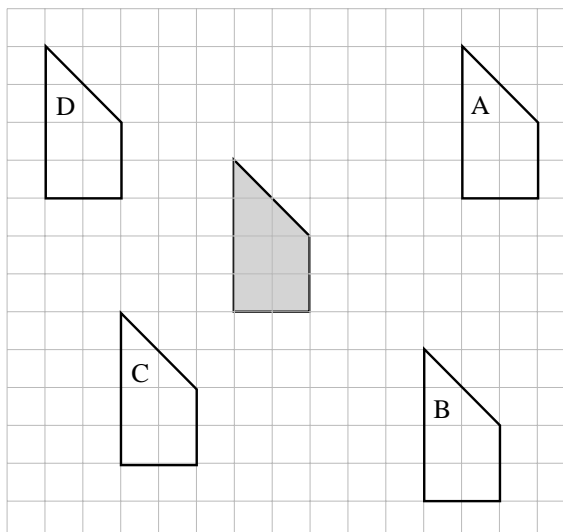
Solution
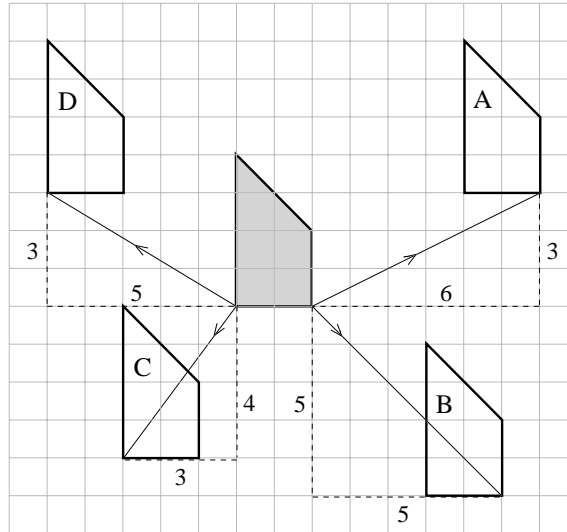
To move to A, the shaded shape is moved 6 units to the right and 3 units up.

This is described by the vector $\begin{pmatrix} 6 \\ 3 \end{pmatrix}$.

To obtain B, the shaded shape is moved 5 units across and 5 units down.

This is described by the vector $\begin{pmatrix} 5 \\ -5 \end{pmatrix}$.

To obtain C, the shaded shape is moved 3 units to the left and 4 units down.

This is described by the vector $\begin{pmatrix} -3 \\ -4 \end{pmatrix}$.

To obtain D, the shaded shape is moved 5 units to the left and then 3 units up.

This is described by the vector $\begin{pmatrix} -5 \\ 3 \end{pmatrix}$.

Worked Example 2

The shape shown in the diagram is to be

translated using the vector $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$.

Draw the image obtained using this translation.
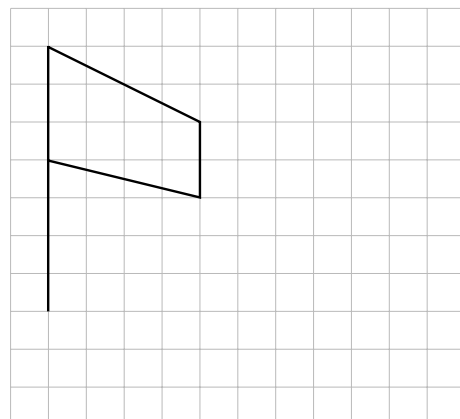
Solution

The vector $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$ describes a translation which moves an object 6 units to the right and

2 units down. This translation can be applied to each point of the original.
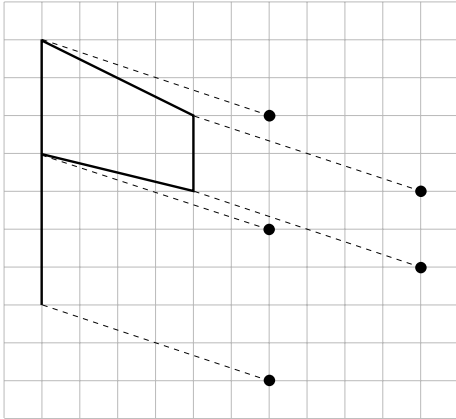
The points can then be joined to give the

translated image.
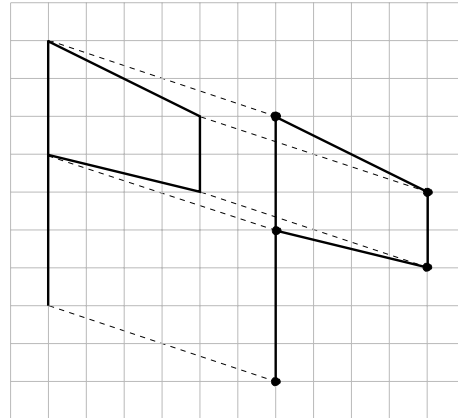
Exercises

1.   The shaded shape has been moved to each of the other positions shown by a
      translation. Give the vector used for each translation.
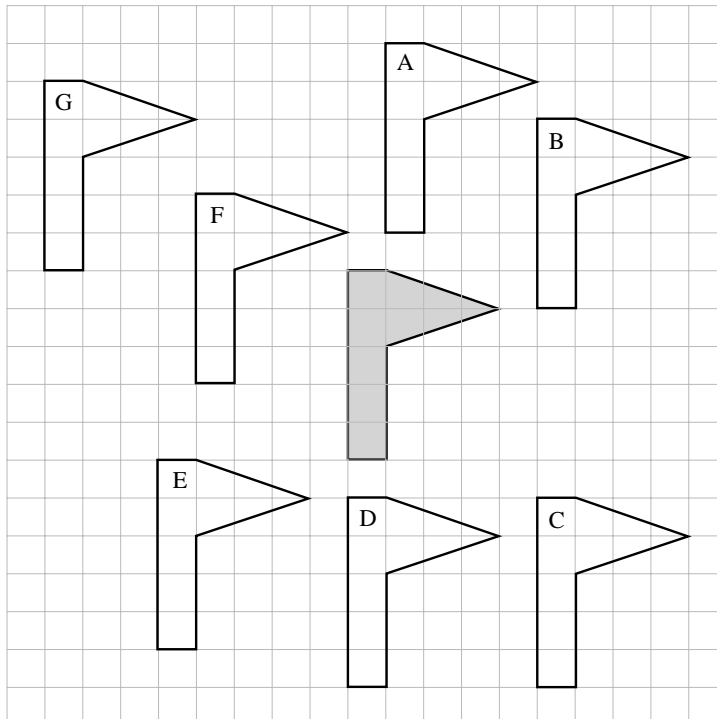
2. Describe the translation which moves:

    (a)   A → C

    (b)   C → B

    (c)   F → E

    (d)   B → D

    (e)   D → B

    (f)   E → C

    (g)   C → D

    (h)   A → F
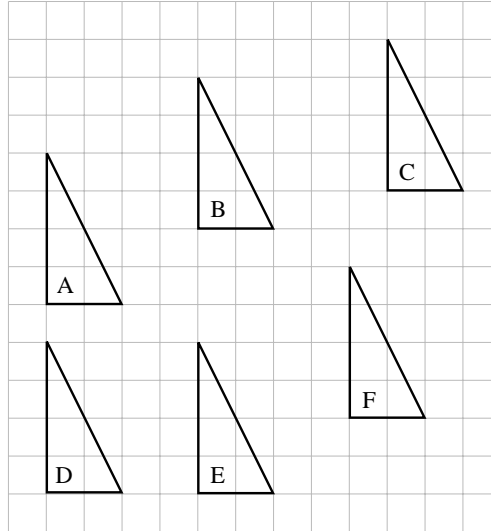
3. Draw the shape shown and its image when translated using each of the following vectors.

    (a) $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ (b) $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

    (c) $\begin{pmatrix} -2 \\ -5 \end{pmatrix}$ (d) $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$
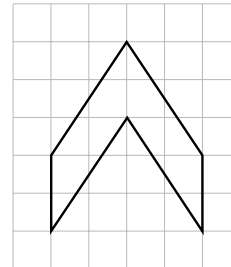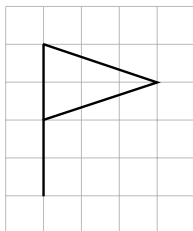
4. (a)   Draw the shape shown.

    (b)   Translate using the vector $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

    (c)   Translate the image using the vector $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

    (d)   Which vector would be needed to translate the final image back to the position of the original?

5.
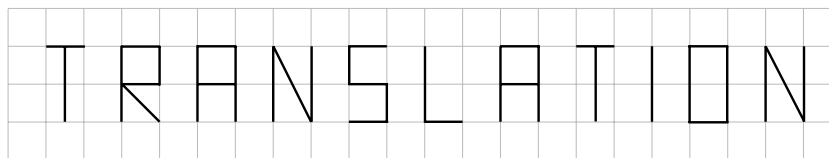
    (a)   Describe a translation which would move one A onto another A.

    (b)   Describe any other translations which would move a letter onto the same letter in a different position.

6. The number 45 can be formed by translation of the lines A and B.
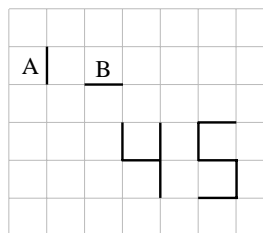
   Describe the translations which need to be applied to A and B to form the number 45.

7. (a) Draw a simple shape.

   (b) Write down the coordinates of each corner of your shape.

   (c) Translate the shape using the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and write down the coordinates of the new shape.

   (d) Compare the coordinates obtained in (b) and (c). How do they change as a result of the translation?

   (e) Repeat (c) and (d) with a translation using the vector $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

8. (a) Draw a simple shape and translate it using the vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

   Then translate the image using the vector $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

   (b) Which single translation would map the original shape to its final position?

   (c) Translate your shape using the vector $\begin{pmatrix} 3 \\ 7 \end{pmatrix}$. Then translate the image using the vector $\begin{pmatrix} -3 \\ -4 \end{pmatrix}$. Which single translation would move the original shape to its final position?

   (d) If a shape was translated using the vector $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and then the vector $\begin{pmatrix} -6 \\ -8 \end{pmatrix}$, which single translation would be equivalent?

9. The points A, B, C and D have coordinates $(4, 7)$, $(2, -6)$, $(-3, -6)$ and $(0, 7)$. Find the vector which would be used to translate:

   (a) A to B   (b) C to D   (c) D to A   (d) A to D.

Just for Fun

*1. By moving only **one** coin in the pattern shown, make one row and one column, each containing 5 coins.*

2. Rearrange the 8 coins to form a square with 3 coins on each side. By rearranging 4 coins, make a square with 4 coins on each side.

14.11 Combined Transformations

An object can be subjected to more than one transformation, so when describing how a shape is moved from one position to another it may be necessary to use two different transformations.

Worked Example 1

Draw the image of the triangle shown if it is first reflected in the line $x = 4$ and then rotated clockwise about the point $(4, 0)$.

Solution

The diagram below shows the line $x = 4$ and the image of the triangle when it has been reflected in this line.
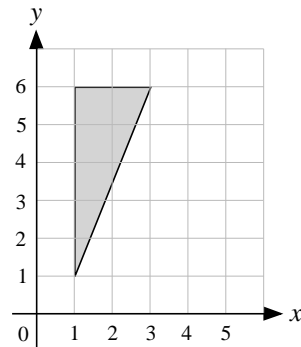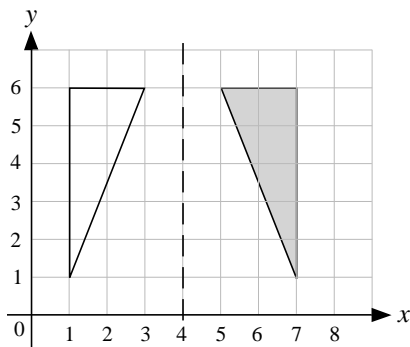
The new image can then be rotated about the point $(4, 0)$, as below.
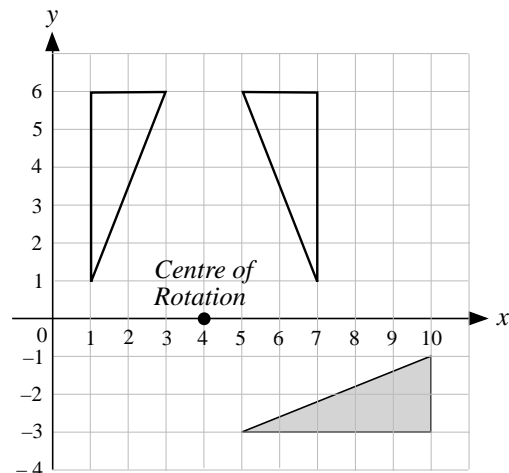
Worked Example 2

Describe two different ways in which the shape marked A can be moved to the position shown at B.
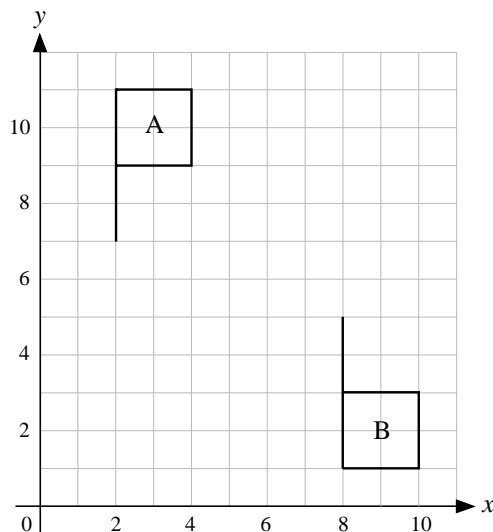
Solution

One way, shown below, is to first translate A using the vector $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$, and then reflect in the line $y = 6$.
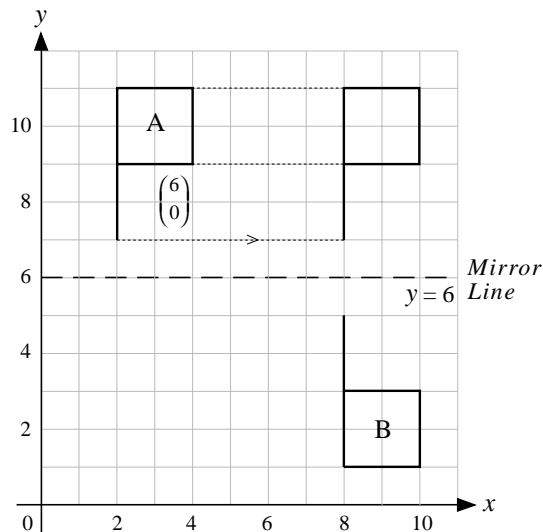
An alternative approach is to rotate shape A through $180°$ around point $(2, 6)$.

This can then be reflected in the line $x = 6$ to obtain B, as shown below.
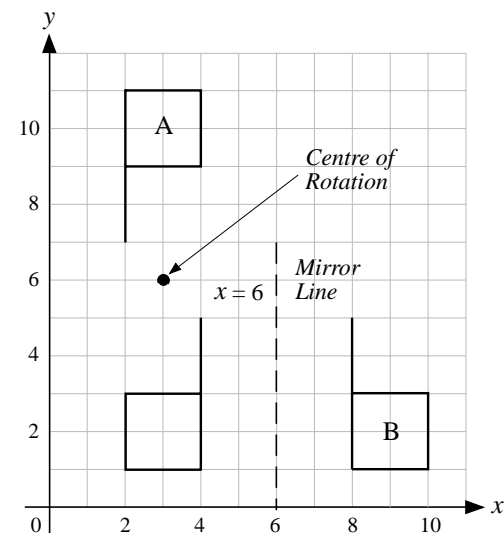
Exercises

1. (a) Draw a set of axes with *x* and *y* values from 0 to 9. Plot the points $(5, 1)$, $(7, 4)$, $(9, 4)$ and $(7, 1)$. Join them to form a single shape.

    (b) Reflect the shape in the line $y = 4$.

    (c) Translate the shape obtained in (b) using the vector $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$.

    (d) Rotate the original shape through $180°$ about the point with coordinates $(5, 4)$

2. (a) Draw a set of axes with *x*-values from 0 to 14 and *y*-values from 0 to 4. Join the points $(1, 4)$ and $(4, 1)$ to form a straight line. Rotate this line through $90°$ clockwise around the point $(4, 1)$.

    (b) Describe two ways in which the shape you have obtained could be transformed into a 'W' shape.

3. The letter P is to be formed by applying a number of transfomations to the solid line. Each transformation maps the solid line onto one of the dashed lines.

    Describe how this could be done using:

    (a) only rotations

    (b) only reflections.
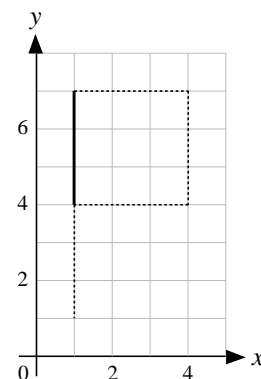
4. (a) Draw a set of axes with *x* values from 0 to 16 and *y* values from −12 to 8.

   (b) Join the points with coordinates (4, 1), (6, 1) and (6, 4) to form a triangle.

   (c) Enlarge this triangle with scale factor 2 using the point (0, 1) as the centre of enlargement.

   (d) Rotate the new triangle through 180° about the point (9, −2).

   (e) Describe fully the transformations which map the final triangle back onto the original.

5. (a) Draw a set of axes with *x* and *y* values from 0 to 8. Plot and join the points (1, 5), (3, 5), (3, 8) and (1, 7).

   (b) Rotate this shape through 90° clockwise around the point (3, 5).

   (c) Then translate the new image using the vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

   (d) Reflect this image in the line $y = 4$.

   (e) How could the final image be mapped back to the position of the original with a single transformation?

6. (a) Draw a set of axes with *x* and *y* values from 0 to 10. Plot the points with coordinates (2, 1), (4, 1) and (2, 4) and join them to form a triangle.

   (b) Reflect this triangle in the line $x = 4$ and then reflect the image in the line $x = 7$.

   (c) Which single transformation would map the original triangle onto its final position?

   (d) Reflect the original triangle in the line $y = 6 - x$ and then reflect this image in the line $y = 11 - x$. Which single transformation would map the final triangle back to its original position?

7. (a) The triangle A can be mapped onto B, C and D using single transformations.

   Describe fully each transformation.

   (b) The triangle A can be mapped onto E, F, G and H using two transformations.

   Describe fully each pair of transformations.
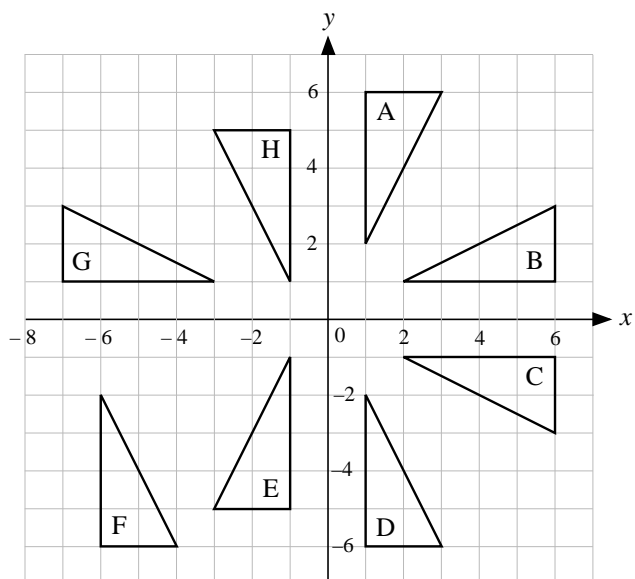
8. (a) (i) Draw a simple shape and reflect it in any vertical line.

    (ii) Reflect the image in any horizontal line.

    (b) Describe two other ways in which the original image could have been mapped onto the final image.

    (c) Repeat (a) and (b) using any two lines which are perpendicular.

    (d) Do you obtain the same result in each case?

9. Copy the diagram below and show the answers to the questions on this diagram. (You are advised to use a pencil.)
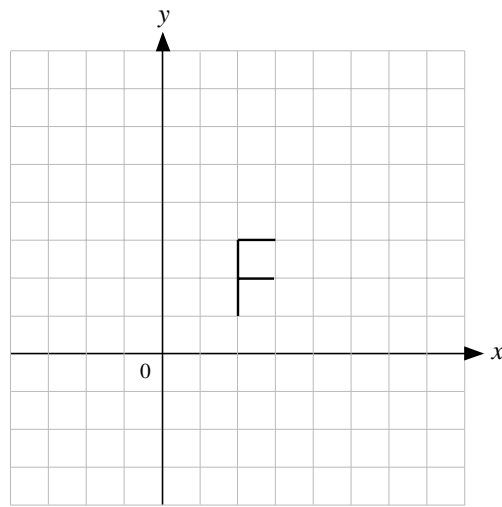
    (a) Draw the reflection of the F in the *x*-axis.

    (b) Rotate the original F through 90° anticlockwise, with O as the centre of rotation. Draw the image.

    (c) Enlarge the original F with centre of enlargement O and scale factor 2.

    (MEG)

10. (a) Triangle A is mapped onto triangle B by means of an anticlockwise rotation, centre the origin, followed by another translation.

    (i) Write down the angle of rotation.

    (ii) Find the column vector of the translation.
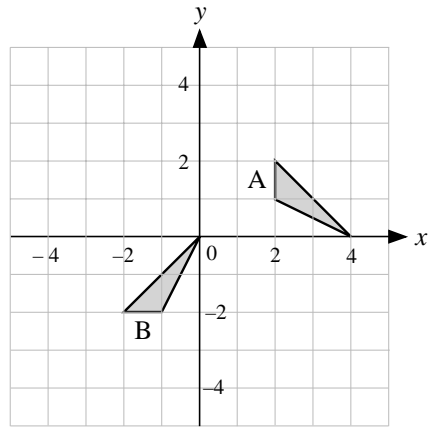
    (b) Triangle A may be mapped onto triangle B by means of a single rotation.

    Find the coordinates of the centre of rotation.

    (c) Triangle B is reflected in the line $y = -2$ to form triangle C. Describe the single transformation which would map triangle A onto triangle C.

    (MEG)

14.12 Congruence

Two shapes are said to be *congruent* if they are identical in every way.

Shapes which are different sizes but which have the same shape are said to be *similar*.

These triangles are *congruent*.
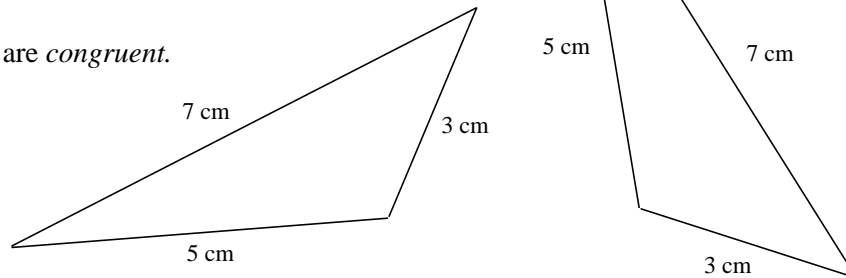
The triangle below is similar to the triangles above but because it is a different size it is not congruent to the triangles above.
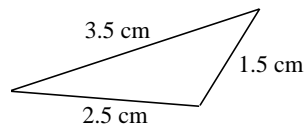
There are four tests for congruence which are outlined below.

TEST 1 (Side, Side, Side)
If all three sides of one triangle are the same as the lengths of the sides of the second triangle, then the two triangles are congruent.
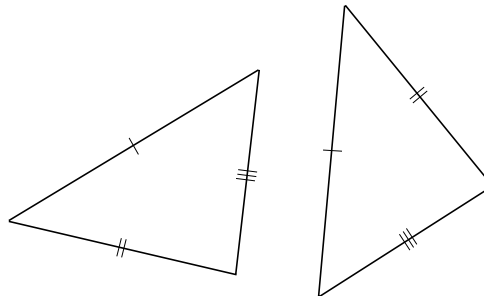
This test is referred to as *SSS*.

TEST 2 (Side, Angle, Side)
If two sides of one triangle are the same length as two sides of the other triangle and the angle *between* these two sides is the same in both triangles, then the triangles are congruent.
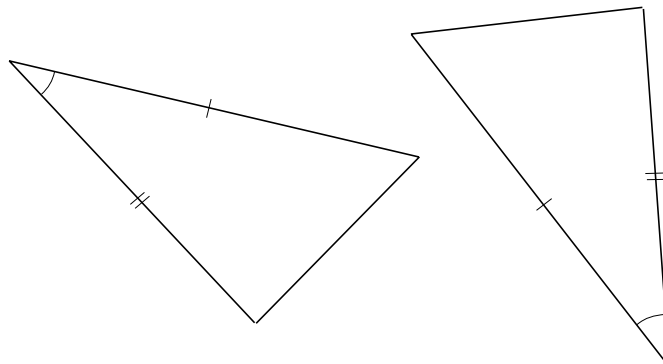
This test is referred to as *SAS*.

TEST 3 (Angle, Angle, Side)

If two angles and the length of
one side are the same in both
triangles then they are congruent.

This test is referred to as *AAS*.
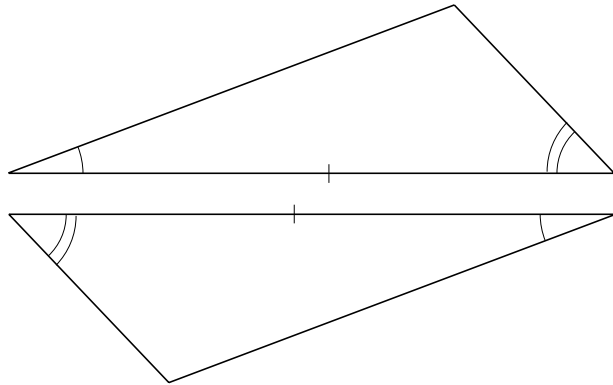
TEST 4 (Right angle, Hypotenuse, Side)

If both triangles contain a right angle,
have hypotenuses of the same length
and one other side of the same length,
then they are congruent.

This test is referred to as *RHS*.
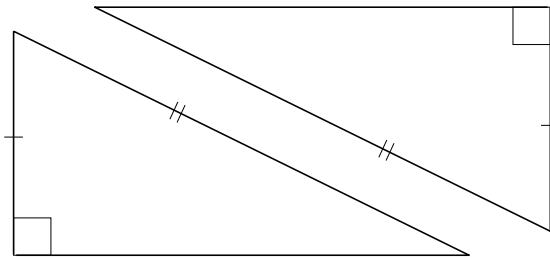
Worked Example 1

Which of the triangles below are congruent to the the triangle ABC?
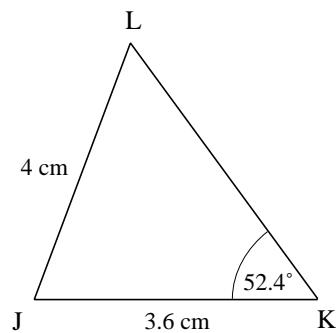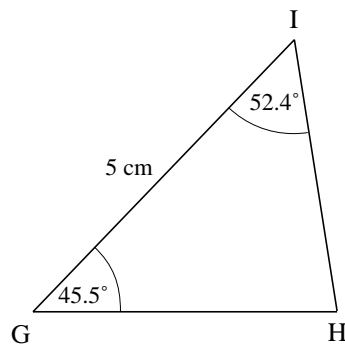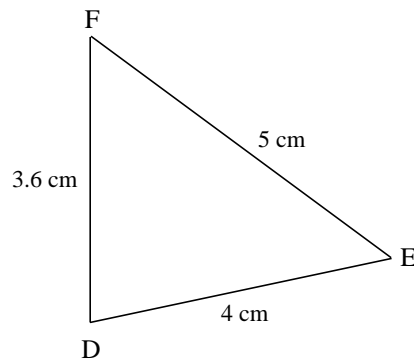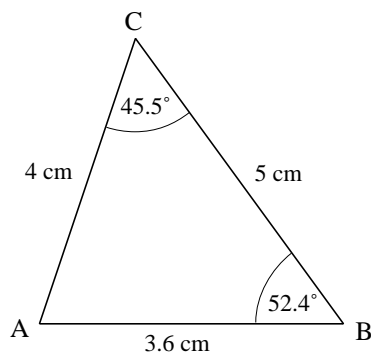
Solution

Consider first the triangle DEF:

$$AB = DE$$
$$BC = EF$$
$$AC = DF$$

As the sides are the same in both triangles the triangles are congruent. *(SSS)*

Consider the triangle GHI:

$$BC = HI$$
$$A\hat{B}C = G\hat{H}I$$
$$A\hat{C}B = G\hat{I}H$$

As the triangles have one side and two angles the same, they are congruent. *(AAS)*

Consider the triangle JKL: Two sides are known but not the angle between, so there is insufficient information to show that the triangles are congruent.

Worked Example 2

ABCDF is a square and $BC = EF$.

Find the pairs of congruent triangles in the diagram.

Solution
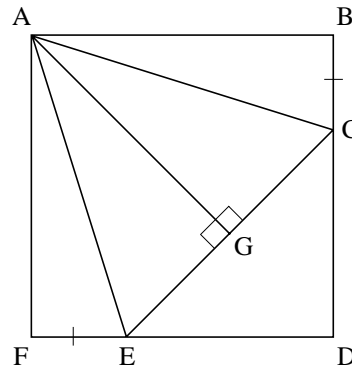
Consider the triangles ABC and AFE:

$AB = AF$	(ABDF is a square.)
$BC = FE$	(This is given in the question.)
$A\hat{B}C = A\hat{F}E = 90°$	(They are corners of a square.)

The triangles ABC and AFE have two sides of the same length and also have the same angle between them, so these triangles are congruent. *(SAS)*

Consider the triangles ACG and AEG:

|		
$AC = AE$	($\triangle$ ABC and $\triangle$ AFE are congruent.)
$AG = AG$	(They are the same line.)
$E\hat{G}A = C\hat{G}A = 90°$	(This is given in the question.)

Both triangles contain right angles, have the same length hypotenuse and one other side of the same length. So the triangles are congruent. *(RHS)*

Investigation

1.   *How many straight lines can you draw to divide a square into two congruent parts?*

2.   *How many lines can you draw to divide a rectangle into two congruent parts?*

3.   *Can you draw two straight lines through a square to divide it into four congruent quadrilaterals which are **not** parallelograms?*

Exercises

1.   Identify the triangles below which are congruent.

A

5 cm   8 cm

C   4 cm   B

F   5 cm   D

4 cm   8 cm

E

G

69°

6 cm   4.61 cm

45°

I   6.08 cm   H

K

66°

4.61 cm   6.08 cm

J   L

O

10 cm

85°   20°   N

M

R

75°   10 cm

20°   Q

P

T

4.61 cm

45°

S   6 cm   U
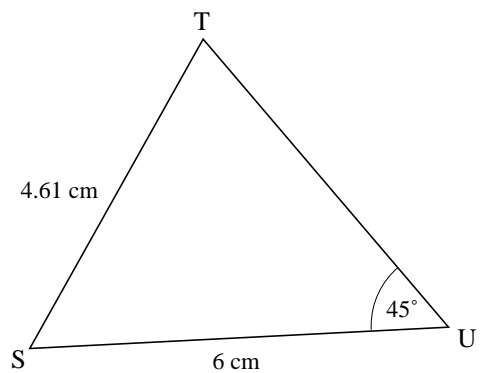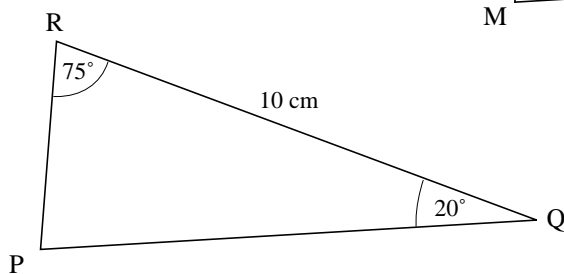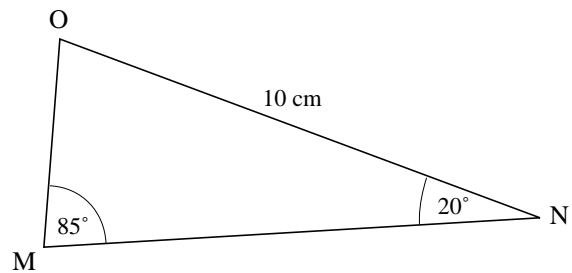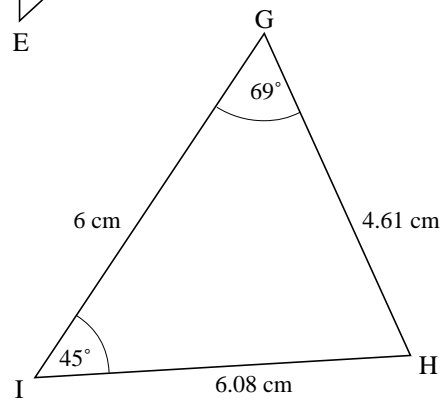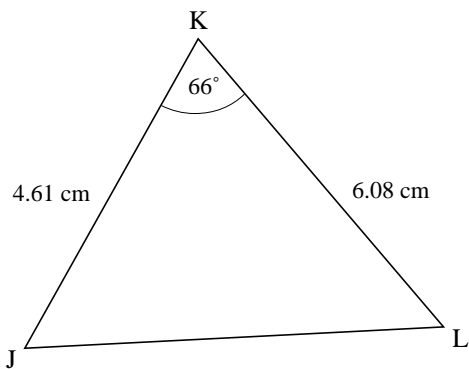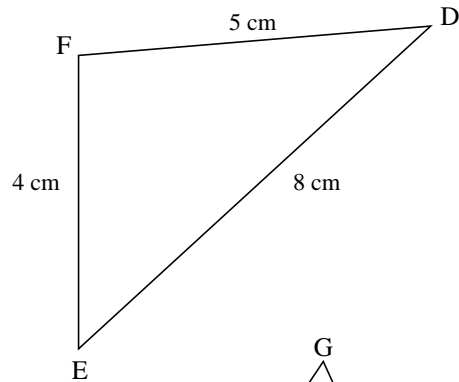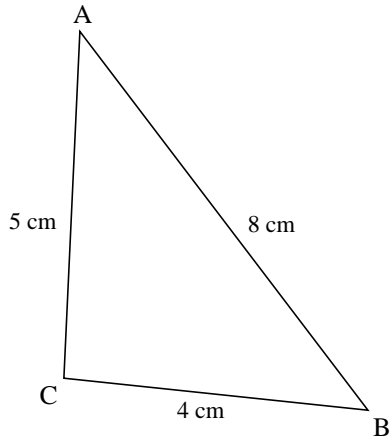
2. If O is the centre of the circle, prove that the triangles OAB and OCB are congruent.
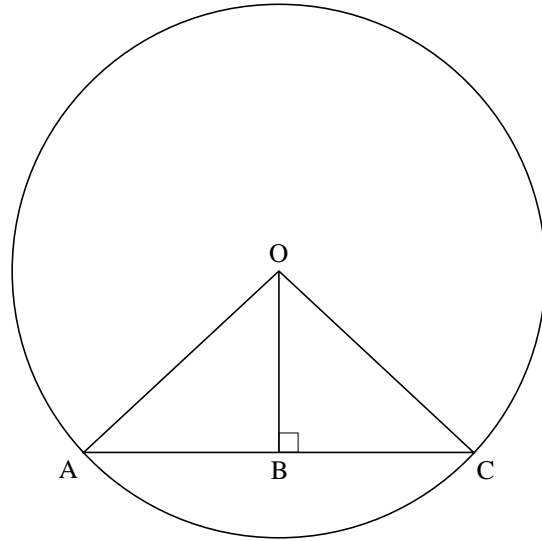
3.
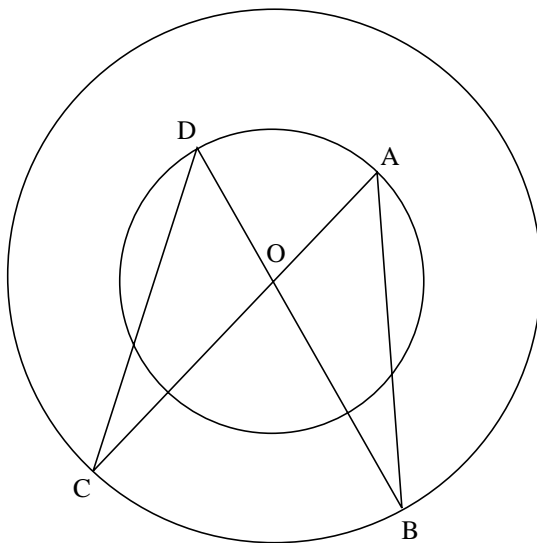
If O is the centre of both circles, prove that the triangles OAB and ODC are congruent.

4. If O is the centre of the circle, prove that the triangles OAB and OCD are congruent.
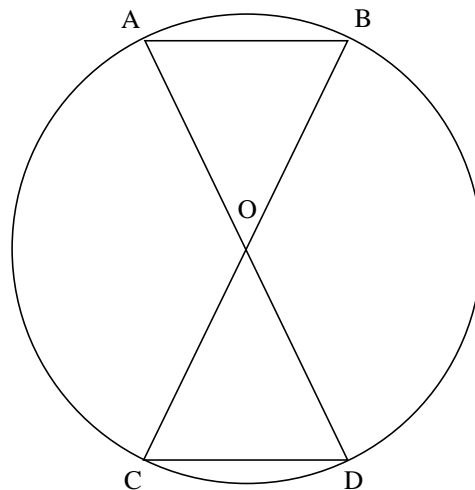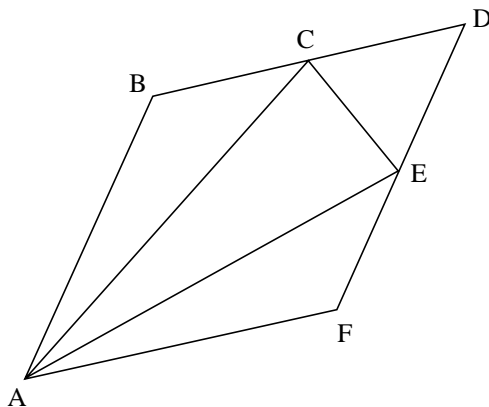
When BC = EF, this rhombus contains two congruent triangles.

Identify the triangles and prove that they are congruent.

6.   If O is the centre of the circle
      and AD = BC, show that ABD
      and CDB are congruent triangles.
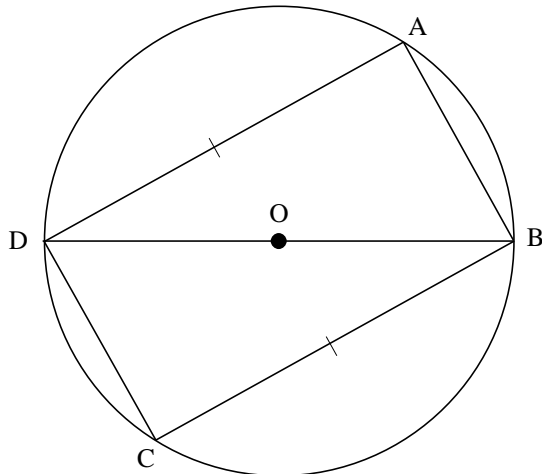
7.   Two triangles have sides of lengths 8 cm and 6 cm, and contain an angle of 30°.
      Show that it is possible to draw 4 different triangles, none of which are congruent,
      using this information.

8.   The diagram shows the parallelogram
      ABCD with diagonals which cross at X.
      Prove that the parallelogram contains
      2 pairs of congruent triangles.
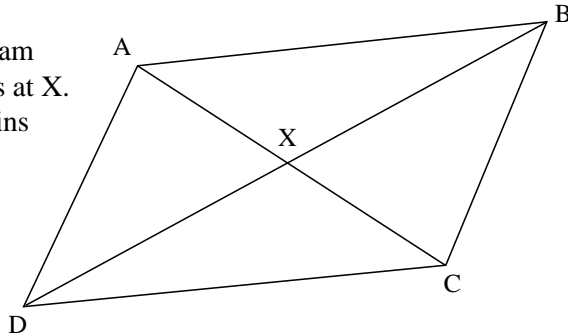
9.   The diagram shows a cuboid.
      The mid-point of the side DC is X.

      (a)   Show that the triangles AHG
             and FAD are congruent.

      (b)   Show that the triangles ADX
             and BCG are congruent.
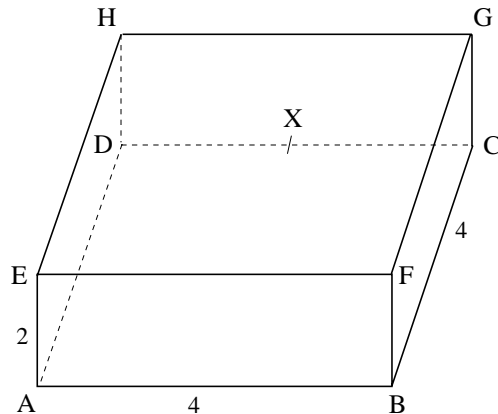
Investigation

1.   In how many ways can you cut a square piece of cake into two congruent parts?

*2.   How can you use eight straight lines of equal length to make a square and four
       congruent equilateral triangles?*

Just for Fun

*Study the diagram opposite and then find the ratio of the
area of the large triangle to that of the small triangle.*
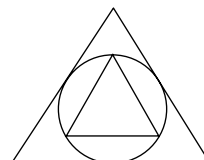
14.13 Similarity

Similar shapes have the same shape but may be different sizes. The two rectangles shown below are similar – they have the same shape but one is much smaller than the other.

2 cm

3 cm

6 cm

9 cm
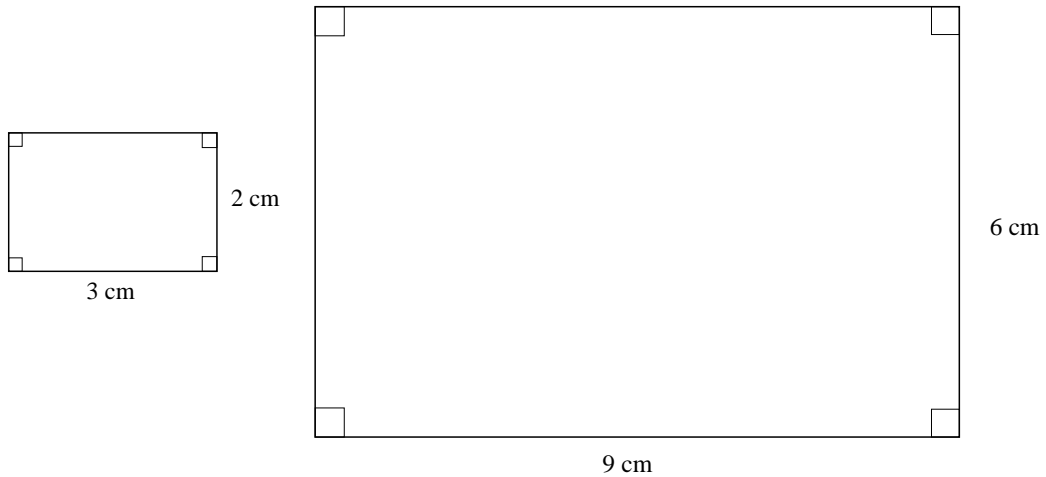
They are similar because they are both rectangles and the sides of the larger rectangle are three times longer than the sides of the smaller rectangle.

It is interesting to compare the area of the two rectangles. The area of the smaller rectangle is 6 cm^2 and the area of the larger rectangle is 54 cm^2, which is nine times $\left(3^2\right)$ greater.

Note

In general, if the length of the sides of a shape are increased by a factor *k*, then the area is increased by a factor k^2.

These two triangles are *not* similar.

3

5

4

4

5

6
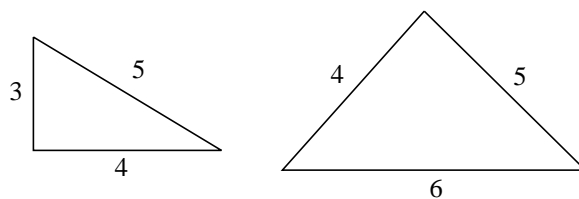
The lengths of the sides of the triangles are not in the same ratio and so the triangles are not similar. For two triangles to be similar, they must have the same internal angles, as shown below.

7.71   30°

50°

6

3.92   100°

15.42   30°

50°

12

7.84   100°
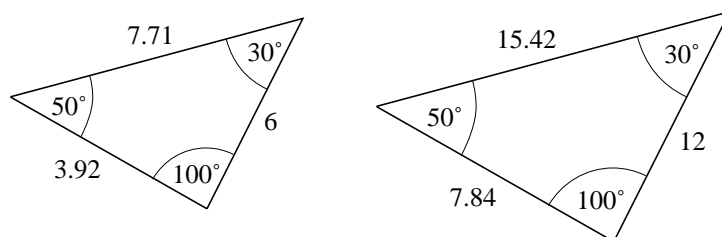
The diagrams below show 3 similar cubes.

1 cm
1 cm
1 cm

2 cm
2 cm
2 cm

3 cm
3 cm
3 cm
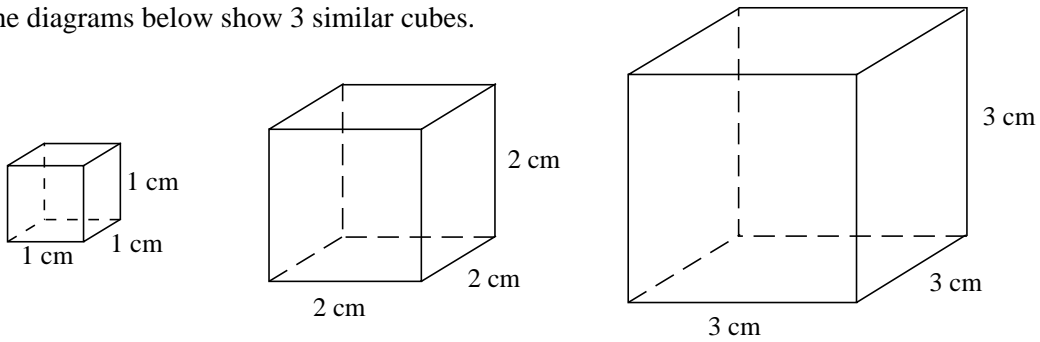
Length of side	Area of one face	Surface Area	Volume
1 cm	1 cm²	6 cm²	1 cm³
2 cm	4 cm²	24 cm²	8 cm³
3 cm	9 cm²	54 cm²	27 cm³

The table gives the lengths of sides, area of one face, total surface area and volume.

Comparing the larger cube with the 1 cm cube we can note that:

For the 2 cm cube

The lengths are	2	times greater.
The areas are	$4 = 2^2$	times greater.
The volume is	$8 = 2^3$	times greater.

For the 3 cm cube

The lengths are	3	times greater.
The areas are	$9 = 3^2$	times greater.
The volume is	$27 = 3^3$	times greater.

Note

If the lengths of a solid are increased by a factor, k, its surface area will increase by a factor k^2 and its volume will increase by a factor k^3.

Worked Example 1

(a)   Which of the triangles shown below are similar?

A

50°
6.13 cm
9.04 cm
60°
70°
8 cm

B

12.26 cm
18.08 cm
16 cm
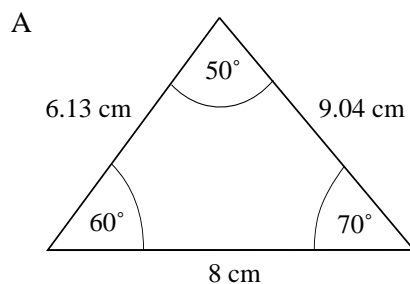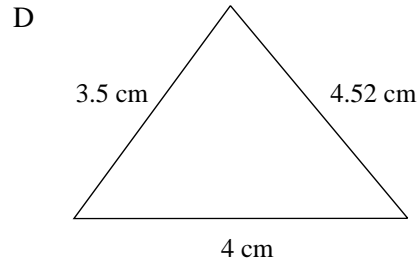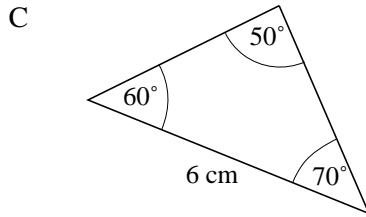
(b) How do the areas of the triangles which are similar compare?

Solution

(a) *First compare triangles A and B*.

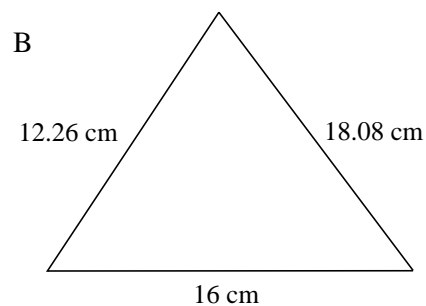Here all the lengths of the sides are twice the length of the sides of triangle A, so the two triangles are similar.

Then compare triangles A and C.

Here all the angles are the same in both triangles, so the triangles must be similar.

Finally, compare triangles A and D.

Note that $4 = \dfrac{1}{2} \times 8$ and $4.52 = \dfrac{1}{2} \times 9.04$, but $3.5 \neq \dfrac{1}{2} \times 6.13$.

So these triangles are *not* similar.

(b) The lengths of triangle B are 2 times greater than the lengths of triangle A, so the area will be $2^2 = 4$ times greater.

The lengths of triangle C are $\dfrac{3}{4}$ of the lengths of triangle A.

So the area will be $\left(\dfrac{3}{4}\right)^2 = \dfrac{9}{16}$ of the area of triangle A.

The ratio of the areas of triangles $C : A : B$ can be written as:

$$\dfrac{9}{16} : 1 : 4$$

or $\qquad 9 : 16 : 64.$

Worked Example 2

(a) Explain why the triangles ABE and ACD are similar.

(b) Find the lengths of x and y.

(c) Find the ratio of the area of ABE to BCDE.

Solution

(a) As the lines BE and CD are parallel,

$$A\hat{B}E = A\hat{C}D$$

and $\qquad A\hat{E}B = A\hat{D}C.$

As the corner, A, is common to both triangles,

$$D\hat{A}C = E\hat{A}B.$$

So the three angles are the same in both triangles and therefore they are similar.

(b)   Comparing the sides BE and CD, the lengths in the larger triangle are 1.5 times the lengths in the smaller triangle. Alternatively, it can be stated that the ratio of the lengths is $2 : 3$.
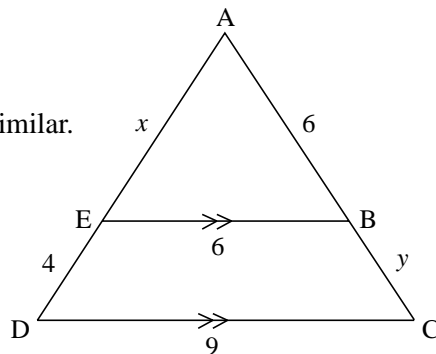
So AC will be 1.5 times bigger than AB.

$$\begin{aligned} AC &= 1.5 \times 6 \\ &= 9. \end{aligned}$$

$$\text{So} \qquad y = 3.$$

In the same way, $\qquad AD = 1.5 \times AE,$

$$\text{so} \qquad 4 + x = 1.5x$$

$$4 = 0.5x$$

$$x = 8.$$

(c)   As the lengths are increased by a factor of 1.5 or $\dfrac{3}{2}$ for the larger triangle, the areas

will be increased by a factor of 1.5^2 or $\left(\dfrac{3}{2}\right)^2$. We can say that the ratio of the areas

of the triangles is $1 : 2.25$ or $1 : \dfrac{9}{4}$ or $4 : 9$.
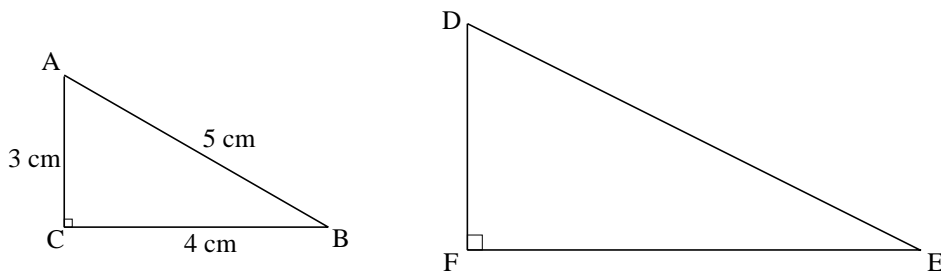
If the area of the triangle ABE is $4k$, then the area of the triangle ACD is $9k$ and hence the area of the quadrilateral BCDE is $5k$.

So the ratio of the area of ABE to BCDE is $4 : 5$.

Worked Example 3

The diagrams show two similar triangles.

If the area of the triangle DEF is 26.46 cm^2, find the lengths of its sides.

Solution

If the lengths of the sides of the triangle DEF are a factor k greater than the lengths of the sides of the triangle ABC, then its area will be a factor k^2 greater than the area of ABC.

$$\text{Area of ABC} = \frac{1}{2} \times 4 \times 3$$

$$= 6 \text{ cm}^2.$$

$$\text{So} \quad 6 \times k^2 = 26.46$$

$$k^2 = 4.41$$

$$k = \sqrt{4.41}$$

$$= 2.1.$$

So the lengths of the sides of the triangle DEF will be 2.1 times greater than the lengths of the sides of the triangle ABC.

$$\text{DE} = 2.1 \times 5$$

$$= 10.5 \text{ cm}$$

$$\text{DF} = 2.1 \times 3$$

$$= 6.3 \text{ cm}$$

$$\text{EF} = 2.1 \times 4$$

$$= 8.2 \text{ cm}.$$

Worked Example 4

A can has a height of 10 cm and has a volume of 200 cm^3. A can with a similar shape has a height of 12 cm.

(a)   Find the volume of the larger can.

(b)   Find the height of a similar can with a volume of 675 cm^3.

Solution

(a)   The lengths are increased by a factor of 1.2, so the volume will be increased by a factor of 1.2^3.

$$\text{Volume} = 200 \times 1.2^3$$

$$= 345.6 \text{ cm}^3.$$

(b)   If the lengths are increased by a factor of k, then the volume will be increased by k^3.

$$675 = 200 \times k^3$$

$$k^3 = 3.375$$

$$k = \sqrt[3]{3.375}$$

$$= 1.5.$$

So the height must be increased by a factor of 1.5, to give

$$\text{height} = 1.5 \times 10$$

$$= 15\text{ cm}.$$

Exercises

1. Which of the triangles below are similar?

A
5 cm 78.5°
57.1°
6 cm
7 cm 44.4°

B
60°
8 cm
4 cm
30°
6.92 cm

C
12 cm
11 cm
14 cm

D
60°
20 cm
30°

E
44.4°
1 cm
78.5° 57.1°

F
56 cm
48 cm
44 cm
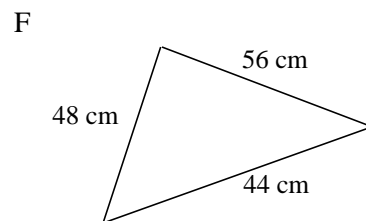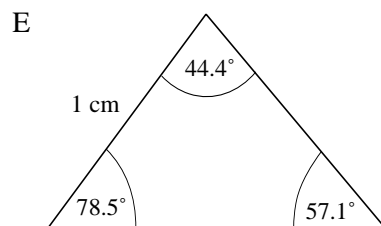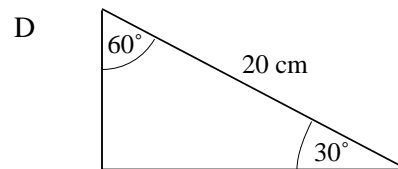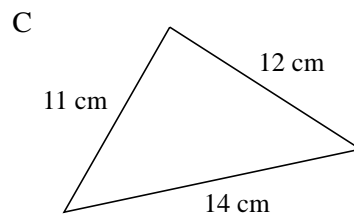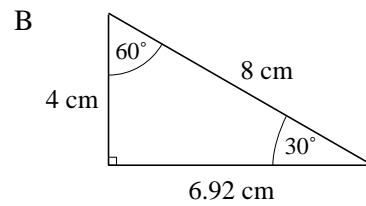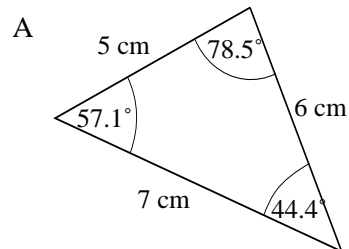
2. The diagram shows 2 similar triangles.

D 12.99 E
A 4.33 B
5
60°
C
F
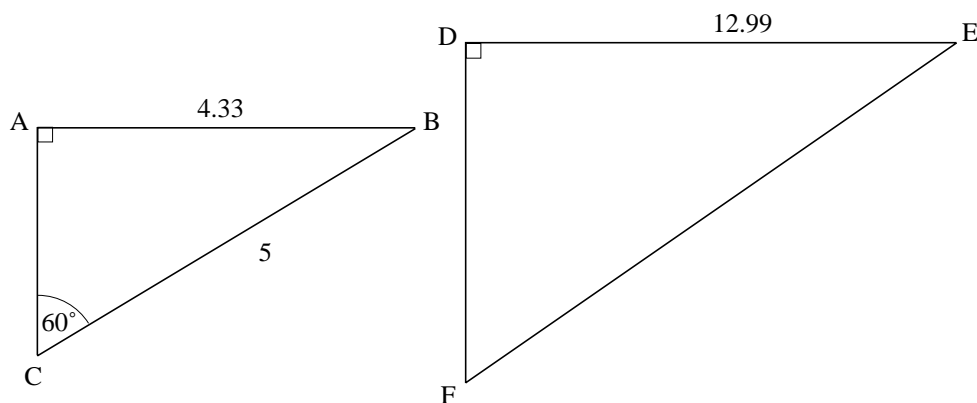
(a) Copy the triangles and label all the angles and the lengths of all the sides.

(b) How do the areas of the two triangles compare?
     (Express your answer as a ratio.)

3.   The diagram shows two similar rectangles, ABCD and EFGH.
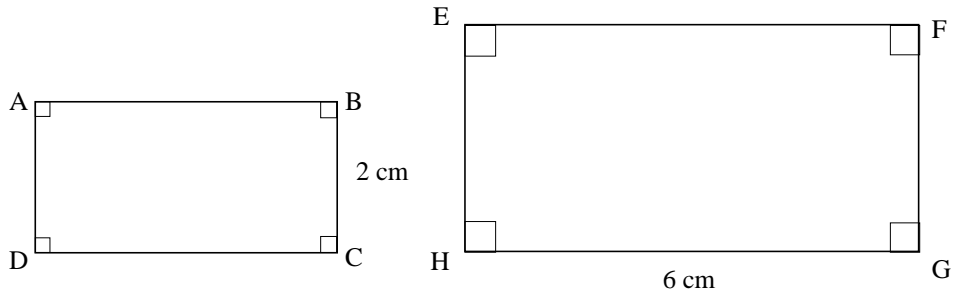
Find the lengths of AB and EH if the ratio of the area of ABCD to the area of EFGH is:

(a)   1 : 4   (b)   4 : 9.

4.   The diagram shows two regular hexagons and   AB = BC.

(a)   What is the ratio of the area of the smaller
        hexagon to the area of the larger hexagon?

(b)   What is the ratio of the area of the smaller
        hexagon to that of the shaded area?
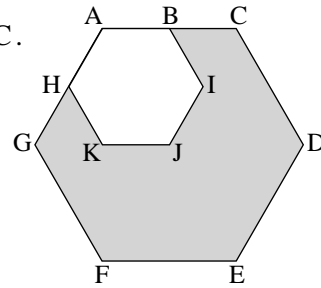
5.   (a)   Explain why the triangles
              ABE and ACD are similar.

      (b)   Find the lengths of:

              (i)   AD

              (ii)   DE

              (iii)   BC
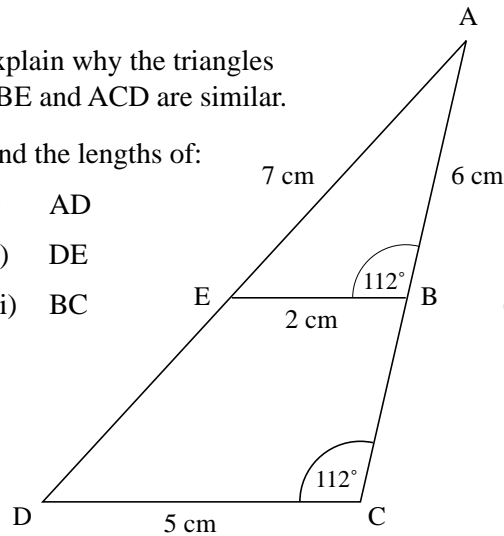
      (c)   Find the ratio of
              the areas of:

              (i)   ABE to ACD

              (ii)   ABE to BCDE.

6.   (a)   Explain why ABE and ACD are similar triangles.

      (b)   What can be deduced about the lines BE and CD?

      (c)   Find the lengths x and y.

      (d)   Find the ratio of the area of the triangle ABE

              to the area of the quadrilateral BCDE.

7. A bottle has a height of 8 cm and a volume of 30 cm^3. Find the volume of similar bottles of heights:

(a) 12 cm   (b) 10 cm   (c) 20 cm.

8. A box has a volume of 50 cm^3 and a width of 6 cm. A similar box has a width of 12 cm.

(a) Find the volume of the larger box.

(b) How many times bigger is the surface area of the larger box?

9. A packet has the dimensions shown in the diagram.

All the dimensions are increased by 20%.
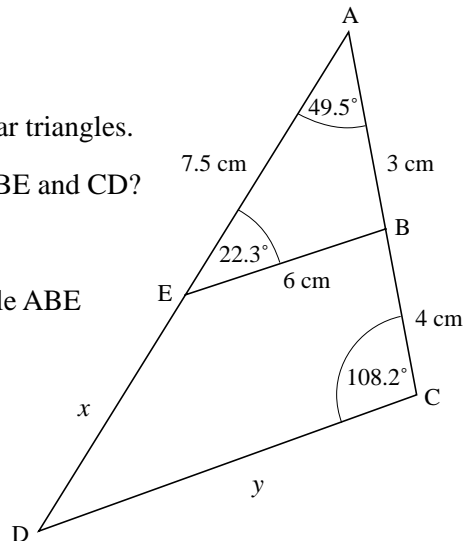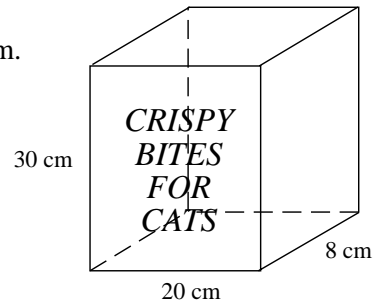
30 cm

CRISPY BITES FOR CATS

8 cm

20 cm

(a) Find the percentage increase in:

(i) surface area   (ii) volume.

(b) Find the percentage increase needed in the dimensions of the packet to increase the volume by 50%.

10. Two similar cans have volumes of 400 cm^3 and 1350 cm^3.

(a) Find the ratio of the heights of the cans.

(b) Find the ratio of the surface areas of the cans.

11. One box has a surface area of 96 cm^2 and a height of 4 cm. A second similar box has a volume of 1728 cm^3 and a surface area of 864 cm^2.

Find:

(a) the height of the larger box   (b) the volume of the smaller box.

12

X

*Diagram **not** accurately drawn*

18.9 cm

A

4.5 cm

O

15 cm
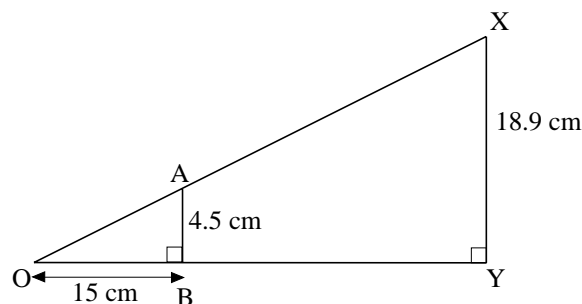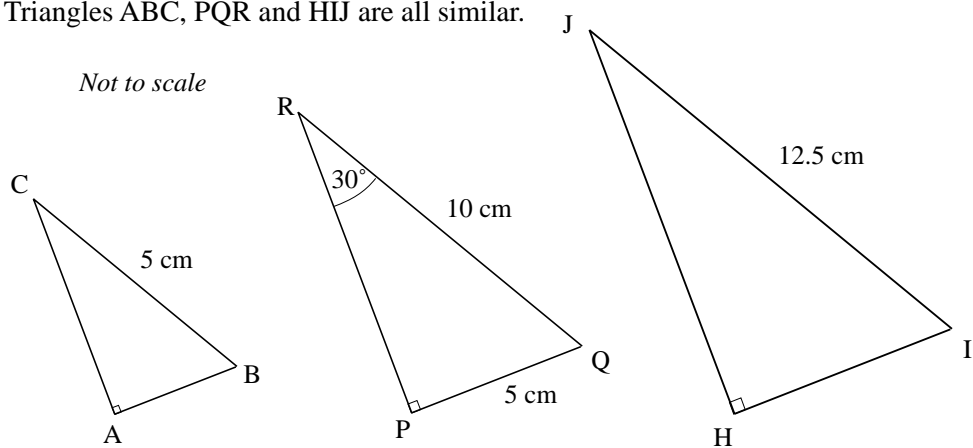
B

Y

(a) Calculate the length of OY.

(b) Calculate the size of angle XOY.

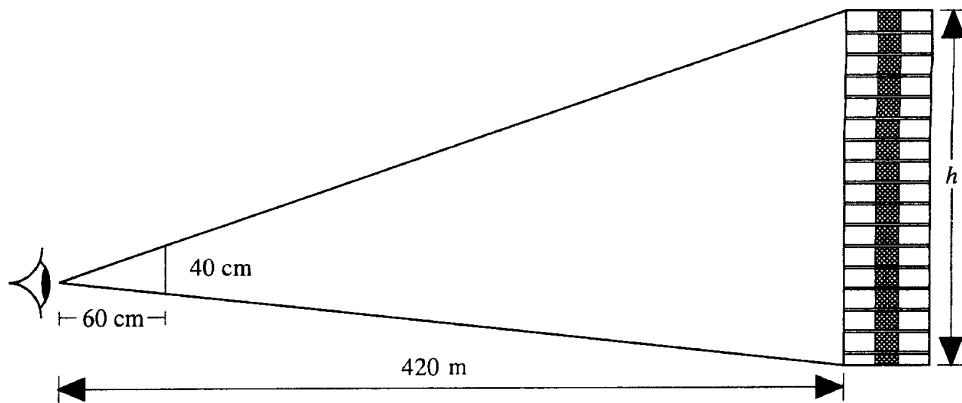*(LON)*

13.   Triangles ABC, PQR and HIJ are all similar.

Not to scale

(a)   Calculate the length of AB.   (b)   What is the size of angle B?

(c)   Calculate the length of HJ.

(SEG)

14.

I stood 420 m away from the tallest building in Singapore. I held a piece of wood 40 cm long at arms length, 60 cm away from my eye. The piece of wood, held vertically, just blocked the building from my view.

Use similar triangles to calculate the height, *h* metres, of the building.

(LON)

15.   AD = 4 cm, BC = 6 cm, angle BCD = 35°.
       BD is perpendicular to AC.

Not to scale
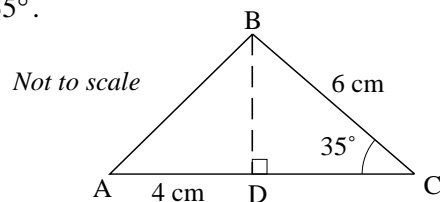
(a)   Calculate BD.

(b)   Calculate angle BAC.

(c)   Triangle A′B′C′ is similar to triangle ABC.
       The area of triangle A′B′C′ is nine times the area of triangle ABC.

       (i)   What is the size of angle A′B′C′?

       (ii)   Work out the length of B′C′.

(SEG)

16. A roof has a symmetrical frame, with dimensions as shown.

AB = BC
PR = AP
A$\hat{\text{T}}$B = 90°

5 m

5 m

Not to scale

B

R   S

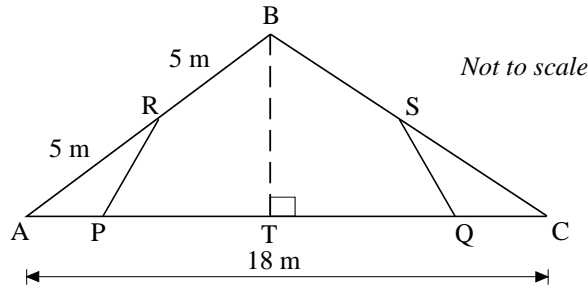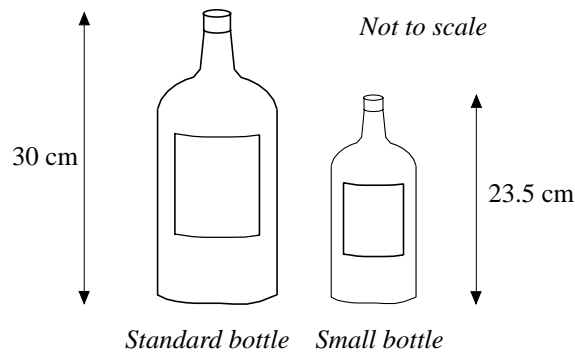A   P   T   Q   C

18 m

(a) (i)   Write down a triangle which is similar to triangle ABC.

(ii) Calculate the length PR.

(b) Calculate the value of angle BAT.

(SEG)

17. Two wine bottles have similar shapes. The standard bottle has a height of 30 cm and the small bottle has a height of 23.5 cm.

Not to scale

30 cm

23.5 cm

Standard bottle   Small bottle

(a) Calculate the ratio of the areas of the bases of the two bottles.
     Give your answer in the form *n* : 1.

(b) What is the ratio of the volumes of the two bottles?
     Give your answer in the form *n* : 1.

(c) Is it a fair description to call the small bottle a 'half bottle'?
     Give a reason for your answer.

(MEG)

18. The normal size and selling price of small and medium toothpaste is shown.

TOOTHPASTE

TOOTHPASTE

Small Size
60 ml
50p

Medium Size
135 ml
£1

A supermarket sells the toothpaste on special offer.

(a) The special offer small size has 20% more toothpaste for the same price. How much more toothpaste does it contain?

(b) The special offer medium size costs 90 pence for 135 ml. What is the special offer price as a fraction of the normal price?

(c) Calculate the number of ml per penny for each of these special offers. Which of these special offers gives better value for money?
You must show your working.

(d) (i) The 60 ml content of the small size has been given to the nearest 10 ml. What is the smallest number of ml it can contain?

(ii) The 135 ml content of the medium size has been given to the nearest 5 ml. What is the smallest number of ml it can contain?

(SEG)

19. (a) Two bottles of perfume are similar to each other. The heights of the bottles are 4 cm and 6 cm. The smaller bottle has a volume of 24 cm^3.

Calculate the volume of the larger bottle.

(b) Two bottles of aftershave are similar to each other. The areas of the bases of these bottles are 4.8 cm^2 and 10.8 cm^2. The height of the smaller bottle is 3 cm. Calculate the height of the larger bottle.

(NEAB)

14.14 Enlargements with Negative Scale Factors

When an enlargement uses a scale factor which is negative, the image will be reversed and turned upside down. The image lines must be drawn from each point on the original through the *centre of enlargement* and then the image can be drawn *beyond* the *centre of enlargement* as shown below.
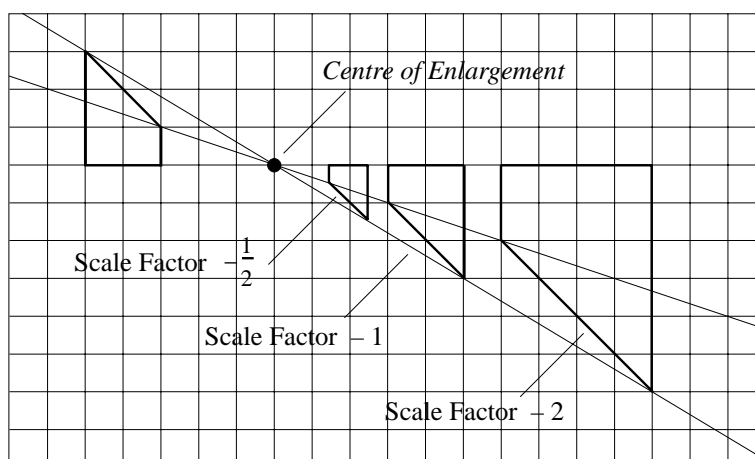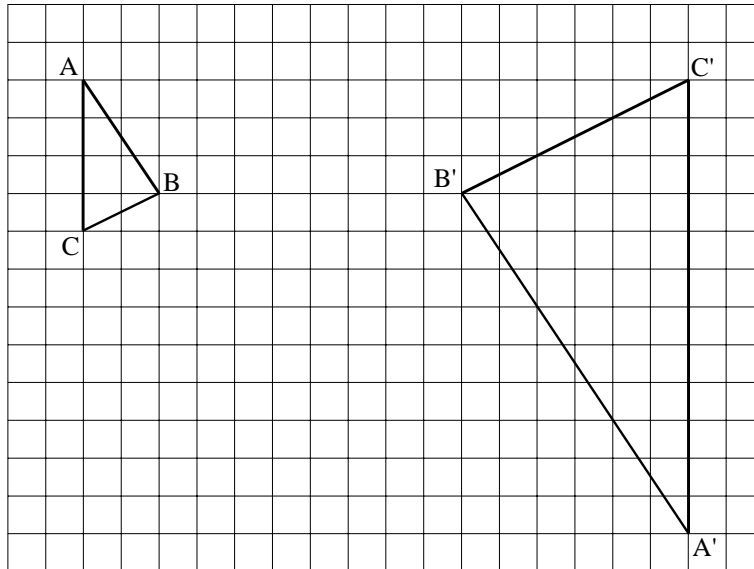

Note

The scale factor affects the lengths of the shapes: for scale factor -2 the lengths are doubled, for scale factor $-\frac{1}{2}$ they are halved and for scale factor -1 they remain the same.
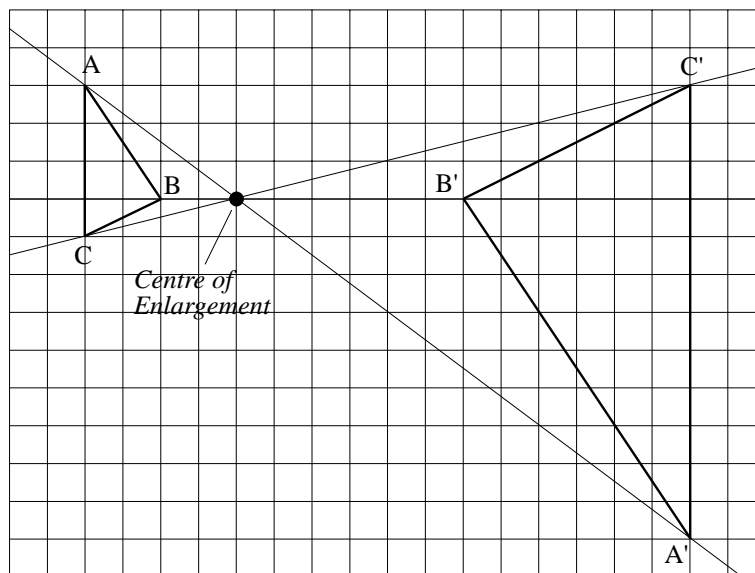
Worked Example 1

The triangle ABC has been enlarged to give the image A′B′C′. Find the centre of enlargement and the scale factor.

Solution

To find the centre of enlargement, join the corresponding corners of the image to the original
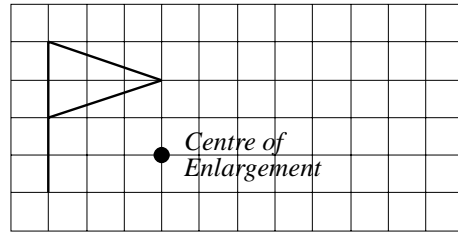

Centre of Enlargement

To find the scale factor, first compare the lengths. For example, AC = 2 cm and A′C′ = 6 cm. The sides of the image are all 3 times longer than the sides of the original. As the image has also been inverted, the scale factor is −3.

Worked Example 2

Enlarge the shape shown in the diagram with scale factor -2 and the *centre of enlargement* marked.

Solution

The diagram shows lines drawn through the centre of enlargement from each corner of the original. Theen the positions can be fixed using:
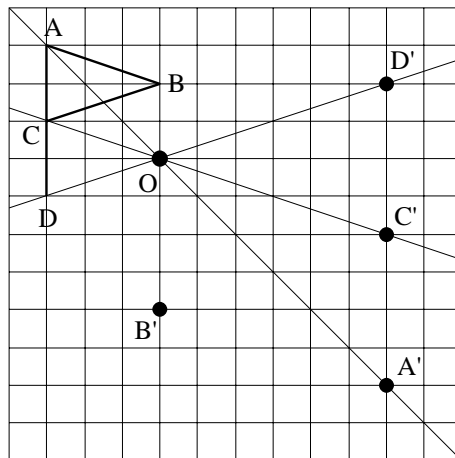
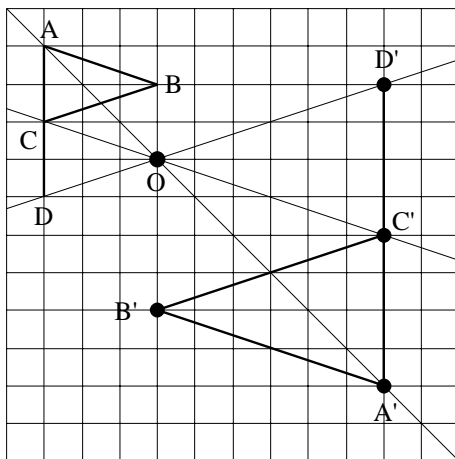$$OA' = 2 \times OA$$
$$OB' = 2 \times OB$$
$$OC' = 2 \times OC$$
$$OD' = 2 \times OD$$

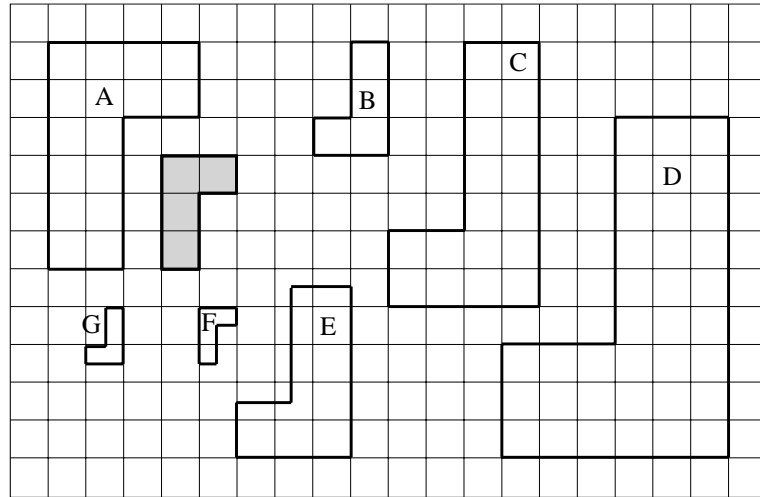and measuring from O away from the original shape gives:

Then points can be joined to give the shape shown below.

Exercises

1. The diagram below shows the original shape (shaded) and the images which could be obtained by enlargements with different scale factors. State the scale factor for each enlargement.

2. The shaded shape has been enlarged to give the other images. For each image find the scale factor and the coordinates of the centre of enlargement.
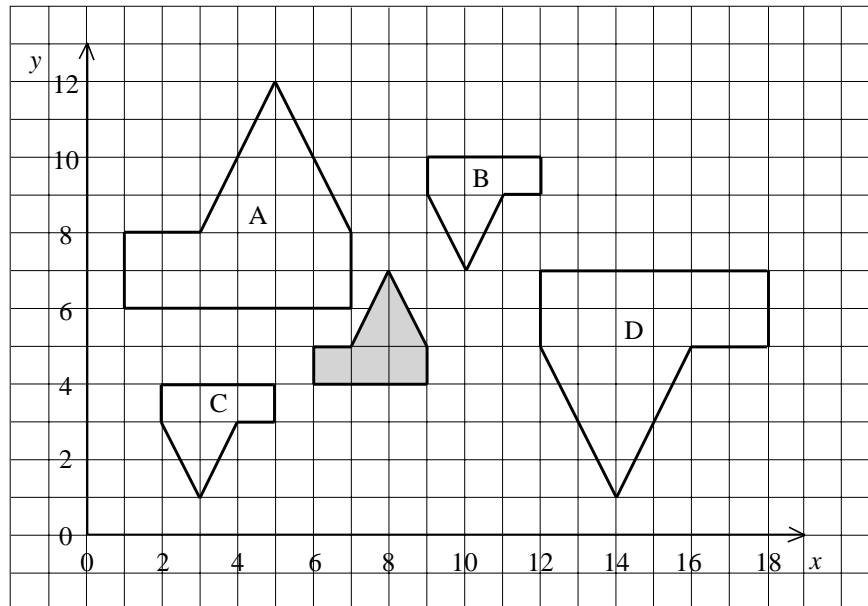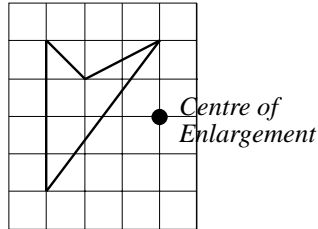

Just for Fun

Copy the diagram and fill in the nine circles with different numbers using the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9 so that each inequality involved is valid.
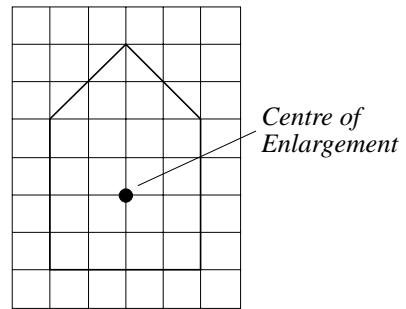
Try to find as many solutions as you can.

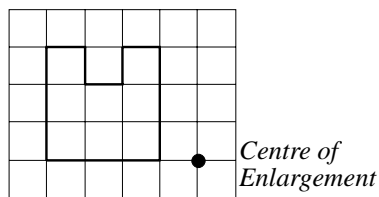3.   Copy each diagram below. Enlarge each shape using the scale factor and centre of enlargement given.
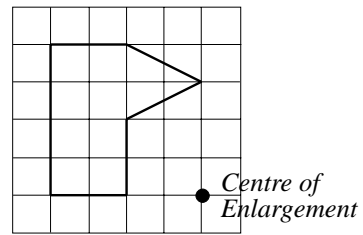
(a)

Scale Factor – 2

(b)

Centre of Enlargement

Scale Factor $-\frac{1}{2}$

(c)

Centre of Enlargement

Scale Factors – 1 and – 2

(d)

Centre of Enlargement
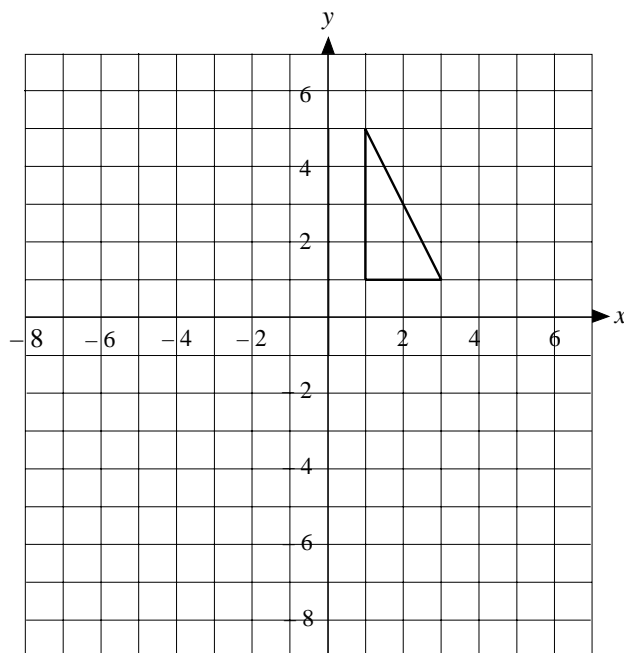
Scale Factors $-\frac{1}{2}$ and – 2

4.   (a)   Copy the axes and triangle shown below.

(b)   Draw the image obtained when the original is enlarged with:
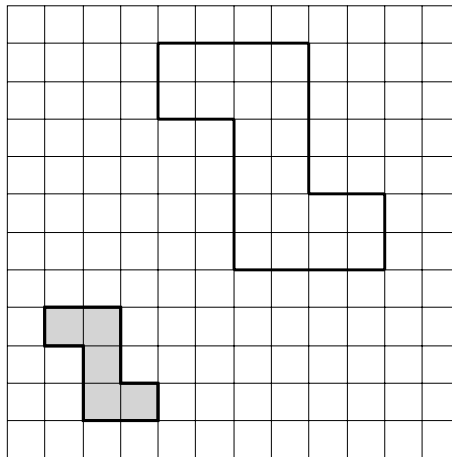
(i)   scale factor -2,   centre of enlargement $(-1, 1)$

(ii)   scale factor $-1\frac{1}{2}$,   centre of enlargement $(2, 0)$

(iii)   scale factor $-\frac{1}{2}$,   centre of enlargement $(-3, 3)$

(iv)   scale factor -1,   centre of enlargement $(3, 3)$.

5. (a)   Copy the diagram and show how the
          larger image can be obtained from
          the shaded shape by a single enlargement.

    (b)   Show how the larger shape can be
          obtained by two successive enlargements
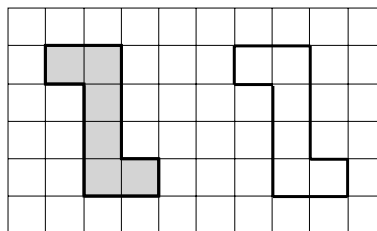   starting with the shaded shape.

          Show the centre of enlargement and
          state the scale factor for both enlargements.
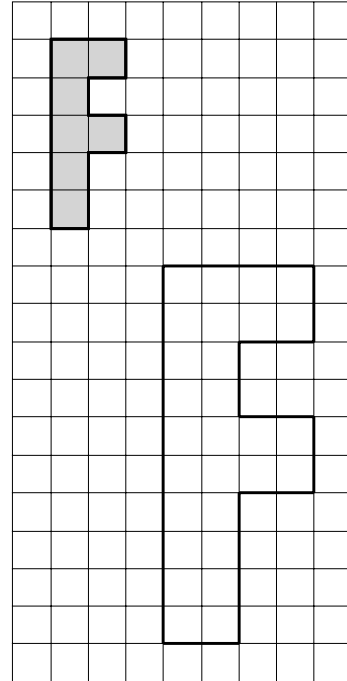
6. The shaded shape is enlarged to give the other shape.

    (a)   Copy the diagram and find two possible centres of enlargement.
          State the scale factor for each of the two centres.

    (b)   The shaded shape below has been enlarged to give the other shape.

          Find the centre of enlargement. Explain why there is only one possible
          centre of enlargement.

7. (a)   Draw a single shape and enlarge with scale factors -1, -2 and -3 using $(0, 0)$ as the centre of enlargement.

    (b)   List the coordinates of the original and the three images. How do the coordinates change as a result of these enlargements?

    (c)   Find the area of each shape. How is the area affected by each enlargement?

    (d)   Would your answers to (b) and (c) be the same if the centre of enlargement was not $(0, 0)$? Use a different centre of enlargement for some enlargements to illustrate your answer.

8. Triangle P is mapped to triangle Q by an enlargement of scale factor -0.5.
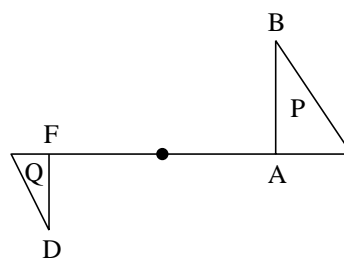
    If AB is of length 6.4 cm, how long is FD?

    (LON)

Just for Fun

The diagram opposite shows four equilateral triangles formed by using 9 toothpicks.

By removing 3 toothpicks and rearranging the figure, can you form 4 congruent triangles?