Answers

1.1 True or False

1. (a) Sometimes true
   (b) False (the maximum number of days in a year - on Earth - is 366)
   (c) Always true (d) False (they both have 30 days)
   (e) Always true (f) True, for Games in recent years
   (g) Sometimes true (century years are not leap years unless they can be divided by 400, e.g. 1900 was not a leap year but 2000 was a leap year)

2. (a) False (angles on a line add up to $180°$) (b) True (c) True
   (d) True - a square is a special type of rectangle. However, the converse is false, not every rectangle is a square.
   (e) False (the circumference of a circle is only *approximately* 3 times the diameter)
   (f) True

3. (a) True (b) False (for example, $5 - 3 = 2$ but $3 - 5 = -2$) (c) True
   (d) False (for example, $\frac{8}{2} = 4$ but $\frac{2}{8} = 0.25$)
   (e) False (for example, $(3 + 4)^2 = 49$ but $3^2 + 4^2 = 25$)
   (f) False (for example $(7 - 4)^2 = 9$ but $7^2 - 4^2 = 33$) (g) True
   (h) True (provided both sides are defined, i.e. provided $y \neq 0$)

1.2 Proof

1. If N is the first number then the other must be $N + 1$ because they are consecutive.

   Adding the two numbers gives $N + (N + 1) = 2N + 1$.

   Being a multiple of 2, $2N$ must be even. Adding on 1 will give an odd total.
   This proves that the sum of the two consecutive numbers must be odd.

2. If N is the first number then the other must be $N + 2$ because they are consecutive odd numbers.

   Adding the two numbers gives $N + (N + 2) = 2N + 2 = 2(N + 1)$.

   Being a multiple of 2, $2(N + 1)$ must be even.
   This proves that the sum of two consecutive odd numbers must be even.

3. Being even, the first of the two consecutive numbers must be a multiple of 2. We can therefore write it as $2N$, where N is a whole number. Being consecutive, the other even number must be $2N + 2$.
   Multiplying the two numbers gives a product equal to

   $$2N \times (2N + 2) = 2N \times 2(N + 1) = 2 \times 2 \times N \times (N + 1) = 4N(N + 1)$$

   which is 4 times $N(N + 1)$.
   This proves that the product of two consecutive even numbers must be a multiple of 4.

4. (a) If the first of the two consecutive odd numbers is x then $x - 1$ will be even.

This means that $x - 1 = 2p$ where p is a whole number, so $x = 2p + 1$.

Since they are consecutive odd numbers, the other number must be

$$x + 2 = (2p + 1) + 2 = 2p + 3$$

Adding the two consecutive odd numbers gives a total of

$$(2p + 1) + (2p + 3) = 4p + 4 = 4(p + 1)$$

which is 4 times $(p + 1)$.

This proves that the sum of two consecutive odd numbers must be a multiple of 4.

(b) If the consecutive odd numbers are $(2p + 1)$ and $(2p + 3)$, as in part (a), then their product is

$$(2p + 1)(2p + 3) = 4p^2 + 2p + 6p + 3$$
$$= 4p^2 + 8p + 3$$

$\times$	$2p$	$+1$
$2p$	$4p^2$	$+2p$
$+3$	$+6p$	$+3$

Adding on 1 gives $(4p^2 + 8p + 3) + 1 = 4p^2 + 8p + 4 = 4(p^2 + 2p + 1)$ which is 4 times $(p^2 + 2p + 1)$.

This proves that if you multiply two consecutive odd numbers, and then add 1, the result must be a multiple of 4.

5. (a) The total is always 3 times the middle number.

(b) If N is the first number then the other two numbers must be $N + 1$ and $N + 2$.

Adding gives a total of

$$N + (N + 1) + (N + 2) = 3N + 3 = 3(N + 1)$$

which is 3 times the middle number.

This proves that the sum of 3 consecutive numbers is always equal to 3 times the middle number.

6. (a) The total is always 5 times the middle number.

(b) If N is the first number then the other four numbers must be $N + 1$, $N + 2$, $N + 3$ and $N + 4$.

Adding gives a total of

$$N + (N + 1) + (N + 2) + (N + 3) + (N + 4) = 5N + 10 = 5(N + 2)$$

which is 5 times the middle number.

This proves that the sum of 5 consecutive numbers is always equal to 5 times the middle number.

7. (a) The total is always even.

(b) If N is the first number then the other three numbers must be $N + 1$, $N + 2$ and $N + 3$.

Adding gives a total of

$$N + (N + 1) + (N + 2) + (N + 3) = 4N + 6 = 2(2N + 3)$$

which is 2 times $(2N + 3)$, so must be even.

This proves that the sum of 4 consecutive numbers is always even.

N.B. Unlike questions 5 and 6, the total is not a multiple of 4. It is, however, always equal to 4 times the mean of the two middle numbers.

8. If the numbers are N, $N+1$ and $N+2$ then

the sum of the squares $= N^2 + (N+1)^2 + (N+2)^2$

$$= N^2 + \left(N^2 + 2N + 1\right) + \left(N^2 + 4N + 4\right)$$

$$= 3N^2 + 6N + 5$$

so subtracting 2 from the sum of the squares gives

$$\left(3N^2 + 6N + 5\right) - 2$$

$$= 3N^2 + 6N + 3$$

$$= 3\left(N^2 + 2N + 1\right)$$

which is 3 times $\left(N^2 + 2N + 1\right)$.

This proves that 2 less than the sum of the squares of 3 consecutive numbers is always a multiple of 3.

N.B. The proof above for question 8 shows that, in fact, 2 less than the sum of the squares of 3 consecutive numbers is always equal to 3 times the square of the middle number.

9. (a)

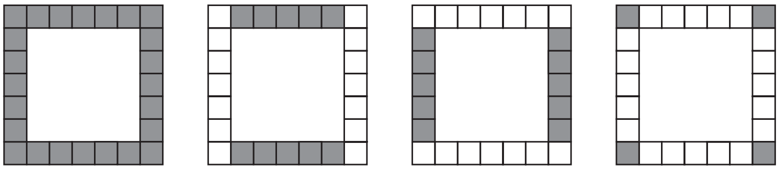
Size of Pond (metres)	*Number of Paving Slabs*
1×1	8
2×2	12
3×3	16
4×4	20
5×5	24
6×6	28
7×7	32
8×8	36
9×9	40
10×10	44

(b) The formula linking the number of tiles, N, to the side length, L, of the pond is

$$N = 4L + 4$$

(c) In the following figure, if the diagram on the left represents a pond with sides of length L then the slabs surrounding it consist of 2 'horizontal' strips of L slabs, 2 'vertical' strips of L slabs, and 4 corner slabs, Hence the total number of slabs is

$$N = L + L + L + L + 4 = 4L + 4$$



$N \quad = \quad L + L \quad + \quad L + L \quad + \quad 4$

10. If N is the original number then

The instruction	*produces*
double your number	$2N$
add 5	$2N + 5$
multiply the number you now have by itself	$(2N + 5)^2 = 4N^2 + 20N + 25$
subtract 25	$4N^2 + 20N$
divide by 4	$N^2 + 5N = N(N + 5)$
divide by your original number	$N(N + 5) \div N = N + 5$ = 5 more than the original number

This proves that Roger's trick always works.

1.3 Algebraic Identities

1. (a) $7(x - 8) - 3(x - 20) = 4(x + 1)$ (A)

If $x = 1$,

LHS of (A) $= 7(1 - 8) - 3(1 - 20) = [7 \times (-7)] - [3 \times (-19)]$

$= -49 + 57 = 8$

RHS of (A) $= 4(1 + 1) = 4 \times 2 = 8$

$\therefore$ statement (A) is true for $x = 1$.

If $x = 3$,

LHS of (A) $= 7(3 - 8) - 3(3 - 20) = [7 \times (-5)] - [3 \times (-17)]$

$= -35 + 51 = 16$

RHS of (A) $= 4(3 + 1) = 4 \times 4 = 16$

$\therefore$ statement (A) is true for $x = 3$.

If $x = 5$,

LHS of (A) $7(5 - 8) - 3(5 - 20) = [7 \times (-3)] - [3 \times (-15)]$

$= -21 + 45 = 24$

RHS of (A) $= 4(5 + 1) = 4 \times 6 = 24$

$\therefore$ statement (A) is true for $x = 5$.

$\therefore$ $x = 1$, $x = 3$ and $x = 5$ all satisfy equation (A).

(b) *Proof of the statement* $7(x-8) - 3(x-20) \equiv 4(x+1)$ (A)

<u>*Proof*</u> LHS of (A) $\equiv 7(x-8) - 3(x-20)$

$\equiv 7x - 56 - 3x + 60$ *(multiplying out the brackets)*

$\equiv 4x + 4$ *(collecting like terms)*

RHS of (A) $\equiv 4(x+1)$

$\equiv 4x + 4$ *(multiplying out the brackets)*

Since both sides of (A) simplify to $4x + 4$, it follows that

LHS of (A) $\equiv$ RHS of (A) for any value of x.

$\therefore$ $7(x-8) - 3(x-20) \equiv (4x+1)$

2. (a) $x^3 - 9x^2 + 23x = 15$ (B)

If $x = 1$,

LHS of (B) $= 1^3 - 9 \times 1^2 + 23 \times 1 = 1 - 9 + 23 = 15 =$ RHS of (B)

$\therefore$ statement (B) is true for $x = 1$.

If $x = 3$,

LHS of (B) $= 3^3 - 9 \times 3^2 + 23 \times 3 = 27 - 81 + 69 = 15 =$ RHS of (B)

$\therefore$ statement (B) is true for $x = 3$.

If $x = 5$,

LHS of (B) $= 5^3 - 9 \times 5^2 + 23 \times 5 = 125 - 225 + 115 = 15 =$ RHS of (B)

$\therefore$ statement (B) is true for $x = 5$.

$\therefore$ $x = 1$, $x = 3$ and $x = 5$ all satisfy equation (B).

(b) If $x = 4$,

LHS of (B) $= 4^3 - 9 \times 4^2 + 23 \times 4 = 64 - 144 + 92 = 12 \neq$ RHS of (B)

$\therefore$ statement (B) is not true for $x = 4$.

Since we have found a value for which statement (B) is not true, statement (B) is not an identity.

3. (a) $8(p-q) + 3(p+q) = 2(p+2q) + 9(p-q)$ (C)

If $p = 10$ and $q = 5$,

LHS of (C) $= 8(10-5) + 3(10+5) = 8 \times 5 + 3 \times 15$

$= 40 + 45 = 85$

RHS of (C) $= 2(10 + [2 \times 5]) + 9(10 - 5) = 2 \times 20 + 9 \times 5$

$= 40 + 45 = 85$

$\therefore$ statement (C) is true for $p = 10$ and $q = 5$.

(b) If $p = 6$ and $q = 4$,

LHS of (C) $= 8(6 - 4) + 3(6 + 4) = 8 \times 2 + 3 \times 10$

$= 16 + 30 = 46$

RHS of (C) $= 2(6 + [2 \times 4]) + 9(6 - 4) = 2 \times 14 + 9 \times 2$

$= 28 + 18 = 46$

$\therefore$ statement (C) is true for $p = 6$ and $q = 4$.

(c) *Proof of the statement* $8(p - q) + 3(p + q) \equiv 2(p + 2q) + 9(p - q)$ (C)

Proof

LHS of (C) $\equiv 8(p - q) + 3(p + q)$

$\equiv 8p - 8q + 3p + 3q$ *(multiplying out the brackets)*

$\equiv 11p - 5q$ *(collecting like terms)*

RHS of (C) $\equiv 2(p + 2q) + 9(p - q)$

$\equiv 2p + 4q + 9p - 9q$ *(multiplying out the brackets)*

$\equiv 11p - 5q$ *(collecting like terms)*

Since both sides of (C) simplify to $11p - 5q$ it follows that LHS of (C) $\equiv$ RHS of (C) for any values of p and q.

$\therefore 8(p - q) + 3(p + q) \equiv 2(p + 2q) + 9(p - q)$

4. *Proof of the statement* $x(m + n) + y(n - m) \equiv m(x - y) + n(x + y)$ (D)

Proof

LHS of (D) $\equiv x(m + n) + y(n - m)$

$\equiv xm + xn + yn - ym$ *(multiplying out the brackets)*

RHS of (D) $\equiv m(x - y) + n(x + y)$

$\equiv mx - my + nx + ny$ *(multiplying out the brackets)*

$\equiv xm + xn + yn - ym$ *(rearranging the order of terms)*

Since both sides of (D) simplify to $xm + xn + yn - ym$, it follows that LHS of (D) $\equiv$ RHS of (D) for any values of m, n, x and y.

$\therefore x(m + n) + y(n - m) \equiv m(x - y) + n(x + y)$

5. (a)

$\times$	x	$+2$
x	x^2	$+2x$
$+10$	$+10x$	$+20$

$(x+2)(x+10) = x^2 + 12x + 20$

(b)

$\times$	x	-5
x	x^2	$-5x$
-4	$-4x$	$+20$

$(x-5)(x-4) = x^2 - 9x + 20$

(c) *Proof of the statement* $(x+2)(x+10) - (x-5)(x-4) \equiv 21x$ (E)

<u>*Proof*</u>

LHS of (E) $\equiv (x+2)(x+10) - (x-5)(x-4)$

$\equiv [x^2 + 12x + 20] - [x^2 - 9x + 20]$ *(from parts (a) and (b))*

$\equiv x^2 + 12x + 20 - x^2 + 9x - 20$ *(removing the brackets)*

$\equiv 21x$ *(collecting like terms)*

$\equiv$ RHS of (E)

$\therefore (x+2)(x+10) - (x-5)(x-4) \equiv 21x$

6. (a)

$\times$	x	$+6$
x	x^2	$+6x$
$+8$	$+8x$	$+48$

so $(x+6)(x+8) = x^2 + 14x + 48$

(b) By part (a), $x^2 + 14x + 48 = (x+6)(x+8)$

If $x \neq -6$, then $(x+6) \neq 0$ so we may divide both sides by $(x+6)$ to get

$$\frac{x^2 + 14x + 48}{x+6} = \frac{(x+6)(x+8)}{x+6} = x + 8$$

$\therefore \dfrac{x^2 + 14x + 48}{x+6} = x + 8$, provided $x \neq -6$.

7. (a) *Proof of the identity* $a^2 - b^2 \equiv (a+b)(a-b)$ (F)

<u>*Proof*</u>

From the multiplication grid

$\times$	a	$+b$
a	a^2	$+ab$
$-b$	$-ab$	$-b^2$

RHS of (F) $\equiv (a+b)(a-b)$

$\equiv a^2 + ab - ab - b^2$

$\equiv a^2 - b^2 =$ LHS of (F)

$\therefore a^2 - b^2 \equiv (a+b)(a-b)$

(b) (i) $81^2 - 80^2 = (81 + 80)(81 - 80) = 161 \times 1 = 161$

(ii) $101^2 - 99^2 = (101 + 99)(101 - 99) = 200 \times 2 = 400$

(iii) $2731^2 - 269^2 = (2731 + 269)(2731 - 269) = 3000 \times 2462 = 7\,386\,000$

(iv) $11.7^2 - 8.3^2 = (11.7 + 8.3)(11.7 - 8.3) = 20 \times 3.4 = 68$

(v) $999\,991^2 - 9^2 = (999\,991 + 9)(999\,991 - 9)$

$$= 1\,000\,000 \times 999\,982 = 999\,982\,000\,000$$

(vi) $75.41^2 - 24.59^2 = (75.41 + 24.59)(75.41 - 24.59) = 100 \times 50.82 = 5082$

8. (a) *Proof of the identity* $m^2 - 1 \equiv (m+1)(m-1)$ (G)

<u>*Proof*</u>

From the multiplication grid

×	m	$+1$
m	m^2	$+m$
-1	$-m$	-1

RHS of (G) $\equiv (m+1)(m-1)$

$\equiv m^2 + m - m - 1$

$\equiv m^2 - 1 \equiv$ LHS of (G)

$\therefore\ m^2 - 1 \equiv (m+1)(m-1)$

(b) *Proof of the identity* $m^4 - 1 \equiv (m^2+1)(m^2-1)$ (H)

<u>*Proof*</u>

From the multiplication grid

×	m^2	$+1$
m^2	m^4	$+m^2$
-1	$-m^2$	-1

LHS of (H) $\equiv (m^2+1)(m^2-1)$

$\equiv m^4 + m^2 - m^2 - 1$

$\equiv m^4 - 1 \equiv$ LHS of (H)

$\therefore\ m^4 - 1 \equiv (m^2+1)(m^2-1)$

(c) *Proof of the identity* $m^4 - 1 \equiv (m^2+1)(m+1)(m-1)$ (I)

<u>*Proof*</u>

LHS of (I) $\equiv m^4 - 1$

$\equiv (m^2+1)(m^2-1)$ *(by part (b))*

$\equiv (m^2+1)[(m+1)(m-1)]$ *(by part (a))*

$\equiv (m^2+1)(m+1)(m-1)$ *(removing the extra brackets)*

$\equiv$ RHS of (I)

$\therefore\ m^4 - 1 \equiv (m^2+1)(m+1)(m-1)$

9. (a) *Proof of the identity* $(x+y)^2 + (x-y)^2 \equiv 2(x^2+y^2)$ (J)

Proof

$\times$	x	$+y$
x	x^2	$+xy$
$+y$	$+xy$	$+y^2$

$\times$	x	$-y$
x	x^2	$-xy$
$-y$	$-xy$	$+y^2$

$(x+y)^2 = x^2 + 2xy + y^2$ $(x-y)^2 = x^2 - 2xy + y^2$

$$\therefore \text{ LHS of (J)} \equiv (x+y)^2 + (x-y)^2$$
$$\equiv \left[x^2 + 2xy + y^2\right] + \left[x^2 - 2xy + y^2\right]$$
$$\equiv 2x^2 + 2y^2$$
$$\equiv 2(x^2+y^2) \equiv \text{RHS of (J)}$$
$$\therefore (x+y)^2 + (x-y)^2 \equiv 2(x^2+y^2)$$

(b) *Proof of the identity* $(x+y)^2 - (x-y)^2 \equiv 4xy$ (K)

Proof

$$\therefore \text{ LHS of (K)} \equiv (x+y)^2 - (x-y)^2$$
$$\equiv \left[x^2 + 2xy + y^2\right] - \left[x^2 - 2xy + y^2\right]$$
$$\equiv x^2 + 2xy + y^2 - x^2 + 2xy - y^2$$
$$\equiv 4xy \equiv \text{RHS of (K)}$$
$$\therefore (x+y)^2 - (x-y)^2 \equiv 4xy$$

10. (a) *Proof of the identity* $a^3 - b^3 \equiv (a-b)(a^2+ab+b^2)$ (L)

$\times$	a^2	$+ab$	$+b^2$
a	a^3	$+a^2b$	$+ab^2$
$-b$	$-a^2b$	$-ab^2$	$-b^3$

Proof

From the multiplication grid

$$\text{RHS of (L)} \equiv (a-b)(a^2+ab+b^2)$$
$$\equiv a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3$$
$$\equiv a^3 - b^3 \equiv \text{LHS of (L)}$$
$$\therefore a^3 - b^3 \equiv (a-b)(a^2+ab+b^2)$$

(b) *Proof of the identity* $a^3 + b^3 \equiv (a+b)(a^2 - ab + b^2)$ (M)

Proof

From the multiplication grid

$\times$	a^2	$-ab$	$+b^2$
a	a^3	$-a^2b$	$+ab^2$
$+b$	$+a^2b$	$-ab^2$	$+b^3$

RHS of (M) $\equiv (a+b)(a^2 - ab + b^2)$

$\equiv a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3$

$\equiv a^3 + b^3 \equiv$ LHS of (M)

$\therefore\ a^3 + b^3 \equiv (a+b)(a^2 - ab + b^2)$

1.4 Geometrical Proof

1. *Proof that* $p = t$

$p + s = 180°$ *(Corollary 3, angles on a line add to* $180°$*)*

$t + u = 180°$ *(Corollary 3, angles on a line add to* $180°$*)*

$u = s$ *(Theorem 5, alternate angles are equal)*

$\therefore\ t + s = 180°$

$\therefore\ p + s = t + s$

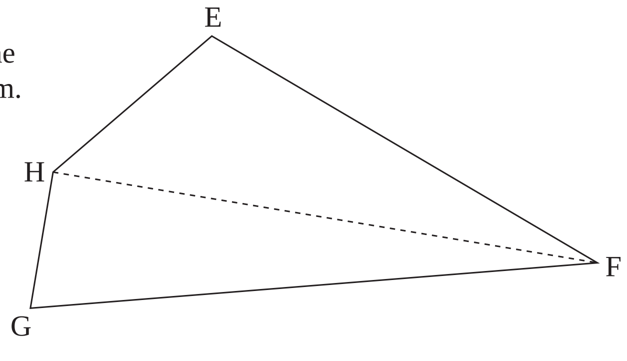
$\therefore\ p = t$

N.B. The proofs that $q = u$, $r = v$ and $s = w$ are similar.

2. *Proof that the angles of the quadrilateral EFGH add up to* $360°$

Proof

If EFGH is a quadrilateral, join the vertices F and H, as in the diagram.



Then

$\angle$EFH + $\angle$FHE + $\angle$HEF = 180° *(by Theorem 6, angle sum in triangle* = 180°*)*

and

$\angle$GFH + $\angle$FHG + $\angle$HGF = 180° *(by Theorem 6, angle sum in triangle* = 180°*)*

Adding these two statements gives

$\angle$EFH + $\angle$FHE + $\angle$HEF + $\angle$GFH + $\angle$FHG + $\angle$HGF = 360°

Rearranging and bracketing gives

($\angle$EFH + $\angle$GFH) + ($\angle$FHE + $\angle$FHG) + $\angle$HEF + $\angle$HGF = 360°

so, combining gives

$\angle EFG + \angle EHG + \angle HEF + \angle HGF = 360°$

and reordering

$\angle HEF + \angle EFG + \angle HGF + \angle EHG = 360°$

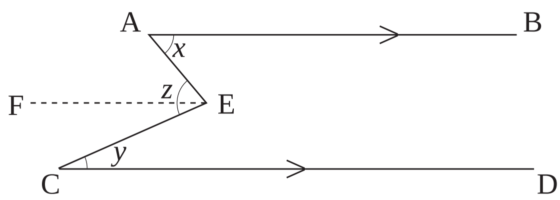
i.e. $\angle E + \angle F + \angle G + \angle H = 360°$

i.e. the angles of the quadrilateral EFGH add up to $360°$.

3. *Proof that* $x + y = z$

<u>*Proof*</u>

Through E, draw a line EF, parallel to AB and CD, as shown.



Then

$\angle AEF = x$ *(by Theorem 5, alternate angles AB parallel to FE)*

$\angle CEF = y$ *(by Theorem 5, alternate angles CD parallel to FE)*

so adding, $\angle AEF + \angle CEF = x + y$

i.e. $\angle AEC = x + y$, i.e. $z = x + y$

4. *Proof that* $\alpha = \beta$

<u>*Proof*</u>

$\alpha = \angle RST = \angle STU$ *(by Theorem 5, alternate angle SR parallel to UT)*

$\beta = \angle VUT = \angle UTS$ *(by Theorem 5, alternate angle UV parallel to ST)*

Since both angles α and β equal angle $\angle UTS$, it follows that $\alpha = \beta$.

5. (a) *Proof that* ΔEDA *and* ΔECB *are congruent*

<u>*Proof*</u>

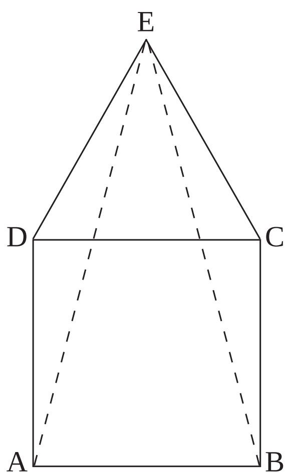
ABCD is a square, so $\angle ADC = 90°$.

Triangle EDC is equilateral, so $\angle EDC = 60°$.

Adding these together, $\angle EDA = 150°$.

By the same reasoning, $\angle ECB = 150°$.

In triangles EDA and ECB



ED = EC *(sides of an equilateral triangle)*

$\angle EDA = \angle ECB$ *(both* $150°$ *as shown above)*

DA = CD *(sides of a square)*

$\therefore$ ΔEDA is congruent to ΔECB *(SAS)*

(b) From the proof in part (a), we can conclude that corresponding sides and angles in the two triangles are equal. In particular, the remaining sides must be equal in length, i.e. AE = BE.

(c) Since ED = DC (because ΔEDC is equilateral) and DC = DA (because ABCD is a square), it follows that ED = DA.

Therefore, ΔEDA is an isosceles triangle with $\angle EDA = 150°$.

Hence $\angle DAE = \angle DEA = \frac{1}{2}(180° - 150°) = 15°$.

Applying the same reasoning to ΔECB, $\angle CBE = \angle CEB = 15°$.

But $\angle DEC = 60°$ because ΔEDC is equilateral. So, subtracting

$$\angle AEB = \angle DEC - \angle DEA - \angle CEB = 60° - 15° - 15° = 30°.$$

But, as noted in part (b), AE = BE. Therefore ΔEAB is isosceles.

Hence the remaining angles of ΔEAB are equal, so

$$\angle EAB = \angle EBA = \frac{1}{2}(180° - 30°) = 75°$$

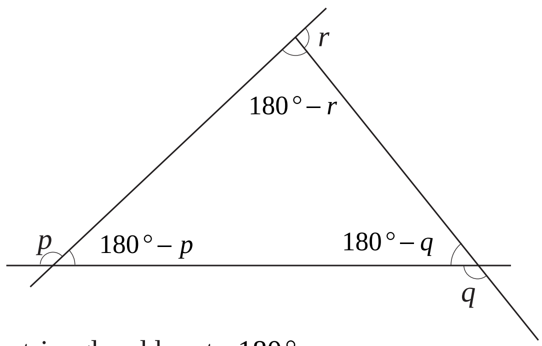
The angles of ΔEAB are therefore 75°, 75° and 30°.

6. (a) *Proof that* $p + q + r = 360°$

<u>*Proof*</u>

By Corollary 3, angles on a straight line add up to 180°.

This means that the angles of the triangle are $180° - p$, $180° - q$ and $180° - r$.



But, by Theorem 6, the angles of a triangle add up to 180°.

$$\therefore\ (180° - p) + (180° - q) + (180° - r) = 180°$$

$$\therefore\ 540° - (p + q + r) = 180°$$

$$\therefore\ p + q + r = 360°$$

(b) *Proof that the triangle is right-angled if, in addition,* $p + q = 3r$

<u>*Proof*</u>

By part (a), $p + q + r = 360°$

But, $p + q = 3r$

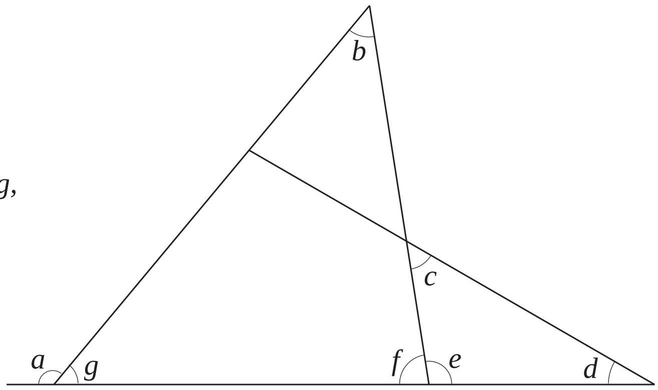
Substituting the second equation into the first gives

$$3r + r = 360°$$

i.e. $4r = 360°$

i.e. $r = 90°$

$\therefore$ the angle $(180° - r) = 90°$, showing that the triangle is right-angled.



7. *Proof that* $a = b + c + d$

<u>*Proof*</u>

We mark three of the other angles in the diagram *e*, *f* and *g*, as shown.

By Theorem 6, the angles of a triangle add up to $180°$.

$\therefore\ c + e + d = 180°$

and $b + f + g = 180°$

Adding these two equations together gives

$$c + e + d + b + f + g = 360°$$

Rearranging the order and bracketing gives

$$b + c + d + (e + f) + g = 360° \qquad (\#)$$

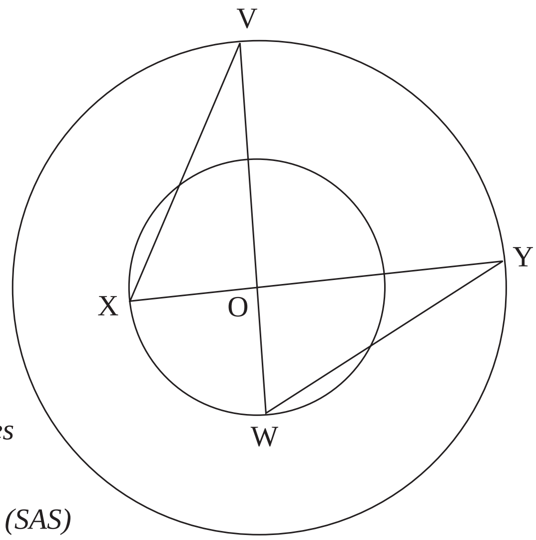
But angles on a straight line add up to $180°$ by Corollary 3, so

$$e + f = 180° \text{ and } a + g = 180°, \quad \text{i.e. } g = 180° - a$$

Substituting these two facts into equation (#) gives

$$b + c + d + 180° + (180° - a) = 360°$$

so $b + c + d - a = 0°$ *i.e.* $a = b + c + d$



8. (a) *Proof that* ΔVOX *and* ΔWOY *are congruent*

<u>*Proof*</u>

In triangles VOX and WOY

OX = OW *(radii of small circle)*

OV = OY *(radii of large circle)*

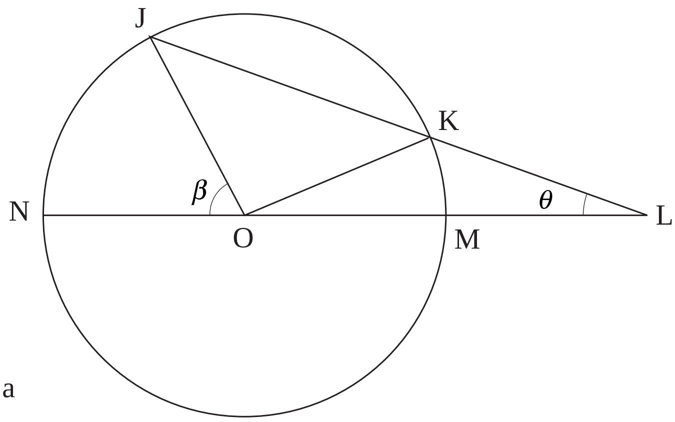
$\angle$VOX = $\angle$YOW *(by Theorem 4, vertically opposite angles are equal)*

$\therefore$ ΔVOX *is congruent to* ΔYOW *(SAS)*

(b) From the proof in part (a), we can conclude that corresponding sides and angles in the two triangles are equal. In particular, it follows that

(i) VX = WY (ii) $\angle$OVX = $\angle$OYW and (iii) $\angle$OXV = $\angle$OWY

9. *Proof that* $\beta = 3\theta$



<u>*Proof*</u>

In ΔKOL,

KL = OK (*given*)

$\therefore$ ΔKOL is isosceles

$\therefore$ $\angle KOL = \angle KLO = \theta$

By Theorem 6, the angles of a triangle add up to $180°$.

$$\therefore \quad \angle KOL + \angle KLO + \angle OKL = 180°$$

i.e. $\theta + \theta + \angle OKL = 180°$ $\quad \therefore \quad \angle OKL = 180° - 2\theta$

But angles on a straight line add up to $180°$ by Corollary 3, so

$\angle OKL + \angle OKJ = 180°$ i.e. $(180° - 2\theta) + \angle OKJ = 180°$

$$\therefore \angle OKJ = 2\theta$$

Triangle OJK is isosceles because OJ = OK = radii of the circle, so

$\angle OJK = \angle OKJ = 2\theta$ i.e. $\angle OJL = 2\theta$

By Corollary 3, angles on a straight line add up to $180°$, so

$\angle JON + \angle JOL = 180°$ i.e. $\beta + \angle JOL = 180°$

$$\therefore \angle JOL = 180° - \beta$$

Adding the angles of ΔJOL now gives

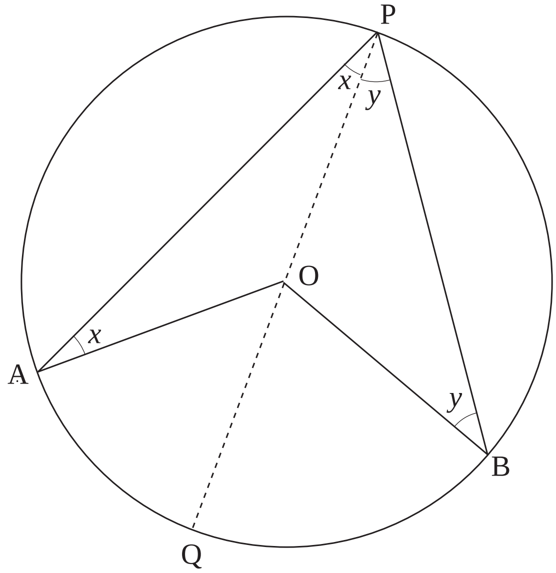
$$\angle JOL + \angle JLO + \angle OJL = 180°$$

i.e. $(180° - \beta) + \theta + 2\theta = 180°$

$$\therefore \quad 180° + 3\theta - \beta = 180°$$

$\therefore 3\theta - \beta = 0°$ i.e. $\beta = 3\theta$

10. (a) The construction is shown in the second diagram



(b) In ΔOAP,

OA = OP (*radii of circle*)

$\therefore$ ΔOAP is isosceles

so by Theorem 7

$\angle OAP = \angle OPA$

These equal angles are labelled as angle *x* on the diagram.

(c) In ΔOBP,

OB = OP *(radii of circle)*

$\therefore$ ΔOBP is isosceles

so by Theorem 7, $\angle$OBP = $\angle$OPB.

These equal angles are labelled as angle y on the diagram.

(d) $\angle APB = \angle OPA + \angle OPB = x + y$

(e) By Theorem 6, the angles of triangle OAP add up to $180°$.

$\therefore \quad \angle AOP + \angle APO + \angle OAP = 180°$

i.e. $\angle AOP + x + x = 180°$ $\quad\therefore\ \angle AOP = 180° - 2x$

Similarly, the angles of triangle OBP add up to $180°$.

$\therefore \quad \angle BOP + \angle BPO + \angle OBP = 180°$

i.e. $\angle BOP + y + y = 180°$ $\quad\therefore\ \angle BOP = 180° - 2y$

(f) By Theorem 1, angles at a point add up to $360°$.

$\therefore \quad \angle AOP + \angle BOP + \angle AOB = 360°$

i.e. $(180° - 2x) + (180° - 2y) + \angle AOB = 360°$

i.e. $360° + \angle AOB - (2x + 2y) = 360°$

$\therefore \quad \angle AOB - (2x + 2y) = 0°$ i.e. $\angle AOB = 2x + 2y$

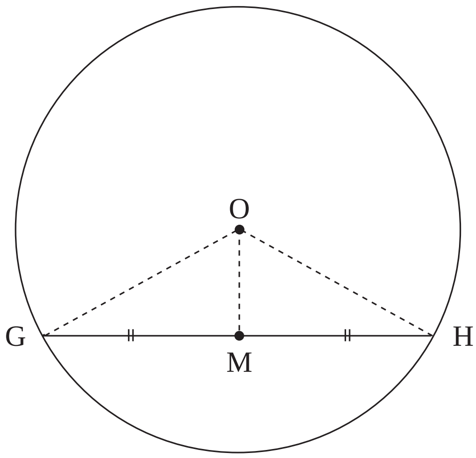
(g) $\angle AOB = 2x + 2y$ *(by part (f))*

$= 2(x + y)$ *(factorising)*

$= 2 \times \angle APB$ *(by part (d))*

$\therefore\ \angle AOB = 2 \times \angle APB$

11. *Proof that* OM *is perpendicular to* GH



<u>*Proof*</u>

In triangles OMG and OMH

OG = OH *(radii of circle)*

OM = OM *(common side)*

GM = HM *(*M *is the midpoint of* GH, *given)*

$\therefore$ ΔOMG is congruent to ΔOMH *(SSS)*

It follows that corresponding angles in the two triangles are equal. In particular, it follows that $\angle OMG = \angle OMH$.

But

$\angle OMG + \angle OMH = 180°$ *(Angles on a straight line add up to* $180°$*, Corollary 3)*

$\therefore\ \angle OMG + \angle OMG = 180°$ i.e. $2 \times \angle OMG = 180°$

$\therefore\ \angle OMG = 90°$ i.e. OM is perpendicular to GH.