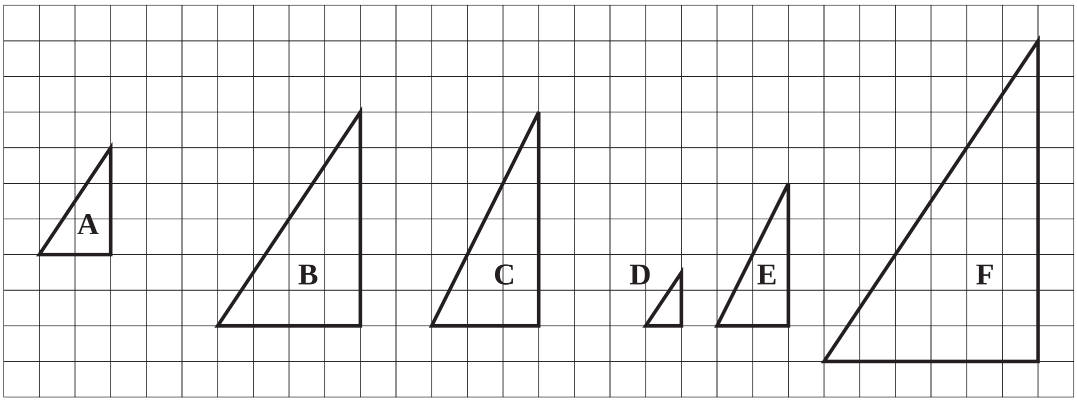
19 Similarity

19.1 Enlargement

An enlargement *increases or decreases* the size of a shape by a multiplier known as the *scale factor*. The *angles* in the shape will *not be changed* by the enlargement.

Example 1

Which of the triangles below are enlargements of the triangle marked A ? State the scale factor of each of these enlargements.



Solution

B is an enlargement of A, since all the lengths are doubled.
The *scale factor* of the enlargement is 2.

C is *not* an enlargement of A.

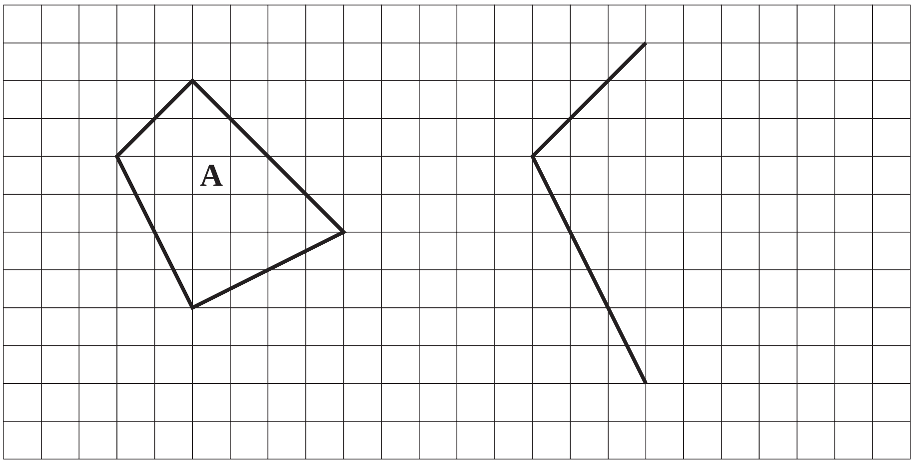
D is an enlargement of A, since all the lengths are halved.
The *scale factor* of the enlargement is $\frac{1}{2}$.

E is *not* an enlargement of A.

F is an enlargement, since all the lengths are trebled.
The *scale factor* of the enlargement is 3.

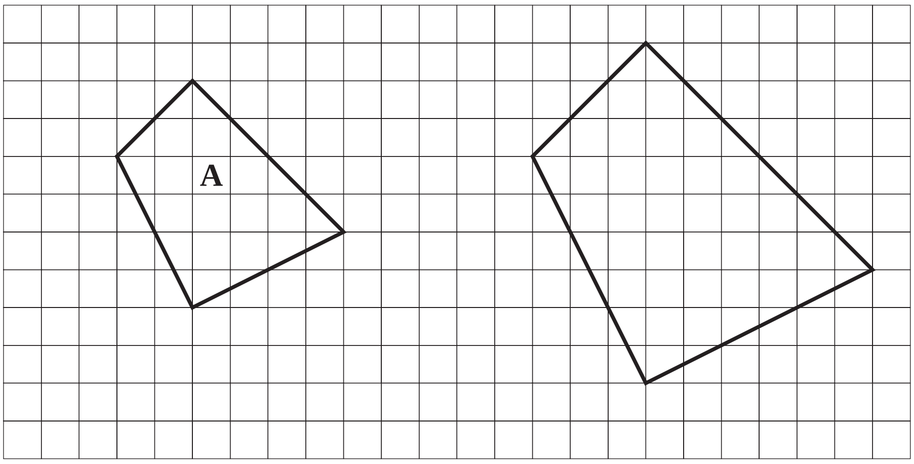
Example 2

Ameer has started to draw an enlargement of the quadrilateral marked A. Copy and complete the enlargement.



Solution

The diagram shows the completed enlargement.

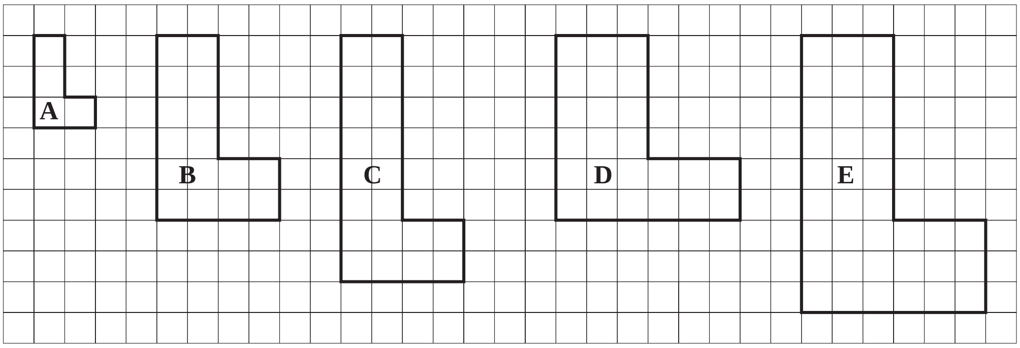


All the lengths have been increased by a factor of $1\frac{1}{2}$.

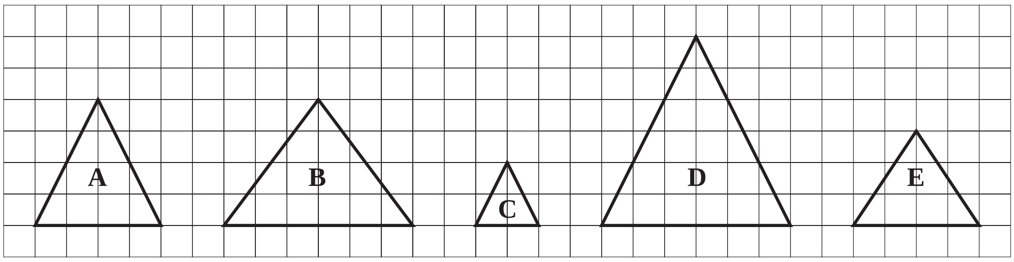
We say that the *scale factor* of the enlargement is $1\frac{1}{2}$.

Exercises

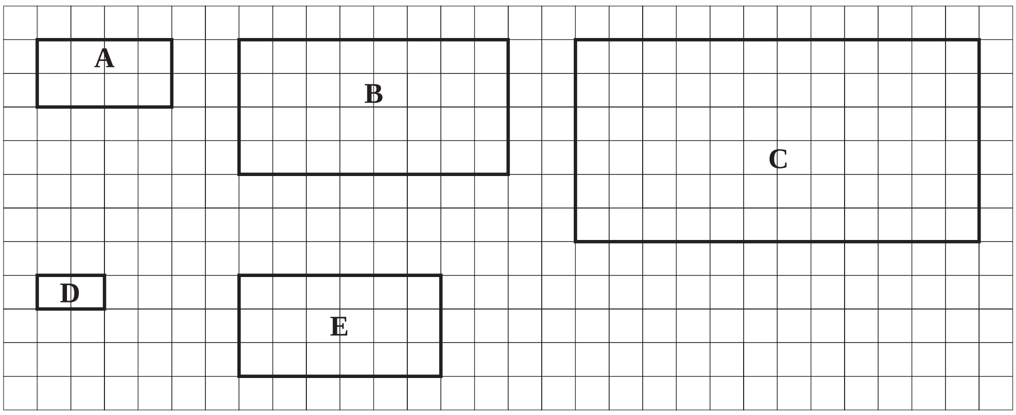
1. Which of the following shapes are enlargements of shape A ? State the scale factor of each of these enlargements.



2. Which of the following triangles are *not* enlargements of the triangle marked A ?



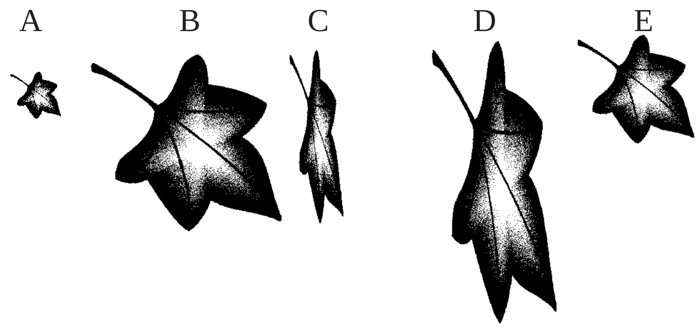
3. The diagram below shows four enlargements of rectangle A. State the scale factor of each enlargement.



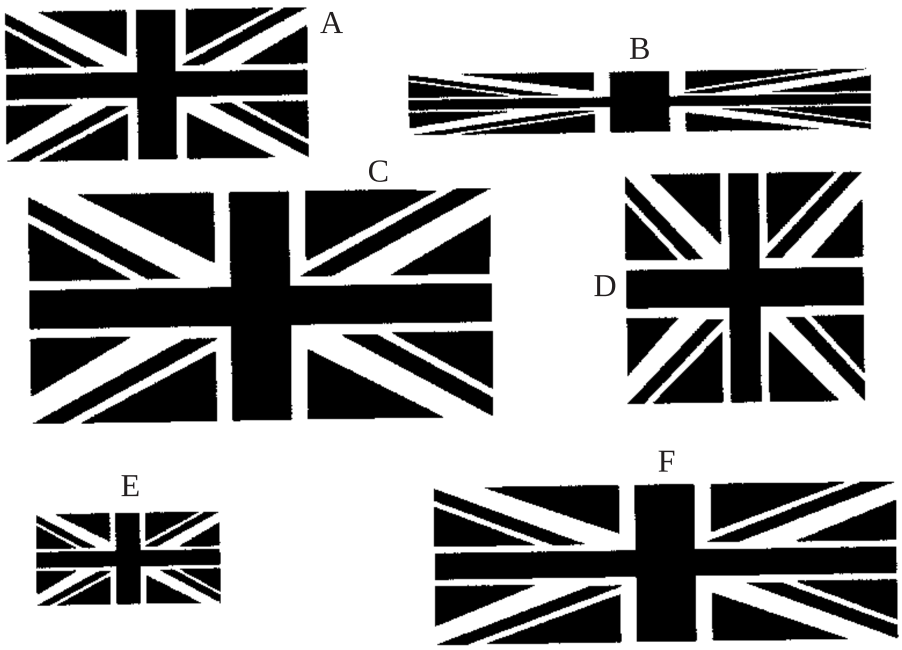
4. Which two signs below are *not* enlargements of sign A ?

A B C D E

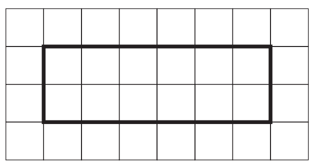
5. Which two of the leaves shown below are enlargements of leaf A ?



6. Which of the flags below are enlargements of flag A ?



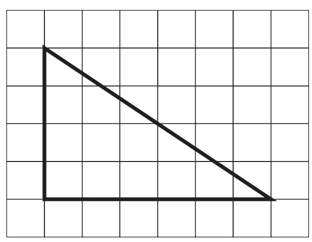
7. Draw enlargements of the rectangle shown with scale factors:



(a) 2 (b) 4

(c) $\frac{1}{2}$ (d) 3

8. Draw enlargements of the triangle shown with scale factors:

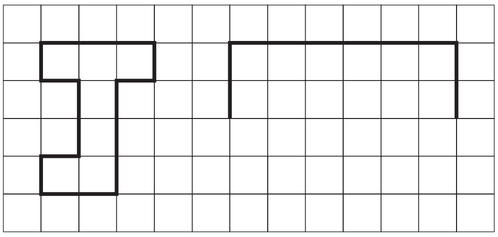


(a) 2

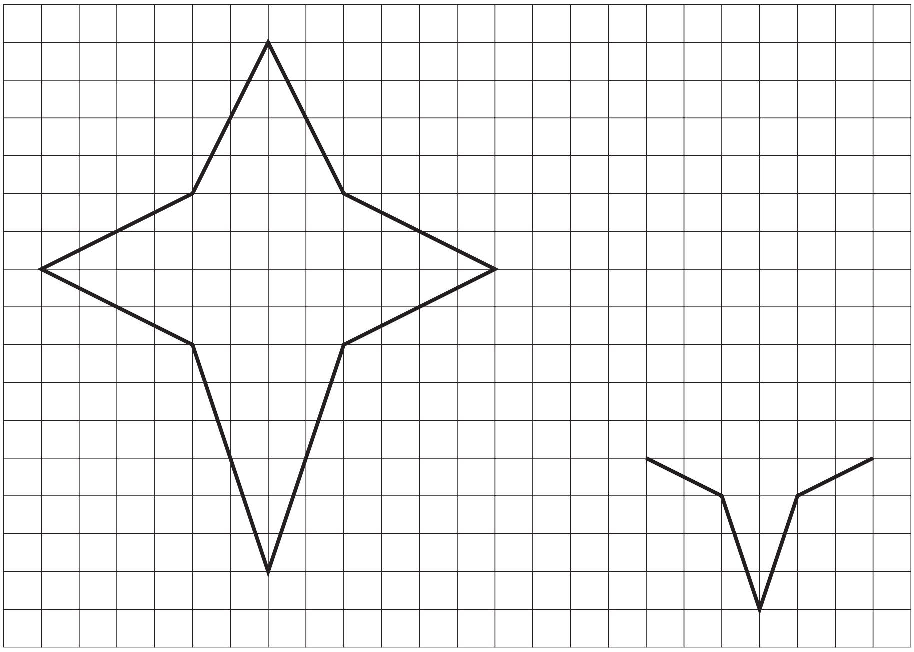
(b) 3

(c) $1\frac{1}{2}$

9. Denise has started to draw an enlargement of the shape below. Copy and complete her enlargement.

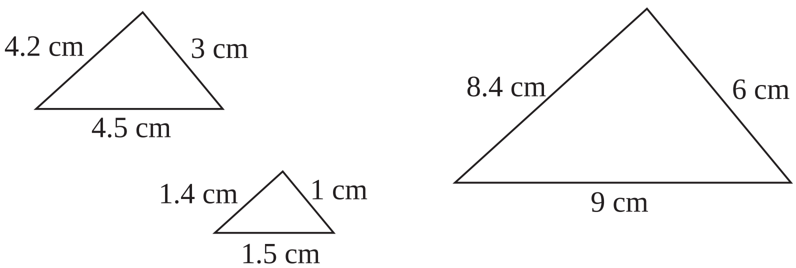


10. Kristian has started to draw an enlargement of the shape below. Copy and complete his enlargement.



19.2 Similar Shapes

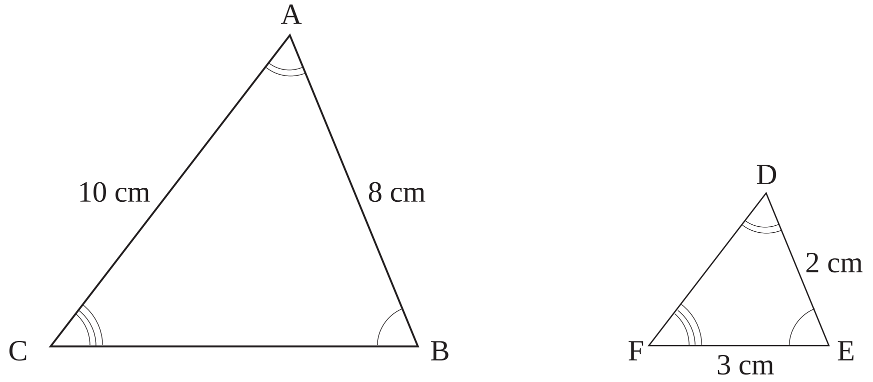
Similar shapes are those which are *enlargements of each other*; for example, the three triangles shown below are similar:



It is possible to calculate the lengths of the sides of similar shapes.

Example 1

The following diagram shows two similar triangles:



Calculate the lengths of the sides BC and DF.

Solution

Comparing the sides AB and DE gives:

$$AB = 4 \times DE$$

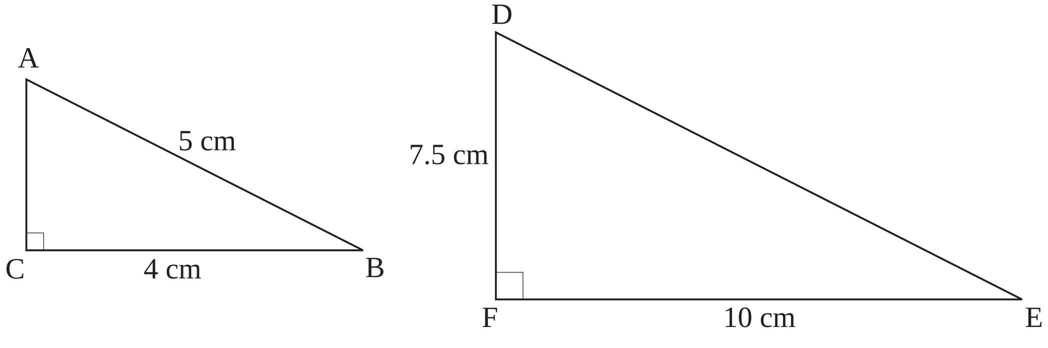
So, all the lengths in the triangle ABC will be 4 times the lengths of the sides in the triangle DEF.

$$\begin{aligned} BC &= 4 \times EF \\ &= 4 \times 3 \\ &= 12 \text{ cm} \end{aligned}$$

$$\begin{aligned} AC &= 4 \times DF \\ 10 &= 4 \times DF \\ DF &= \frac{10}{4} \\ &= 2.5 \text{ cm} \end{aligned}$$

Example 2

The following diagram shows 2 similar triangles:



Calculate the lengths of the sides AC and DE.

Solution

Comparing the lengths BC and EF gives:

$EF = 2.5 \times BC$

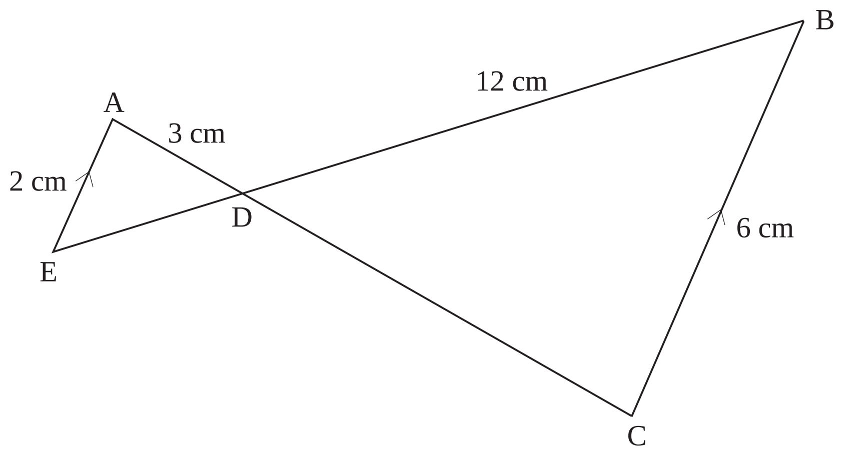
So the lengths in the triangle DEF are 2.5 times longer than the lengths in the triangle ABC.

$$\begin{aligned} DE &= 2.5 \times AB \\ &= 2.5 \times 5 \\ &= 12.5 \text{ cm} \end{aligned}$$

$$\begin{aligned} DF &= 2.5 \times AC \\ 7.5 &= 2.5 \times AC \\ AC &= \frac{7.5}{2.5} \\ &= 3 \text{ cm} \end{aligned}$$

Example 3

In the following diagram, the sides AE and BC are parallel.



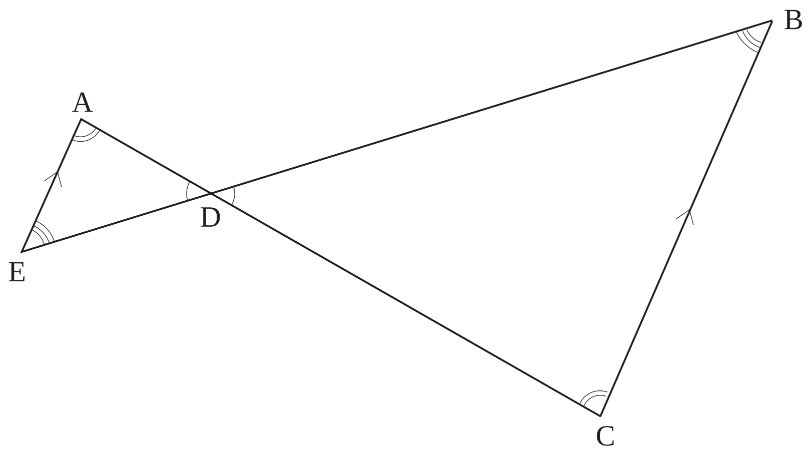
(a) Explain why ADE and CDB are similar triangles.

(b) Calculate the lengths DE and CD.

Solution

(a) $\angle ADE$ and $\angle CDB$ are opposite angles and so are equal.

Because AE and BC are parallel, $\angle DBC = \angle DEA$ and $\angle EAD = \angle BCD$.



As the triangles have angles the same size, they must be similar.

(b) Comparing AE and BC shows that the lengths in the larger triangle are 3 times the lengths of the sides in the smaller triangle, so

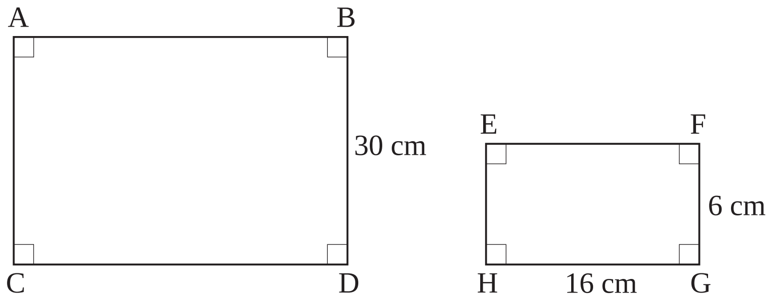
$$\begin{aligned} DC &= 3 \times AD \\ &= 3 \times 3 \\ &= 9 \text{ cm} \end{aligned}$$

and

$$\begin{aligned} BD &= 3 \times DE \\ 12 &= 3 \times DE \\ DE &= \frac{12}{3} \\ &= 4 \text{ cm} \end{aligned}$$

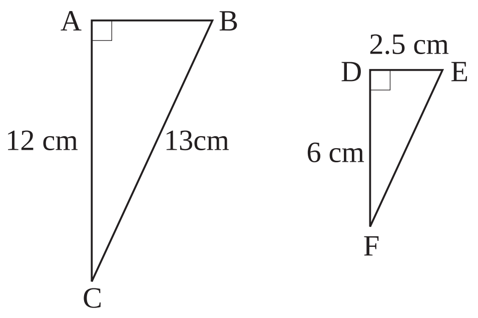
Exercises

1. The following diagram shows two similar rectangles:



Determine the length of the side CD.

2. The following diagram shows two similar triangles:

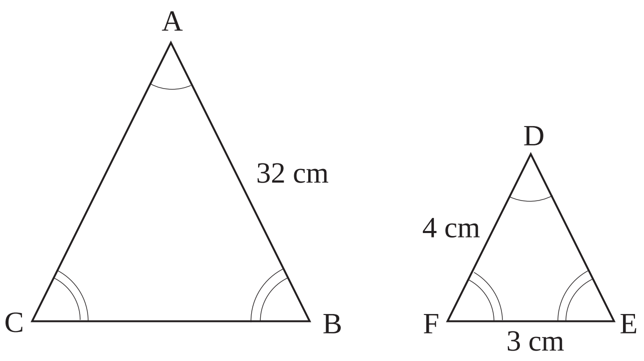


Calculate the lengths of:

(a) A B

(b) E F

3. Two similar isosceles triangles are shown in the diagram below:

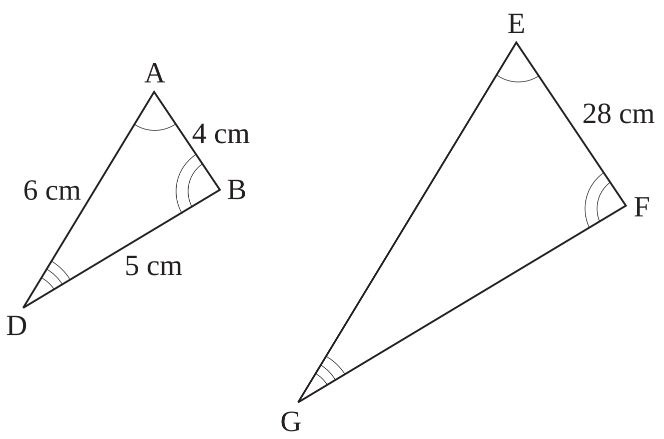


(a) What is the length of D E ?

(b) What is the length of A C ?

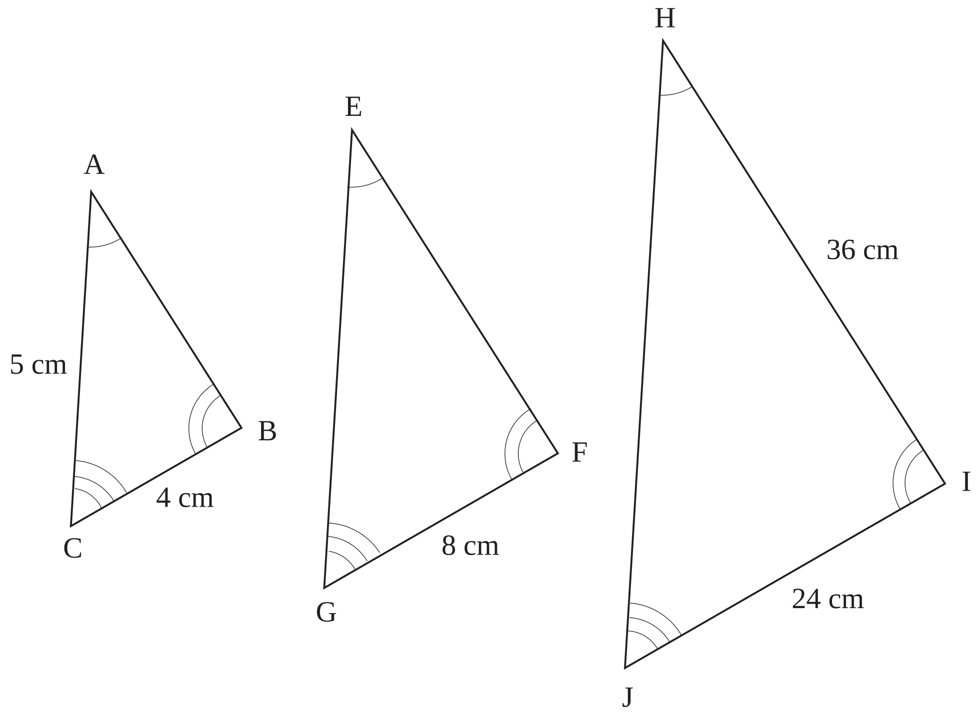
(c) Calculate the length of B C.

4. The following diagram shows two similar triangles:



Calculate the lengths of the sides G E and F G.

5. The following diagram shows three similar triangles:



Calculate the length of:

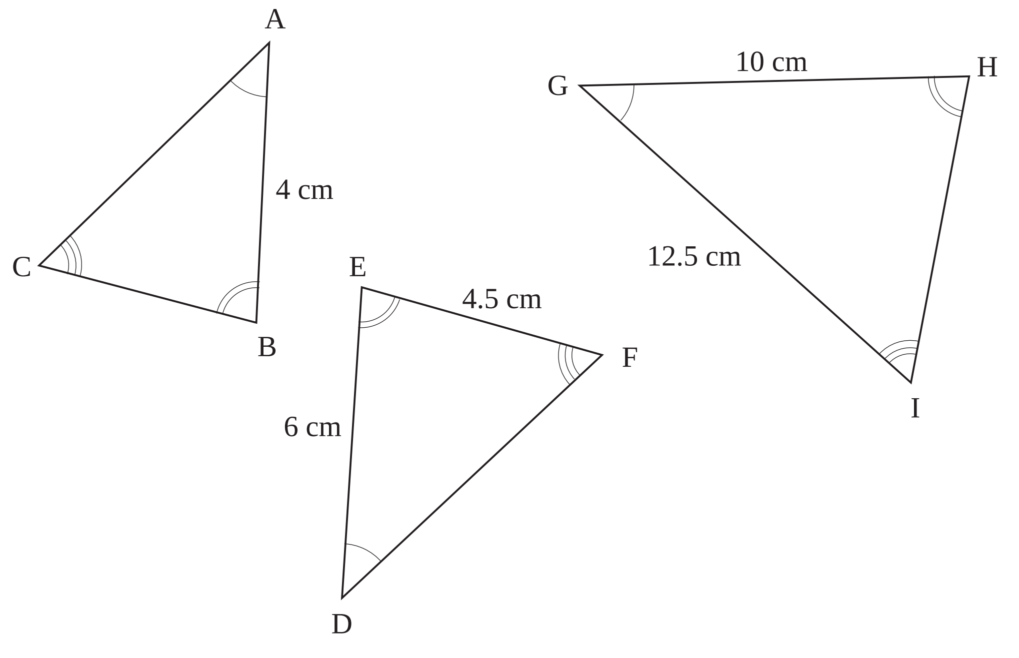
(a) E G

(b) H J

(c) E F

(d) A B

6. The following diagram shows 3 similar triangles:



Calculate the length of the sides:

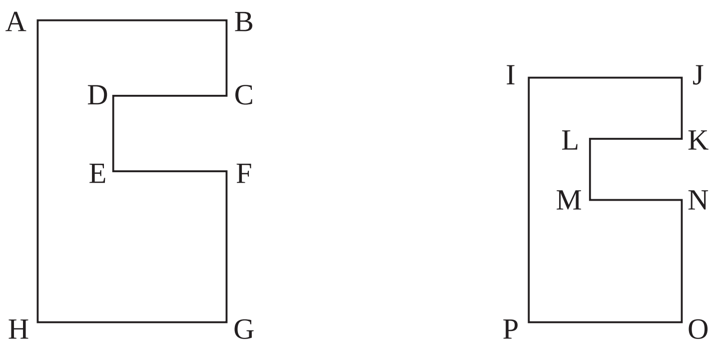
(a) H I

(b) B C

(c) A C

(d) D F

7. The following diagram shows two similar shapes:



The length of the side A B is 6 cm and the length of the side I J is 4 cm.

(a) If A H = 12 cm, calculate the length I P.

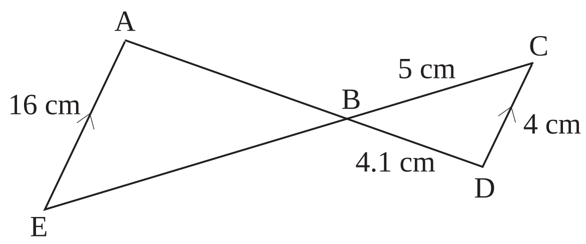
(b) If B C = 3 cm, calculate the length J K.

(c) If D E = B C, determine the length L M.

(d) Calculate the lengths F G and N O.

(e) If M N = 3 cm, determine the length E F.

8. In the diagram below, the lines A E and C D are parallel.



(a) Copy and complete the following statements:

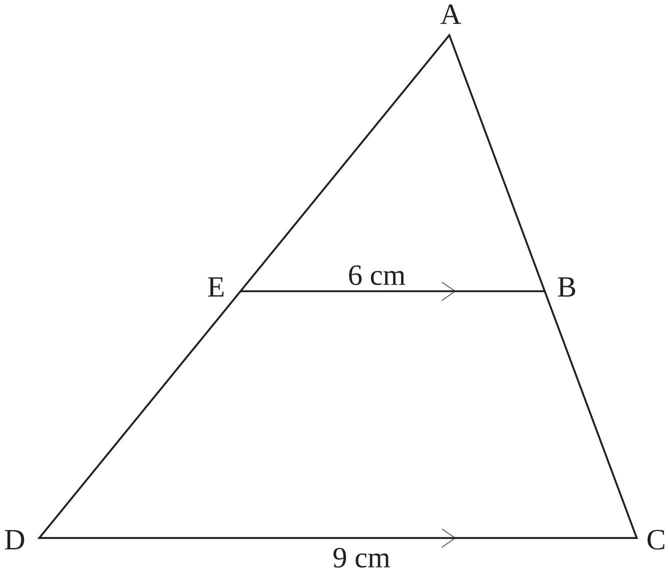
$\angle ABE = \angle$

$\angle BAE = \angle$

$\angle AEB = \angle$

(b) Calculate the lengths of A B and B E.

9. In the diagram shown below the lines B E and C D are parallel.



(a) Explain why the triangles A B E and A C D are similar.

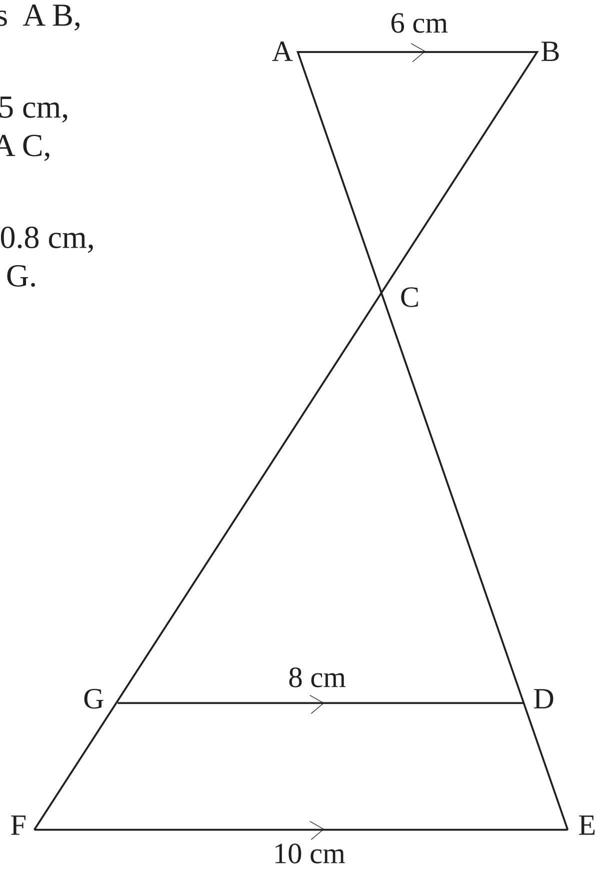
(b) If the length of A B is 4.4 cm, calculate the lengths of A C and B C.

(c) If the length of A D is 13.5 cm, determine the lengths of A E and D E.

10. In the diagram shown, the lines A B, G D and F E are parallel.

(a) If the length of C E is 15 cm, calculate the lengths of A C, C D and D E.

(b) If the length of B C is 10.8 cm, calculate the length of F G.



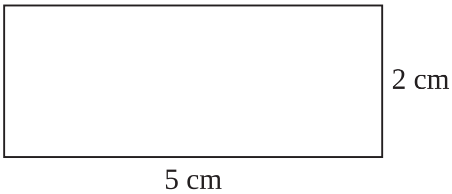
19.3 Line, Area and Volume Ratios

In this section we consider what happens to the area and volume of shapes when they are enlarged.

Example 1

The rectangle shown is enlarged with scale factor 2 and scale factor 3.

What happens to the *area* for each scale factor?



Solution

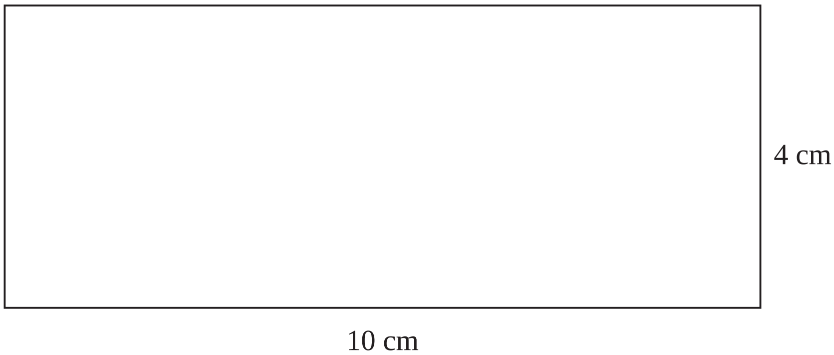
The area of the original rectangle is

$$\begin{aligned} \text{area} &= 5 \times 2 \\ &= 10 \text{ cm}^2 \end{aligned}$$

For an enlargement scale factor 2, the rectangle becomes:

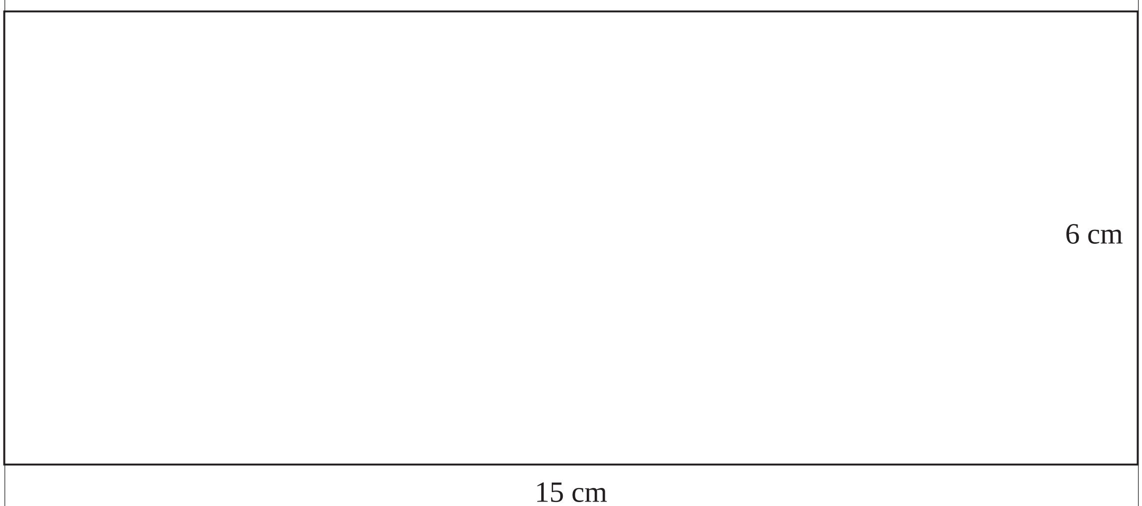
$$\begin{aligned} \text{area} &= 10 \times 4 \\ &= 40 \text{ cm}^2 \end{aligned}$$

The area has increased by a factor of 4, or 2^2.



For an enlargement scale factor 3, the rectangle becomes:

$$\begin{aligned} \text{area} &= 15 \times 6 \\ &= 90 \text{ cm}^2 \end{aligned}$$

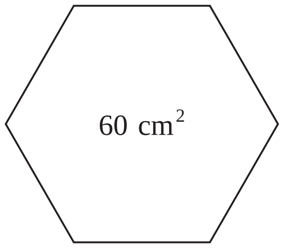


The area has increased by a factor of 9, or 3^2.

> *Note*
> If a shape is enlarged with scale factor k,
> its area is increased by a factor k^2.

Example 2

A hexagon has area 60 cm^2.



Calculate the area of the hexagon, if it is enlarged with scale factor:

(a) 2 (b) 4 (c) 10

Solution

In each case the area will increase by the scale factor squared.

(a) New area $= 2^2 \times 60$

$= 4 \times 60$

$= 240 \text{ cm}^2$

(b) New area $= 4^2 \times 60$

$= 16 \times 60$

$= 960 \text{ cm}^2$

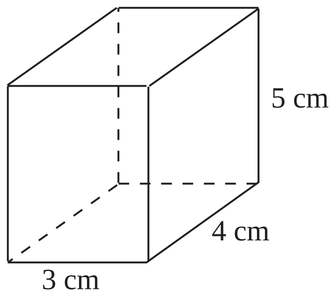
(c) New area $= 10^2 \times 60$

$= 100 \times 60$

$= 6000 \text{ cm}^2$

Example 3

A cuboid has sides of lengths 3 cm, 4 cm and 5 cm.



Calculate the volume of the cuboid, if it is enlarged with scale factor:

(a) 2 (b) 10

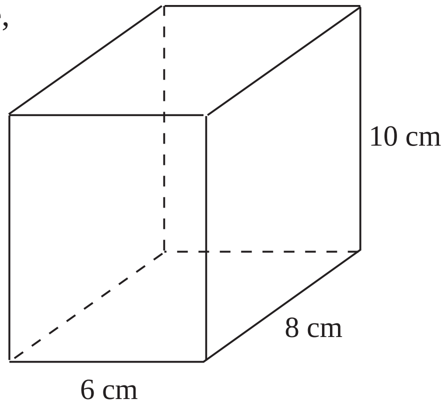
Solution

(a) The dimensions of the cuboid now become,

6 cm, 8 cm and 10 cm.

New volume $= 6 \times 8 \times 10$

$= 480 \text{ cm}^3$



Note that the volume of the original cuboid was 60 cm^3, so the volume has increased by a factor of 8, or 2^3.

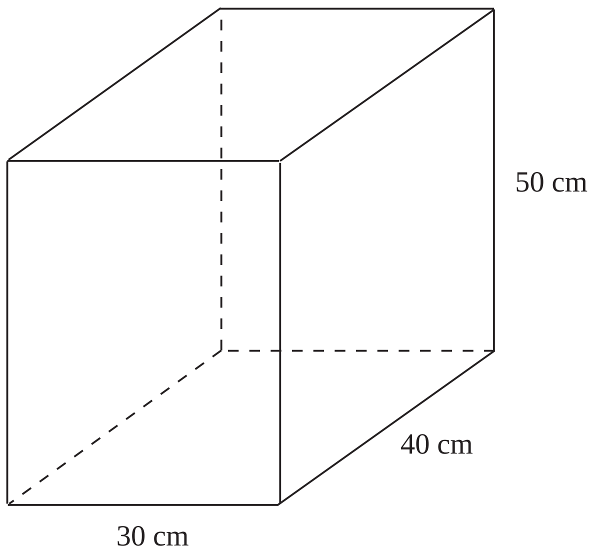
(b) The dimensions of the cuboid now become,

30 cm, 40 cm and 50 cm.

New volume $= 30 \times 40 \times 50$

$= 60\,000 \text{ cm}^3$

Note that this is 1000, or 10^3, times bigger than the volume of the original cuboid.



> *Note*
> If a solid is enlarged with scale factor k, its volume is increased by a factor k^3.

Example 4

A sphere has a volume of 20 cm^3. A second sphere has 4 times the radius of the first sphere. Calculate the volume of the second sphere.

Solution

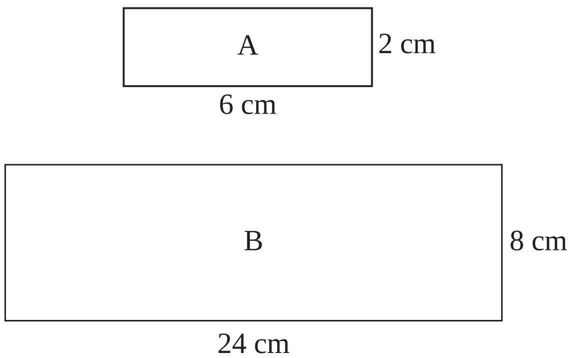
The radius is increased by a factor of 4.

The volume will be increased by a factor of 4^3.

$$\begin{aligned}\text{Volume} &= 20 \times 4^3 \\ &= 20 \times 64 \\ &= 1280 \ \text{cm}^3\end{aligned}$$

Exercises

1. Two rectangles are shown below:



   (a) Calculate the area of each rectangle.

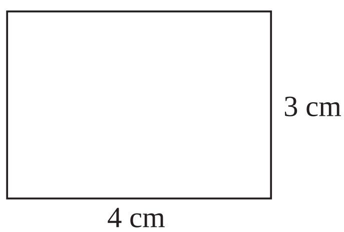
   (b) How many times longer are the sides in rectangle B than those in rectangle A ?

   (c) How many times bigger is the area of rectangle B ?

2. Calculate the area of the rectangle shown if it is enlarged with a scale factor of:

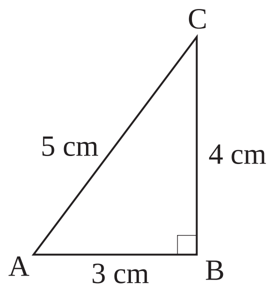
   (a) 2 (b) 3

   (c) 6 (d) 10



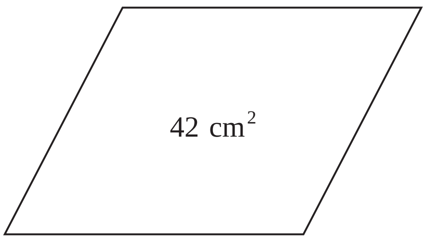
3. The following table gives information about enlargements of the triangle shown, which has an area of 6 cm^2.

Copy and complete the table.



Length of Sides *Base*	*Height*	*Scale Factor*	*Area*	*Area Factor*
3 cm	4 cm	1	6 cm^2	1
		2		
	12 cm			
	16 cm			
15 cm				
		6		
30 cm	40 cm		600 cm^2	100
4.5 cm				

4. The parallelogram shown has an area of 42 cm^2.



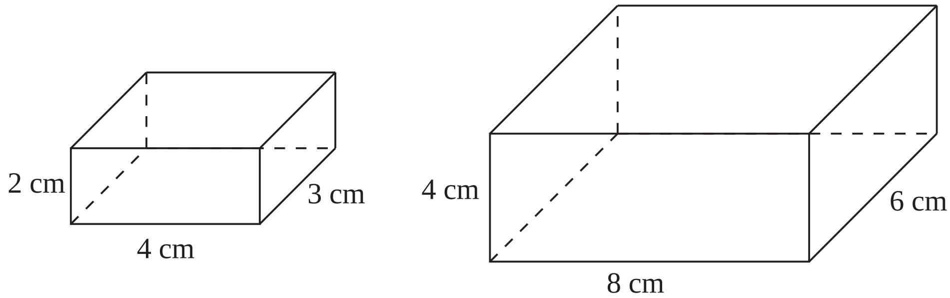
The parallelogram is enlarged with a scale factor of 5.

Calculate the area of the enlarged parallelogram.

5. The area of a circle is 50 cm^2. A second circle has a radius that is 3 times the radius of the first circle. What is the area of this circle?

6. Two similar rectangles have areas of 30 cm^2 and 480 cm^2. Describe how the length and width of the two rectangles compare.

7. (a) Determine the volume of each of the following cuboids:



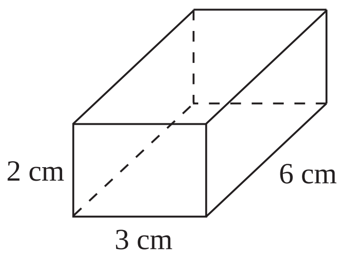
(b) The larger cuboid is an enlargement of the smaller cuboid. What is the scale factor of the enlargement?

(c) How many of the smaller cuboids can be fitted into the larger cuboid?

(d) How many times greater is the volume of the larger cuboid than the volume of the smaller cuboid?

8. A cuboid has dimensions as shown in the diagram.

The cuboid is enlarged to give larger cuboids. Copy and complete the following table:



Dimensions			*Scale Factor*	*Volume*	*Volume Factor*
Width	*Length*	*Height*			
3 cm	6 cm	2 cm	1	36 cm^3	1
6 cm			2		
			4		
		10 cm			
30 cm					

9. A tank has a volume of 32 m^3. It is enlarged with scale factor 3. What is the volume of the enlarged tank?

10. A cylinder has height 10 cm and volume 42 cm^3. An enlargement of the cylinder has height 25 cm. Calculate the volume of the enlarged cylinder.

19.4 Maps and Scale Models

The ideas of how areas and volumes change with enlargement were considered in section 19.3. Here we apply these ideas to maps and scale models.

If a map has a scale $1 : n$, then:
lengths have a scale of $1 : n$
and areas have a scale of $1 : n^2$.

If a model has a scale of $1 : n$, then
lengths have a scale of $1 : n$
areas have a scale of $1 : n^2$
and volumes have a scale of $1 : n^3$.

Note on units

1 km	=	1000 m
	=	100 000 cm
$1\ \text{m}^2$	=	$10\ 000\ \text{cm}^2$
$1\ \text{m}^3$	=	$1\ 000\ 000\ \text{cm}^3$

Example 1

On a map with a scale of $1 : 20\ 000$, a garden has an area of $5\ \text{cm}^2$. Calculate the actual area of the garden.

Solution

$$\begin{aligned} \text{Actual area} &= 5 \times 20000^2 \\ &= 2000000000\ \text{cm}^2 \\ &= 200000\ \text{m}^2 \quad \text{(dividing by 10 000)} \\ &= 0.2\ \text{km}^2 \quad \text{(dividing by 1 000 000)} \end{aligned}$$

Example 2

A map has a scale of $1 : 500$. A small public garden on the map has an area of $14\ \text{cm}^2$. Calculate the actual area of this garden.

Solution

$$\begin{aligned} \text{Actual area} &= 14 \times 500^2 \\ &= 3500000\ \text{cm}^2 \\ &= 350\ \text{m}^2 \end{aligned}$$

Example 3

A model car is made on a scale of 1 : 20.
The length of the model is 24 cm.
The area of the windscreen of the model is 32 cm^2.
The volume of the boot of the model is 90 cm^3.

Calculate the actual:

(a) length of the car,

(b) area of the windscreen,

(c) volume of the boot.

Solution

(a) Actual length $= 24 \times 20$

$= 480$ cm

$= 4.8$ m

(b) Actual area $= 32 \times 20^2$

$= 12\,800 \text{ cm}^2$

$= 1.28 \text{ m}^2$

(c) Actual volume $= 90 \times 20^3$

$= 720\,000 \text{ cm}^3$

$= 0.72 \text{ m}^3$

Exercises

1. A model boat is made to a scale of 1 : 10.
   The length of the model is 40 cm.
   The area of the hull of the model is 500 cm^2.
   The volume of the hull of the model is 3200 cm^3.

   Calculate the actual:

   (a) length of the boat,

   (b) area of the hull,

   (c) volume of the hull.

2. A map has a scale of $1:50\,000$. On the map the area of a lake is 50 cm^2. Calculate the actual area of the lake in:

   (a) cm^2 (b) m^2 (c) km^2

3. A model of a tower block is made with a scale of $1:60$. The volume of the model is $36\,000\ \text{cm}^3$. Calculate the volume of the actual tower block in m^3.

4. A plot of land is represented on a map by a rectangle 2 cm by 5 cm. The scale of the map is $1:40\,000$. Calculate the area of the plot of land in:

   (a) cm^2 (b) m^2 (c) km^2

5. A model of a house is made to a scale of $1:30$.

   The height of the model is 20 cm.

   The area of the roof of the model is 850 cm^2.

   The volume of the model house of 144 400 cm^3.

   Calculate the actual:

   (a) height of the house in m,

   (b) area of the roof in m^2,

   (c) volume of the house in m^3.

6. An aeroplane has a wingspan of 12 m. A model of this plane has a wingspan of 60 cm.

   (a) Calculate the scale of the model.

   (b) The volume of the model is 3015 cm^3. Calculate the volume of the actual aeroplane, in m^3.

   (c) A badge on the model has area 2 cm^2. Calculate the area of the actual badge, in cm^2 and m^2.

7. A forest has an area of 4 cm^2 on a map with a scale of $1:200\,000$. Calculate the actual area of the forest, in km^2.

8. An estate has an area of 50 km^2. What would be the area of the estate on a map with a scale of $1:40\,000$?

9. An indoor sports stadium has 5000 seats surrounding a playing area with an area of 384 m^2. The total volume of the stadium is 3840 m^3. A model is made to a scale of 1 : 80.

   (a) How many seats are there in the model?

   (b) What is the area of the playing surface in the model, in cm^2 ?

   (c) What is the volume of the model, in cm^3 ?

10. A lake has an area of 5 km^2. On a map it is represented by an area of 20 cm^2. What is the scale of the map?