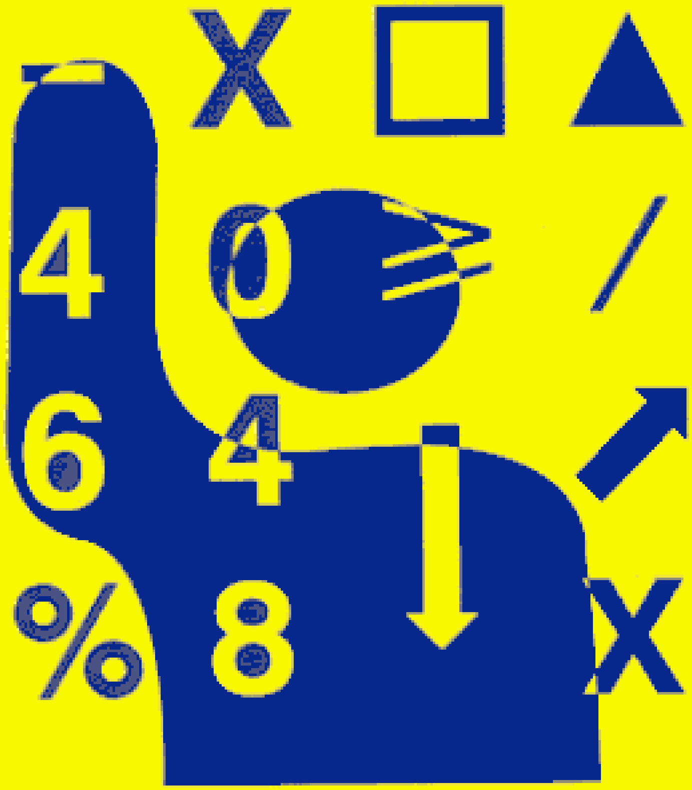
Mathematics Enhancement Programme

Primary Demonstration Project

3B Equations

Help Booklet



Support for Primary Teachers in Mathematics

Primary Project
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Mathematics Enhancement Programme

Help Module 3

EQUATIONS

Part B

Contents of Part B

Contents of Part A

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PREFACE

This is one of a series of *Help Modules* designed to help you gain confidence in mathematics. It has been developed particularly for primary teachers (or student teachers) but it might also be helpful for non-specialists who teach mathematics in the lower secondary years. It is based on material which is already being used in the *Mathematics Enhancement Programme: Secondary Demonstration Project*.

The complete module list comprises:

1.	ALGEBRA	6.	HANDLING DATA
2.	DECIMALS	7.	MENSURATION
3.	EQUATIONS	8.	NUMBERS IN CONTEXT
4.	FRACTIONS	9.	PERCENTAGES
5.	GEOMETRY	10.	PROBABILITY

Notes for overall guidance:

- Each of the 10 modules listed above is divided into 2 parts. This is simply to help in the downloading and handling of the material.
- Though referred to as 'modules' it may not be necessary to study (or print out) each one in its entirely. As with any self-study material you must be aware of your own needs and assess each section to see whether it is relevant to those needs.
- The difficulty of the material in **Part A** varies quite widely: if you have problems with a particular section do try the one following, and then the next, as the content is not necessarily arranged in order of difficulty. Learning is not a simple linear process, and later studies can often illuminate and make clear something which seemed impenetrable at an earlier attempt.
- In **Part B, Activities** are offered as backup, reinforcement and extension to the work covered in Part A. **Tests** are also provided, and you are strongly urged to take these (at the end of your studies) as a check on your understanding of the topic.
- The marking scheme for the revision test includes B, M and A marks. Note that:

  M *marks* are for method;

  A *marks* are for accuracy (awarded only following a correct M mark);

  B *marks* are independent, stand-alone marks.

We hope that you find this module helpful. Comments should be sent to:

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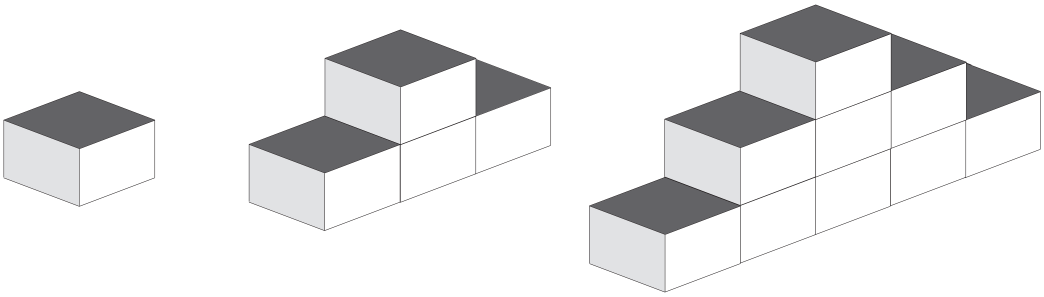
The full range of Help Modules can be found at

www.ex.ac.uk/cimt/help/menu.htm

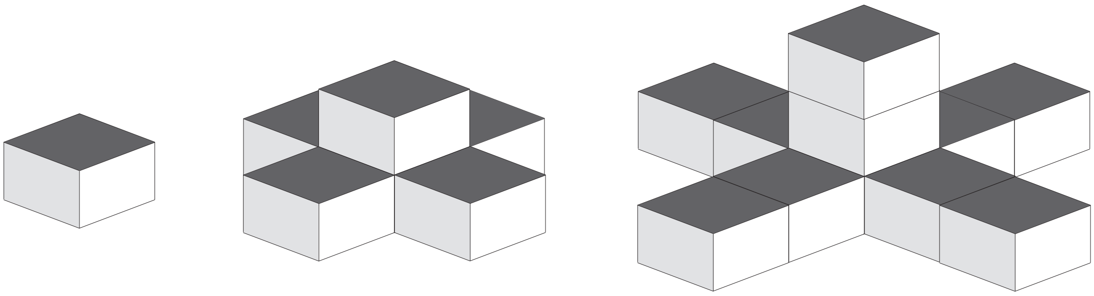
ACTIVITIES

ACTIVITY 3.1 *Monumental Towers*

Ancient civilizations used to make towers for memorials to remember their rulers.
Some might have looked like this ...



Others could have looked like this ...



The height of the tower depended on the importance of the ruler! The more important the ruler, the taller the tower.
Imagine you are the architect for an ancient civilization.

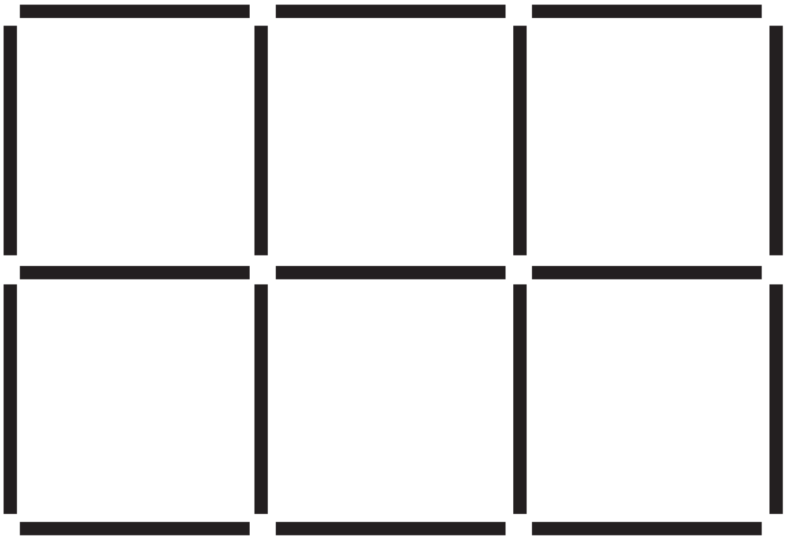
1. Design your own growing pattern of monuments. What sort if tower would you make? Make or draw a few of them.

The rulers were always changing their minds about how high their tower should be.

2. As the architect, you need to know how many bricks you need to make similar towers of any height, How can you work this out for the n^{th} tower for the two designs above?

ACTIVITY 3.2 *Rectangular Grids*

To make a rectangular grid of 2 rows of 3 squares, you would need 17 matches:



1. Investigate the number of matches needed to make rectangular grids of different sizes.

2. Find a rule/formula connecting the size of the grid (rows $\times$ columns) and the number of matches needed (*M*) for

   (a) square $(n \times n)$ grids (b) rectangle $(n \times m)$ grids

   (The grid shown is 2 rows $\times$ 3 columns.)

Extension
Extend your rule to other shapes, e.g. triangles.

ACTIVITY 3.3 *Solving Equations*

Solving equations is a fundamental part of algebra. At first, it might seem as if there are different rules for different types of equations and you might be confused as to which rule applies. In fact, all algebraic manipulations are based on the concept of

balancing equations.

Whichever process is applied to one side of the equation *must* also be applied to the other side. To solve a *general* linear equation of the form

$$\boxed{ax + b = c}$$

where x is the unknown and a, b and c are given constants, we must make x the subject.

The procedure is shown below for both the *general* case and, as an example, $2x + 5 = 9$.

		General formula	*Example*
		$\boxed{ax + b = c}$	$\boxed{2x + 5 = 9}$
Step 1	Subtract b from both sides:	$ax + b - b = c - b$	$2x + 5 - 5 = 9 - 5$
		$ax = c - b.$	$2x = 4$
Step 2	Divide both sides by a:	$\frac{ax}{a} = \frac{c-b}{a}$	$\frac{2x}{2} = \frac{4}{2}$
		$x = \frac{c-b}{a}$	$x = 2.$

Using this method, find the value of x in the following equations.

1. (a) $3x + 7 = 10$ (b) $5x - 4 = 26$ (c) $6x + 7 = 22$

   (d) $4 - x = -3$ (e) $10 - \frac{3x}{2} = 16$ (f) $11x - 52 = 3$

The special cases where either $a = 1$ or $b = 0$ can be solved easily.

2. (a) $x + 3 = 7$ (b) $x - 5 = -4$ (c) $7 - x = -4$

3. (a) $2x = 6$ (b) $3x = -15$ (c) $\frac{x}{2} = 10$ (d) $\frac{5x}{3} = 15$

Extension

1. Find the *general* solution of $\boxed{ax + b = cx + d}$ where a, b, c and d are constants.

2. Solve:

   (a) $2x + 1 = x + 6$ (b) $3x + 2 = 5 - x$ (c) $4 - 7x = 3x + 14$

ACTIVITY 3.4 *Magic Squares*

In *magic squares*, in each row, column and diagonal, the sum of the numbers is always equal to the *magic number for* that square.

1. Here is an example which happens to use 9 consecutive numbers.

   Check that the sum of the numbers in each row, column and diagonal is equal to the magic number, 12.

3	2	7
8	4	0
1	6	5

Solving magic squares

	11	7
9		
	5	10

This magic square is more challenging! The answer may be found by trial and error but really a more systematic method is required.

Let x be the unknown number in *Column 1*, *Row 1*,
y be the unknown number in *Column 1*, *Row 3*,
n be the *magic* number.

x	11	7
9		
y	5	10

Then for *Row 1*, $n = x + 11 + 7 = x + 18$,

and for *Column 1*, $n = x + 9 + y$.

So,

$$x + 18 = x + 9 + y \quad \text{(Divide both sides by } x\text{)}$$

$$18 = 9 + y \quad \text{(Subtract 9 from both sides)}$$

$$y = 9.$$

From *Row 3*, $n = y + 5 + 10$, so $n = 24$. From *Row 1*, $x + 18 = n = 24$, so $x = 6$.

The other two missing numbers can then be found to be 8 (*Column 2*) and 7 (*Column 3*).

2. Use an algebraic approach to solve the following magic squares.

(a)

9	2	
12	8	

(b)

10	3	
5		9
	11	4

(c)

14		12
10		8

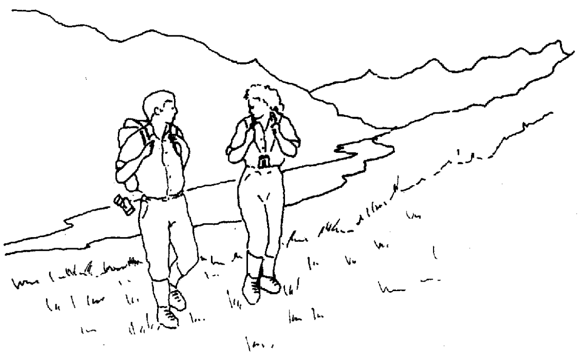
Extension

a	b	
c	d	

For the general magic square opposite:

1. Find an expression for the missing entries in terms of a, b, c, d and n.
2. Form equations for the sums in the two diagonals.
3. Hence solve for the unknowns, c and d, in terms of a, b and n and find the form of a general magic square
4. Use this general form to solve the magic squares in *Problem 2*.

ACTIVITY 3.5 *Hill Walking*



Walking up a hill slows down your pace, but by how much?

We will try to provide a mathematical model to describe this situation, which can then be used to estimate the time it would take to climb a hill.

The time taken (T hours) will generally depend on 4 factors:

1. *Map distance* (horizontal distance travelled) (d miles)
2. *Height of the hill* (vertical distance climbed) (h feet)
3. *Speed of walking horizontally* (x miles per hour)
4. *Speed of climbing vertically* (y feet per hour)

The model will be of the form

$$T = \frac{d}{x} + \frac{h}{y}$$

1. The table below shows some data which was gathered experimentally.

   (a) If $X = \frac{1}{x}$ and $Y = \frac{1}{y}$, use the data opposite to form two equations involving X and Y.

   (b) Solve these equations for X and Y.

   (c) Hence find the values of x and y.

Map Distance (d miles)	*Height of Hill* (h feet)	*Time Taken* (T hours)
12	1500	$5\frac{1}{2}$
15	2000	7

2. Use the model

$$T = \frac{d}{3} + \frac{h}{1000}$$

   to determine how long it might take to climb *Mount Snowdon* (height 3560 feet) if the map distance from your starting point is 3 miles.

3. If it takes you 8 hours to climb a hill and your map distance is 4 miles, estimate the height of the hill.

Extension

1. Apply this test to a local situation, using your own experimental data, to see whether this model works in practice.

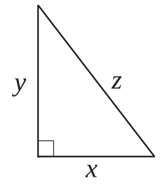
2. Produce your own model for the time it takes to come *down* a hill.

ACTIVITY 3.6 *Diophantine Equations*

Equations which have many *integer* (positive/negative whole numbers) solutions are known as *Diophantine equations*, after the Greek mathematician and philosopher, *Diophantos* of Alexandria (c. 250 A.D.). He is credited with being the founder of modern algebra. The use of symbols to represent numbers was found in his published document, *Arithmetic*.

1. One example of a *Diophantine equation* could be Pythagoras' result that, for any right-angled triangle,

$$x^2 + y^2 = z^2 .$$



Find 3 different integer solutions to $x^2 + y^2 = z^2$.

2. Another example of a *Diophantine equation* is:

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

for some number, n, and where x and y are integers.

(a) For example, when $n = 6$,

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{6} .$$

(i) One possible solution to this equation is $x = 8$, $y = 24$.
Check that these values do indeed give a solution for the equation.

(ii) A second solution is obviously $x = 24$, $y = 8$, but there are many more.
Find in total 17 solutions of this equation.
[*Hint*: Remember that x or y can be a negative integer.]

(b) How many integer solutions can you find for the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{8}$?

Extension

1. Find two solutions, using small integers, to the equation $x^3 + y^3 + z^3 = 3$.

2. An unsolved conjecture is that the *Diophantine equation*,

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{4}{n},$$

has at least *one* solution for x, y and z for any integers, $n > 1$.

Show that this conjecture is true for $n = 2$, 3 and 4.

ACTIVITIES 3.1 – 3.6 Sheet 1 *Notes for Solutions*

Notes and solutions given only where appropriate.

3.1 2. (a) n^2 (b) $5n - 4$

3.2 2. (a) $2n^2 + 2n$ (b) $2nm + n + m$

3.3 1. (a) 1 (b) 6 (c) 2.5 (d) 7 (e) -4 (f) 5

2. (a) 4 (b) 1 (c) 11

3. (a) 3 (b) -5 (c) 20 (d) 9

Extension

1. $\dfrac{d-b}{a-c}$ $(a \neq c)$ 2. (a) 5 (b) $\dfrac{3}{4}$ (c) -1

3.4 2. (a)

9	2	13
12	8	4
3	14	7

$n = 24$

(b)

10	3	8
5	7	9
6	11	4

$n = 21$

(c)

14	7	12
9	11	13
10	15	8

$n = 33$

Extension

1.

a	b	$n-a-b$
c	d	$n-c-d$
$n-a-c$	$n-b-d$	$a+b+c+d-n$

2. $n - 2a - b - c + d = 0, \quad -2n + 2a + b + c + 2d = 0$

3. $d = \dfrac{n}{3}, \quad c = \dfrac{4n}{3} - 2a - b$

4.

a	b	$n-a-b$
$\dfrac{4n}{3} - 2a - b$	$\dfrac{n}{3}$	$-\dfrac{2n}{3} + 2a + b$
$-\dfrac{n}{3} + a + b$	$\dfrac{2n}{3} - b$	$\dfrac{2n}{3} - a$

ACTIVITIES 3.1 – 3.6 Sheet 2 *Notes for Solutions*

3.5 1. (a) $5.5 = \dfrac{12}{x} + \dfrac{1500}{y} \Rightarrow 5.5 = 12X + 1500Y \quad \left(X = \dfrac{1}{x},\ Y = \dfrac{1}{y}\right)$

$7 = \dfrac{15}{x} + \dfrac{2000}{y} \Rightarrow 7 = 15X + 2000Y$

(b) $X = \dfrac{1}{3},\quad Y = \dfrac{1}{1000}$

(c) $x = 3,\ y = 1000.$

2. $T = \dfrac{3}{3} + \dfrac{3560}{1000} = 1 + 3.56 = 4.56$ – almost 5 hours

3. $8 = \dfrac{4}{3} + \dfrac{h}{1000} \Rightarrow h = 6666.7$ feet

3.6 1. There is an infinite number of solutions;
e.g. 9, 40, 41 or 6, 8, 10 (although this is really 3, 4, 5).

3.

x	12	8	24	3	–6	2	–3	4	–12	9	18	10	15	5	–30	7	42
y	12	24	8	–6	3	–3	2	–12	4	18	9	15	10	–30	5	42	7

4.

x	16	4	–8	12	24	6	–24	10	40	7	–56	9	72
y	16	–8	4	24	12	–24	6	40	10	–56	7	72	9

Extensions

1. $x = y = z = 1$ is an obvious solution;
$x = y = 4,\ z = -5$ provides another solution
(or $x = z = 4,\ y = -5$; or $x = -5,\ y = z = 4$)

2. For example, if $n = 3$, $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = \dfrac{4}{3}$ has solution $x = 3,\ y = z = 2$.

TESTS

Tests 3.1 and 3.2 Mental Practice

Answer these questions as quickly as you can, but without the use of a calculator.

Test 3.1

1. Whe value of $x + 2y$ when (a) $x = 3$ and y = 2
   (b) $x = 5$ and $y = -1$?
2. What is the value of $x^2 - y^2$ when $x = 4$ and $y = 2$?
3. Simplify $2x + y + 3x - 2y$.
4. Expand $2(x - 4)$.
5. Solve for x: (a) $x - 3 = 7$
   (b) $3x = 12$
   (c) $2x + 1 = 9$
6. Solve for x and y: $x + y = 3$
   $x - y = 1$

Test 3.2

1. Whe value of $2x - y$ when (a) $x = 3$ and y = 2
   (b) $x = 1$ and $y = -3$?
2. What is the value of $x^2 + y^2$ when $x = 3$ and $y = 4$?
3. Simplify $y - x + 3y - 2x$
4. Expand $3(x + 3)$.
5. Solve for x: (a) $x + 2 = 5$
   (b) $4x = 12$
   (c) $3x - 1 = 14$
6. Solve for x and y: $x + 2y = 5$
   $x + y = 3$

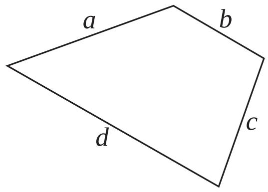
Test 3.3 Revision

60 minutes are allowed.

1. The perimeter length of a quadrilateral is given by:

$$P = a + b + c + d$$

where a, b, c and d are lengths of the four sides.



Find the perimeter length when

(a) $a = 1, \ b = 2, \ c = 3, \ d = 4$

(b) $a = b = c = d = 4$ *(2 marks)*

2. The speed of a falling stone in metres per second is given by the formula $s = u + 10t$.

Find the value of s when $u = 5$ and $t = 6$. *(2 marks)*

3. The distance, d, travelled by a train is given by the formula

$$d = 5(u + v)$$

Calculate d when $u = 2$ and $v = 5$. *(2 marks)*

4. Simplify the following expressions:

(a) $x + 2x + 3x$

(b) $2x + 4y - x - 3y$

(c) $x^2 + 4x + 2x^2 - 2x$ *(3 marks)*

5. Expand the following:

(a) $3(x + 3)$ (b) $5(x + y)$

(c) $4(x - 2y)$ (d) $x(1 - 2x)$ *(4 marks)*

6. Solve each of the following equations:

(a) $x + 3 = 5$ (b) $6 - x = 2x$

(c) $\dfrac{x}{4} = 3$ (d) $2x - 3 = 7$

(e) $\dfrac{x}{2} + 3 = -2$ *(10 marks)*

Test 3.3

7. (a) The perimeter of a rectangle is given by

$$P = 2(L + B)$$

where L and B are the lengths of the sides.

Find P when $L = 17$ and $B = 13$. *(2 marks)*

(b) Solve the equation

$$4x - 5 = x + 1$$ *(3 marks)*

8. A book of 20 postage stamps contains only 17p and 22p stamps. The total value of the stamps is £4.

(a) There are n stamps at 22p.

(i) Write an expression, in terms of n, for the total value of the 22p stamps. *(1 mark)*

(ii) Write an expression, in terms of n, for the number of 17p stamps. *(1 mark)*

(b) The number of 22p stamps, n, can be found by solving the equation:

$$22n + 17(20 - n) = 400.$$

How many 22p stamps are in the book? *(2 marks)*

9. The cost, C pence, of advertising in the local newspaper is worked out using the formula

$$C = 20n + 30$$

where n is the number of words in the advertisement.

(a) Annelise puts in an advertisement of 15 words.

Work out the cost. *(2 marks)*

(b) The cost of Debbie's advertisement is 250 pence.

(i) Use the formula to write down an equation in n. *(1 mark)*

(ii) Solve the equation to find the number of words in Debbie's advertisement. *(2 marks)*

10. The cost, S pound, of a chest of drawers with d drawers may be calculated using the formula

$$S = 29 + 15d$$

(a) Calculate the cost of a chest of drawers with 3 drawers. *(1 mark)*

Another chest of drawers costs £119.

(b) Calculate the number of drawers this chest has. *(2 marks)*

Test 3.3

11. The air temperature, T °C, outside an aircraft flying at a height of h feet is given by the formula

$$T = 26 - \frac{h}{500}$$

An aircraft is flying at a height of 27 000 feet.

(a) Use the formula to calculate the air temperature outside the aircraft. *(2 marks)*

The air temperature outside an aircraft is –52 °C.

(b) Calculate the height of the aircraft. *(2 marks)*

12. Use a trial and improvement method to solve the equation

$$p^3 - 5p = 15$$

Give your answer correct to two decimal places.

The first trial has been done for you.

p	$p^3 - 5p$	
3	12	Too low

(*3 marks*)

13. Solve $3x + y = 4$

$y = x + 2$ (*3 marks*)

Tests 3.1 and 3.2 Answers

Test 3.1

1. (a) 7
   (b) 3

2. 12

3. $5x - y$

4. $2x - 8$

5. (a) 10
   (b) 4
   (c) 4

6. $x = 2$, $y = 1$

Test 3.2

1. (a) 4
   (b) 4

2. 25

3. $4y - 3x$

4. $3x + 9$

5. (a) 3
   (b) 3
   (c) 5

6. $x = 1$, $y = 2$

Test 3.3 Answers

1.	(a) 10 (b) 16		B1	B1		(*2 marks*)
2.	65 m/s		M1	A1		(*2 marks*)
3.	35		M1	A1		(*2 marks*)
4.	(a) $6x$ (b) $x + y$ (c) $3x^2 + 2x$	B1	B1	B1		(*3 marks*)
5.	(a) $3x + 9$ (b) $5x + 5y$		B1	B1		
	(c) $4x - 8y$ (d) $x - 2x^2$		B1	B1		(*4 marks*)
6.	(a) $x = 2$ (b) $x = 2$ (c) $x = 12$	B2	B2	B2		
	(d) $x = 5$ (e) $x = -10$		B2	B2		
						(*10 marks*)
7.	(a) P = $2(17 + 13) = 60$		M1	A1		
	(b) $3x = 6$, $x = 2$		M2	A1		(*5 marks*)
8.	(a) (i) $22n$ (ii) $(20 - n)$		B1	B1		
	(b) $5n = 60$, $n = 12$		M1	A1		(*4 marks*)
9.	(a) £3.30		M1	A1		
	(b) (i) $250 = 20n + 30$ (ii) 11	B1	M1	A1		(*5 marks*)
10.	(a) 374 (b) 6 drawers	B1	M1	A1		(*3 marks*)
11.	(a) $T = -28°$C (b) 39 000 feet	M1	A1	B2		(*4 marks*)
12.	$4 \Rightarrow 44 \Rightarrow$ too high		M1			
	$x \approx 3.13$		A2	(A1 for answer correct to 1 d.p.)		
						(*3 marks*)
13.	$3x + (x + 2) = 4$		M1			
	$4x = 2$, therefore $x = 0.5$, $y = 2.5$		A1	A1		(*3 marks*)

(TOTAL MARKS 50)