STRAND G: Relations, Functions and Graphs

G4 *Functions*

Text

Contents

G4 Functions

G4.1 Functions, Mappings and Domains

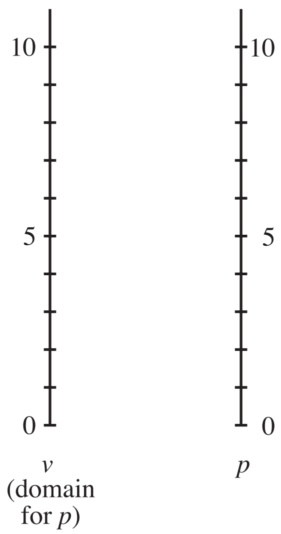
You are already familiar with the idea of a function. Here we make the concept more formal with a number of definitions that we will introduce. We illustrate the definitions using examples, including the formula that relates pressure to volume:

$$p = \frac{k}{v}$$

Here p is the pressure of a gas, v is its volume and k is a constant.

Worked Example 1

If $p = \dfrac{10}{v}$, use the mapping diagram opposite to show how v maps to p for $1 \le v \le 10$.



Solution

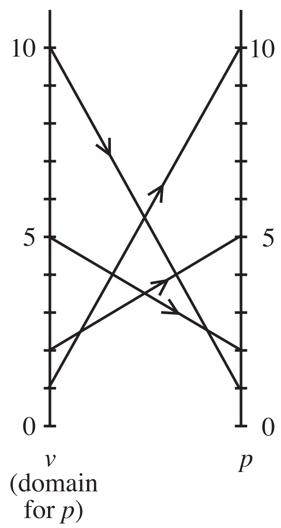
Clearly if $v = 1, \quad p = \dfrac{10}{1} = 10$

$v = 2, \quad p = \dfrac{10}{2} = 5$

$v = 5, \quad p = \dfrac{10}{5} = 2$

$v = 10, \quad p = \dfrac{10}{10} = 1$

We can illustrate this mapping.



In fact, for any value $1 \le v \le 10$, we can map v to p.

For example,

$$v = 4, \quad p = \frac{10}{4} = \frac{5}{2}, \text{ etc.}$$

Notes

1. The set of values of v, defined here as $1 \le v \le 10$, is called the DOMAIN of the mapping.
2. The set of values of p that v maps onto is known as the RANGE; here the range is also $1 \le p \le 10$.
3. For every value of v in its domain, there is a unique value of p. This is called a 1 : 1 mapping

Worked Example 2

For the function $y = f(x) = x^2$, $-5 \le x \le 5$, complete a similar mapping diagram.

Solution

Here, notice that

$$f(-5) = (-5)^2 = 25$$

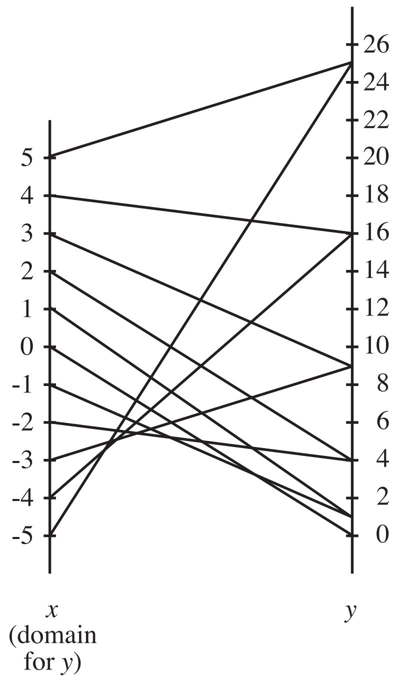
$$f(-2) = (-2)^2 = 4$$

$$f(0) = 0$$

$$f(5) = 25$$

etc.

We can complete the mapping opposite for some of the values.



Notes

1. In this example, apart from 0 (which maps to 0), there are two values which both map to the same value. For example, $+2$ and -2 both map to 4.
   This is NOT a 1 : 1 mapping.
2. As well as the notation $y = f(x)$ for a function, we can also use the mapping notation

$$f : x \to x^2$$

Worked Example 3

If f is defined by

$$f : x \to x^3 - x$$

what are the values of

(a) $f(-1)$ (b) $f(0)$ (c) $f(1)$ (d) $f(5)$?

Solution

(a) $f(-1) = (-1)^3 - (-1) = -1 + 1 = 0$

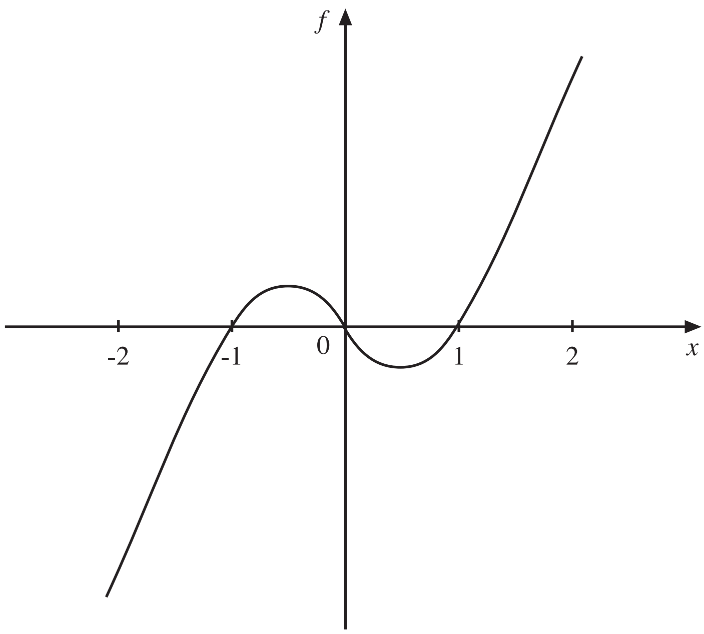
(b) $f(0) = 0^3 - 0 = 0$

(c) $f(1) = 1^3 - 1 = 1 - 1 = 0$

(d) $f(5) = 5^3 - 5 = 125 - 5 = 120$

Note

This is not a 1 : 1 mapping as $f(-1) = f(0) = f(1)$.
This can be seen in a sketch of the function, as below.



Exercises

1. $f : x \rightarrow 2x - 1,\ 1 \le x \le 6$

   (a) Complete a mapping diagram for integer values in the domain.

   (b) What is the range of this function?

   (c) Is this a 1 : 1 mapping?

2. $f : x \rightarrow \frac{1}{2}x^2 - 1,\ -3 \le x \le 3$

   (a) Complete a mapping diagram for integer values in the domain.

   (b) What is the range?

   (c) Is this a 1 : 1 mapping?

3. (a) If f is defined by

   $$f : x \rightarrow \frac{x+1}{x-1} \qquad (x \ne 1)$$

   what are the values of

   (i) $f(-2)$ (ii) $f(-1)$ (iii) $f(0)$

   (iv) $f(2)$ (v) $f(3)$?

   (b) Why is the domain restricted by $x \ne 1$?

4. Given that $h(x) = \frac{x^2 - 16}{x - 2}$,

   calculate

   (a) $h(-2)$

   (b) the values of x for which $h(x) = 0$.

5. $f : x \rightarrow \frac{3 - x}{1 - x}$

   (a) What is the value of

   (i) $f(-2)$ (ii) $f(0)$ (iii) $f(2)$?

   (b) Find the value of x for which $f(x) = 4$.

6. f is a function defined by the equation

$$f : x \to x^2 + 2$$

Find the value of

(a) $f(2)$ (b) $f(-1)$ (c) $f(0)$

(d) $f(a^2)$ (e) $f(1-a)$

(where a is a constant real number).

7. g is defined by the equation

$$g : x \to \frac{1}{x} \qquad (x \neq 0)$$

Find the value of

(a) $g(1)$ (b) $g(-1)$ (c) $g(0.01)$

(d) $g(a^2)$ (e) $g(1-a)$

(where a is a constant real number).

8. $f : x \to 1 - x^2$

(a) What are the values of

(i) $f(-2)$ (ii) $f(-1)$ (iii) $f(0)$

(iv) $f(1)$ (v) $f(2)$

(b) Find the value of x for which $f(x) = -8$.

9. Given that $m * 1 = m^2 - lm$,

(a) evaluate $5 * 3$

(b) solve for g given that $g * 4 = -3$.

G4.2 Composite Functions

The next concept to cover is that of the *composite function*, that is, a 'function of a function'.

Worked Example 1

The two functions f and g are defined by

$$f : x \to 4x - 3$$

$$g : x \to 2x - 1$$

(a) Find $f(g(x))$ and $g(f(x))$,

(b) What are the values of $f(g(0))$ and $g(f(0))$?

(c) Are there any solutions of $f(g(x)) = g(f(x))$?

Solution

(a) $f(g(x)) = 4(g(x)) - 3$ since, for example, $f(a) = 4a - 3$, etc.

$= 4(2x - 1) - 3$

$= 8x - 4 - 3$

$= 8x - 7$

$g(f(x)) = 2(f(x)) - 1$

$= 2(4x - 3) - 1$

$= 8x - 6 - 1$

$= 8x - 7$

(b) $f(g(0)) = 8 \times 0 - 7 = -7$

$g(f(0)) = 8 \times 0 - 7 = -7$

(c) As $f(g(x)) = g(f(x))$, then this is satisfied by all values of x.

Note

The result $f(g(x)) = gf(x)$ is not generally true, but depends on the values of the coefficients in the functions (see Question 6 in Exercises).

Note

We usually write, for short,

$$f(g(x)) = fg(x)$$

and

$$g(f(x)) = gf(x)$$

but you must remember what this notation means.

Worked Example 2

The functions f and g are defined by

$$f : x \to x^2 + 1$$

$$g : x \to x - 1$$

Will $fg(x) = gf(x)$ for any value of x ?

Solution

$$fg(x) = f(g(x)) = (g(x))^2 + 1$$
$$= (x-1)^2 + 1$$
$$= x^2 - 2x + 1 + 1$$
$$= x^2 - 2x + 2$$

$$gf(x) = g(f(x)) = f(x) - 1$$
$$= x^2 + 1 - 1$$
$$= x^2$$

If $fg(x) = gf(x)$, then

$$x^2 - 2x + 2 = x^2$$
$$-2x + 2 = 0$$
$$2x = 2$$
$$x = 1$$

So $fg(x) = gf(x)$ when $x = 1$.

Exercises

1. The functions f and g are defined as

$$f : x \to \frac{2x-1}{x+3} \qquad g : x \to 2x - 1$$

What is the value of

(a) $g(2)$

(b) $f(-2)$

(c) $fg(2)$

(d) $gf(-2)$?

2. If $f(x) = x - 1$ and $g(x) = x^3$,

(a) find (i) $fg(x)$

(ii) $gf(x)$

(b) Does $fg(x) = gf(x)$ have any solutions?

3. The functions f and g are defined by

$$f : x \to \frac{1}{x} \qquad\qquad g : x \to x + 1$$

What is

(a) $fg(x)$ (b) $gf(x)$?

4. Find the composite function $fg(x)$ and $gf(x)$ if

$$f(x) = 1 + \frac{1}{x}, \qquad g(x) = x^2$$

The function $h(x) = gf(x) - fg(x)$.

Determine $h(1)$ and $h(-1)$.

5. $$f : x \to x + 3 \qquad\qquad g : x \to x - 3$$

What is

(a) $fg(x)$ (b) $gf(x)$?

6. If

$$f(x) = ax + b, \qquad g(x) = cx + d$$

where a, b, c, d are constants, show that $fg(x) = gf(x)$ only when $ad + b = cb + d$.

G4.3 Inverse Functions

You have probably already met the formula for changing temperatures in degrees Celsius, °C, to degrees Fahrenheit, °F. It is

$$F = \frac{9}{5}C + 32 \qquad (1)$$

You can regard this as a 1 : 1 mapping and it can be transformed to give C in terms of F, as follows:

$$F - 32 = \frac{9}{5}C \qquad \text{(taking } -32 \text{ from each side)}$$

$$\frac{5}{9}(F - 32) = C \qquad \text{(multiplying by } \frac{5}{9}\text{)}$$

That is, $$C = \frac{5}{9}(F - 32) \qquad (2)$$

If we write equations (1) and (2) in our functions notation, writing C for x in (1) and F for x in (2),

we have

$$f : x \to \frac{9}{5}x + 32$$

$$g : x \to \frac{5}{9}(x - 32)$$

Worked Example 1

Show that $fg(x) = gf(x) = x$ for the functions above.

Solution

$$\begin{aligned} fg(x) = f(g(x)) &= \frac{9}{5}g(x) + 32 \\ &= \frac{9}{5}\left\{\frac{5}{9}(x - 32)\right\} + 32 \\ &= x - 32 + 32 \\ &= x \end{aligned}$$

and

$$\begin{aligned} gf(x) = g(f(x)) &= \frac{5}{9}(f(x) - 32) \\ &= \frac{5}{9}\left(\frac{9}{5}x + 32 - 32\right) \\ &= \frac{5}{9} \times \frac{9}{5}x \\ &= x \end{aligned}$$

Note

If functions f and g are such that

$$fg(x) = x = gf(x)$$

we say that g is the ***inverse*** of f and denote this by

$$f^{-1}(x) = g(x) \quad \text{or} \quad f^{-1} = g$$

Similarly, f is the ***inverse*** of g, so

$$g^{-1}(x) = f(x) \quad \text{or} \quad g^{-1} = f$$

We can see how to find inverse functions in the next Worked Example.

Worked Example 2

If $f(x) = \dfrac{1}{1-x} + 2 \qquad (x \neq 1)$, find its inverse function and state its domain.

Solution

We write $y = \dfrac{1}{1-x} + 2$ and, as with the temperature conversion above, use algebraic manipulation to write x as a function of y.

Starting with $y = \dfrac{1}{1-x} + 2,$ take -2 from each side to give

$$y - 2 = \frac{1}{1-x} + 2 - 2$$

$$= \frac{1}{1-x}$$

Multiply both sides by $(1-x)$ to give

$$(1-x)(y-2) = (1-x) \times \frac{1}{(1-x)}$$

$$(1-x)(y-2) = 1$$

Now divide both sides by $(y-2)$ to give

$$\frac{(1-x)(y-2)}{(y-2)} = \frac{1}{(y-2)}$$

$$1 - x = \frac{1}{y-2}$$

or, multiplying throughout by -1,

$$x - 1 = \frac{1}{2-y}$$

and adding 1 to both sides, gives

$$x - 1 + 1 = \frac{1}{2-y} + 1$$

or

$$x = 1 + \frac{1}{2-y}$$

This is the inverse function, which we could write as

$$f^{-1}(y) : y \rightarrow 1 + \frac{1}{2-y}$$

Interchanging y and x (it is just a variable) gives

$$f^{-1}(x) = x \to 1 + \frac{1}{2 - x}$$

or

$$f^{-1}(x) = 1 + \frac{1}{2 - x}$$

The domain of $f^{-1}(x)$ is $x \neq 2$ (as the function is not defined at $x = 2$).

Note

1. We can check values of

$$f(x) = \frac{1}{1 - x} + 2 \text{ and } f^{-1}(x) = 1 + \frac{1}{2 - x}$$

For example,

$$f(2) = \frac{1}{1 - 2} + 2 = \frac{1}{-1} + 2 = -1 + 2 = 1$$

and $$f^{-1}(1) = 1 + \frac{1}{2 - 1} = 1 + \frac{1}{1} = 1 + 1 = 2$$

Similarly,

$$f(4) = \frac{1}{1 - 4} + 2 = -\frac{1}{3} + 2 = \frac{5}{3}$$

whilst

$$f^{-1}\left(\frac{5}{3}\right) = 1 + \frac{1}{\left(2 - \frac{5}{3}\right)} = 1 + \frac{1}{\left(\frac{1}{3}\right)} = 1 + 3 = 4$$

So we see that, for these values,

$$f^{-1} f(x) = x$$

2. In general,

$$f^{-1} f(x) = f^{-1}(f(x)) = f^{-1}\left(\frac{1}{1 - x} + 2\right), \text{ using the } f \text{ formula}$$

$$= f^{-1}\left(\frac{1 + 2(1 - x)}{(1 - x)}\right)$$

$$= f^{-1}\left(\frac{1 + 2 - 2x}{1 - x}\right)$$

$$= f^{-1}\left(\frac{3-2x}{1-x}\right), \text{ and using the } f^{-1} \text{ formula}$$

$$= 1 + \frac{1}{2 - \left(\frac{3-2x}{1-x}\right)}$$

$$= 1 + \frac{1}{\frac{2(1-x)-(3-2x)}{1-x}}$$

$$= 1 + \frac{(1-x)}{(2-2x-3+2x)}$$

$$= 1 + \frac{(1-x)}{-1}$$

$$= 1 - 1 + x$$

$$= x$$

So this proves that f and f^{-1} are inverse functions.

Worked Example 3

Two functions, g and h, are defined as

$g : x \to \dfrac{2x+3}{x-4}$ and

$h : x \to \dfrac{1}{x}$.

Calculate

(a) the value of $g(7)$

(b) the value of x for which $g(x) = 6$.

Write expressions for

(c) $hg(x)$

(d) $g^{-1}(x)$

Solution

(a) $g(7) = \dfrac{2 \times 7 + 3}{7 - 4} = \dfrac{17}{3}$

(b) $6 = \dfrac{2x + 3}{x - 4} \Rightarrow 6(x - 4) = 2x + 3$

$$6x - 24 = 2x + 3$$

$$4x = 27$$

$$x = \frac{27}{4}$$

<u>Check</u> $g\left(\dfrac{27}{4}\right) = \dfrac{2 \times \dfrac{27}{4} + 3}{\dfrac{27}{4} - 4}$

$$= \frac{54 + 12}{27 - 16}$$

$$= \frac{66}{11}$$

$$= 6$$

(c) $hg(x) = h(g(x))$

$$= \frac{1}{g(x)}$$

$$= \frac{1}{\left(\dfrac{2x + 3}{x - 4}\right)}$$

$$= \frac{x - 4}{2x + 3}$$

(d) To find $g^{-1}(x)$, we write

$$y = \frac{2x + 3}{x - 4}$$

and find x as a function of y; that is

$$y(x - 4) = 2x + 3$$

$$yx - 4y = 2x + 3$$

$$yx - 2x = 3 + 4y$$

$$x(y - 2) = 3 + 4y$$

$$x = \frac{3+4y}{y-2} \Rightarrow g^{-1}(y) = \frac{3+4y}{y-2}$$

So $\quad g^{-1}(x) = \dfrac{3+4x}{x-2} \quad$ (replacing y by x).

Exercises

1. Find the inverse function for each of these functions. In each case, state the domain of the inverse.

   (a) $f(x) = x + 2$

   (b) $f(x) = 4x - 2$

   (c) $f(x) = x$

   (d) $f(x) = \dfrac{3}{x} \qquad (x \neq 0)$

   (e) $f(x) = \dfrac{1}{x+2} \qquad (x \neq -2)$

2. If $f: x \rightarrow 4x - 3$, find $f^{-1}(x)$ and check that

   $$f\,f^{-1}(x) = f^{-1}f(x) = x$$

3. The functions f and g are defined by

   $f(x) = \dfrac{1}{2}x + 5 \qquad g(x) = x^2$

   Evaluate

   (a) $g(3) + g(-3)$

   (b) $f^{-1}(6)$

   (c) $f\,g(2)$

4. The functions f and g are defined as:

   $f(x) = \dfrac{2x-1}{x+3} \qquad g(x) = 4x - 5$

   Determine:

   (a) $g(3)$

   (b) $f\,g(2)$

   (c) $f^{-1}(x)$

5. The function f is defined by

$$f : x \rightarrow \frac{1}{x} - 4$$

and has domain all x $(x \neq 0)$.

(a) Find

(i) $f\left(\frac{1}{4}\right)$

(ii) $f(1)$

(iii) $f^{-1}(0)$

(b) Determine the inverse function $f^{-1}(x)$ and use it to calculate

(i) $f^{-1}(0)$

(ii) $f^{-1}(-3)$

(c) Show that $f\,f^{-1}(x) = x$.

G4.4 Transformations of Graphs of Functions

There are 4 basic transformations of the graph of a function that are considered in this section. These are explored in the following worked examples and then summarised.

Worked Example 1

The function f is defined as $f(x) = x^2$. Plot graphs of each of the following and describe how they are related to the graph of $y = f(x)$:

(a) $y = f(x) + 2$

(b) $y = f(x + 1)$

(c) $y = f(2x)$

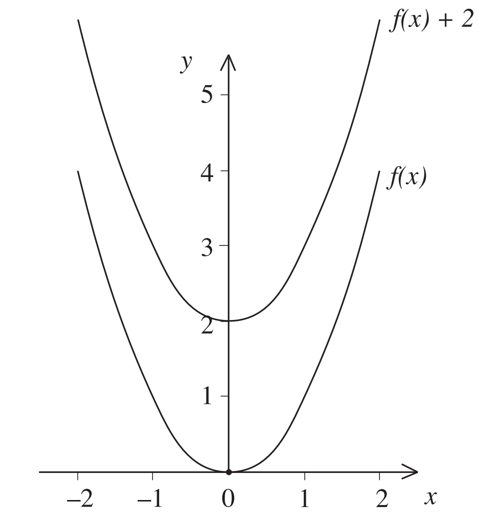
(d) $y = 2f(x)$

Solution

The table below gives the values needed to plot these graphs.

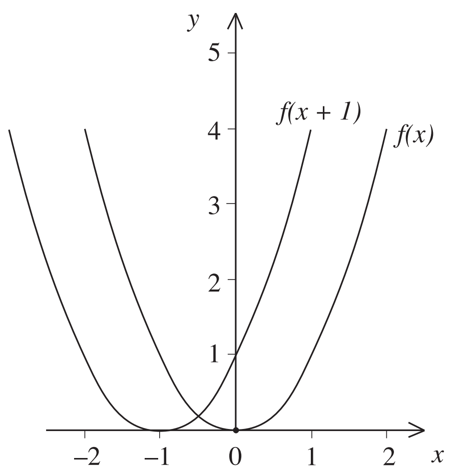
x	–2	–1	0	1	2
$f(x)$	4	1	0	1	4
$f(x) + 2$	6	3	2	3	6
$f(x + 1)$	1	0	1	4	9
$f(2x)$	16	4	0	4	16
$2f(x)$	8	2	0	2	8

The graphs below show how each graph relates to $f(x)$.



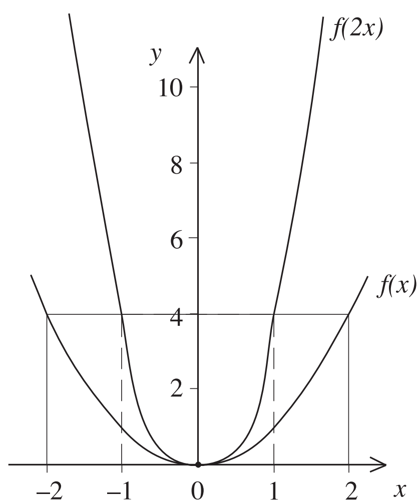
The graph of $y = f(x)$ is mapped onto the graph of $y = f(x) + 2$ by translating it up 2 units.

In general $f(x) + a$ moves a curve up a units and $f(x) - a$ moves it down a units, where a is a positive number.



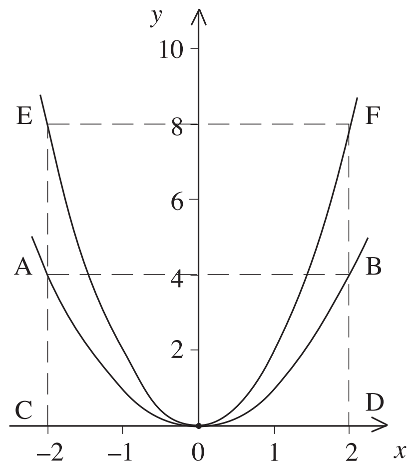
The graph of $y = f(x)$ is mapped onto $f(x + 1)$ by a translation of 1 unit to the left.

In general $f(x + a)$ translates a curve a units to the left and $f(x - a)$ translates a curve a units to the right, where a is a positive number.



The curve for $f(2x)$ is much steeper than for $f(x)$. This is because the curve has been compressed by a factor of 2 in the x-direction. Compare the rectangles ABCD and EFGH.

In general the curve of $y = f(kx)$ will be compressed by a factor of k in the x-direction where $k > 1$.



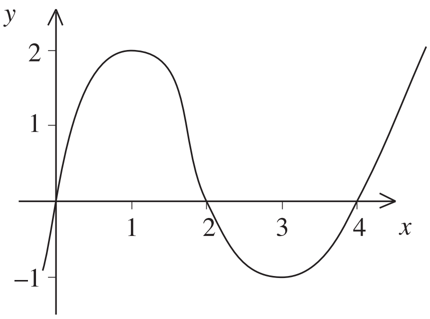
Here the curve $y = f(x)$ has been stretched by a factor of 2 in the vertical or y-direction to obtain the curve $y = 2f(x)$. Compare the rectangles ABCD and CDFE.

> In general the curve of $y = kf(x)$ stretches the graph of $y = f(x)$ by a factor of k in the y-direction, where $k > 1$.

Note that if k is negative and $k < -1$ the curve will be stretched and reflected in the x-axis while if $-1 < k < 1$, it is compressed.

Worked Example 2

The graph below shows $y = g(x)$.



On separate diagrams show:

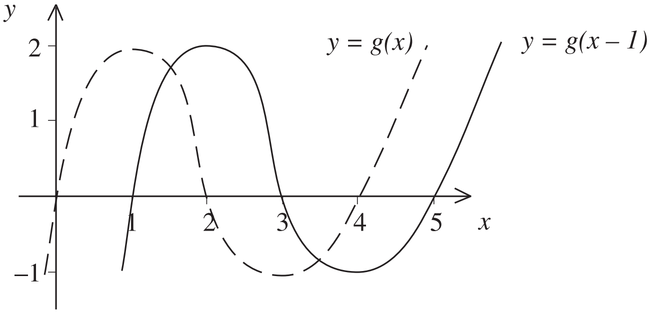
(a) $y = g(x)$ and $y = g(x - 1)$

(b) $y = g(x)$ and $y = g(2x)$

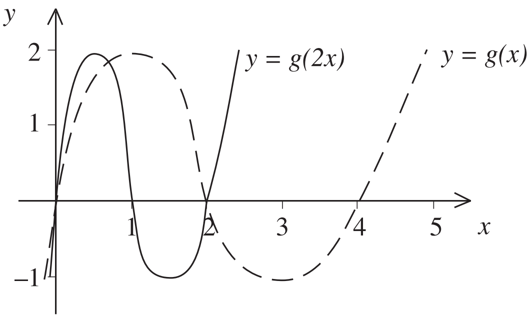
(c) $y = g(x)$ and $y = 3g(x)$

Solution

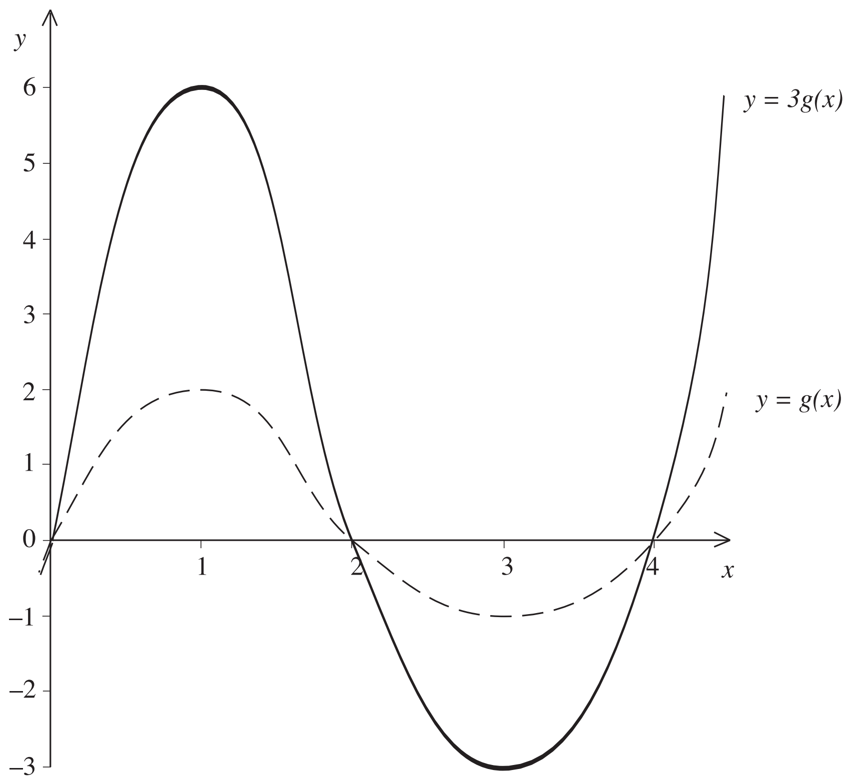
(a) To obtain $y = g(x - 1)$ translate $y = g(x)$ 1 unit to the right.



(b) To obtain $y = g(2x)$ compress $y = g(x)$ by a factor of 2 horizontally.

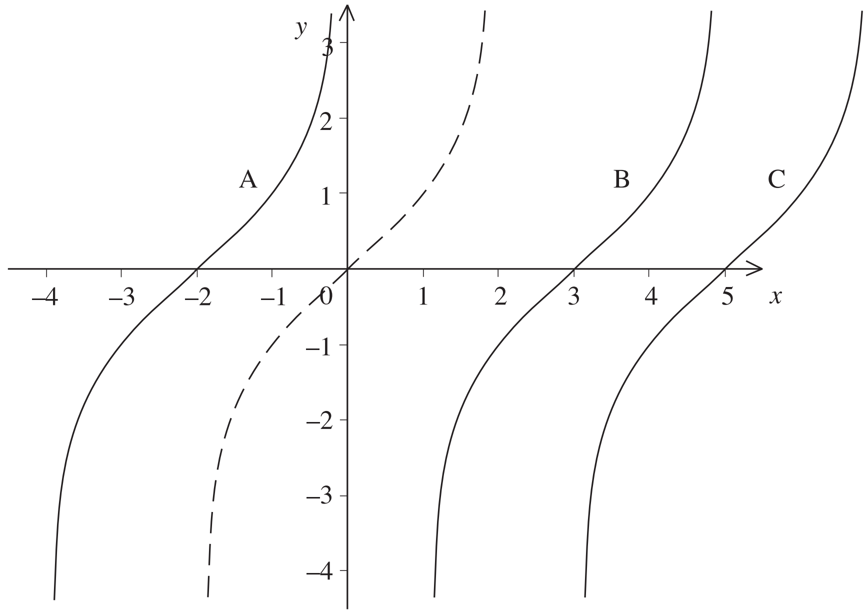


(c) To obtain the graph of $y = 3g(x)$ stretch the graph by a factor of 3 vertically.

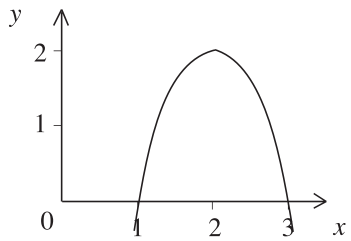


Exercises

1. The graph below shows $y = f(x)$ by a dashed curve.
   Write down the equation of each other curve.



2. The graph below shows $y = h(x)$



   On separate diagrams show:

   (a) $y = h(x)$, $y = h(x) + 1$ and $y = h(x) - 2$

   (b) $y = h(x)$ and $y = 2h(x)$

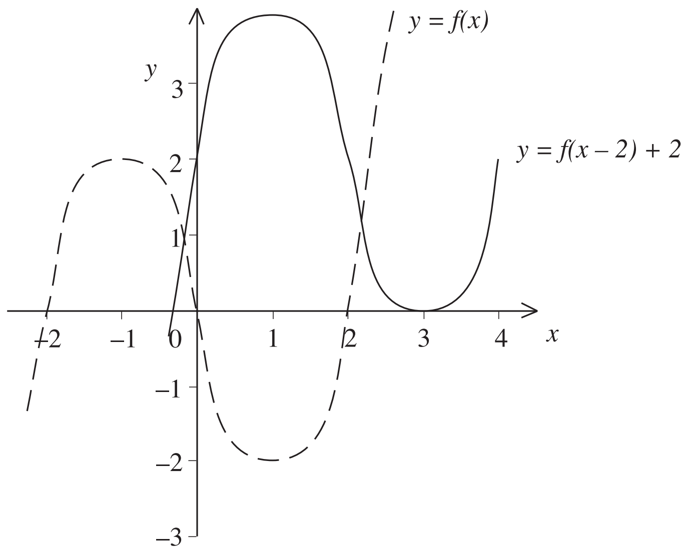
   (c) $y = h(x)$ and $y = 3h(x)$

   (d) $y = h(x)$ and $y = h(2x)$

3. On the same set of axes sketch the curves;

$$y = x^2,\ y = (x+3)^2,\ y = (x-4)^2 \text{ and } y = (x+1)^2.$$

4. The graph below shows $y = f(x)$ and $y = f(x-2)+2$.



(a) Describe how to obtain the curve for $y = f(x-2)+2$ from the curve for $y = f(x)$.

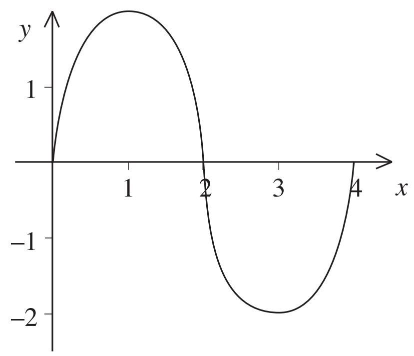
(b) On a set of axes sketch $y = f(x)$, $y = f(x-2)-1$ and $y = f(x-1)+1$.

5. On the same set of axes sketch

$$y = x^2,\ y = (x-2)^2 + 1,\ y = (x-3)^2 - 1 \text{ and } y = (x+3)^2 - 2.$$

6. Draw the graphs of $y = x^2$, $y = 3x^2$, $y = -x^2$ and $y = -3x^2$. Describe how they compare.

7. The graph below shows $y = g(x)$.



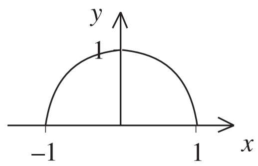
On separate sets of axes plot:

(a) $y = g(x)$ and $y = -g(x)$

(b) $y = g(x)$ and $y = -2g(x)$

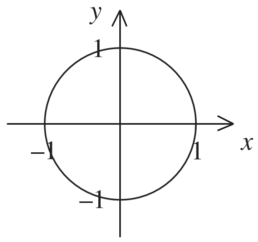
(c) $y = g(x)$ and $y = -\frac{1}{2}g(x)$

8. The function $f(x)$ is such that the graph of $y = f(x)$ produces a graph as shown below, in the shape of a semi-circle.

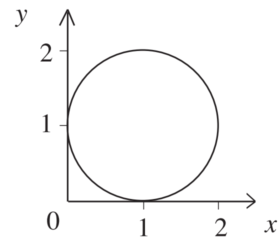


List the pairs of functions that should be plotted to produce the circles below.

(a)

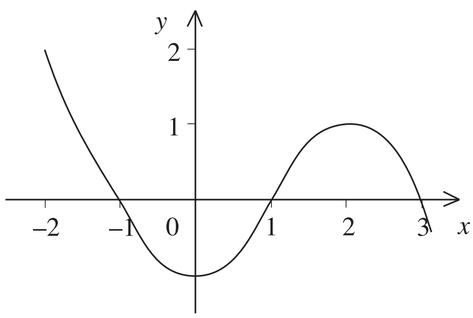


(b)



9. (a) Draw the graphs of $y = f(x)$ and $y = f(-x)$ if $f(x) = x^3$, and describe how the graphs are related.

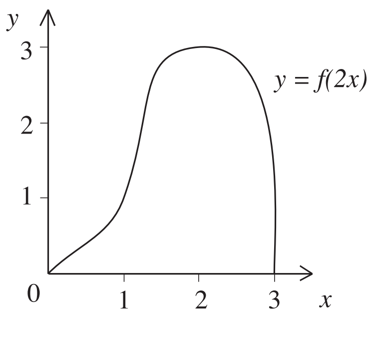
(b) The graph below shows $y = g(x)$.



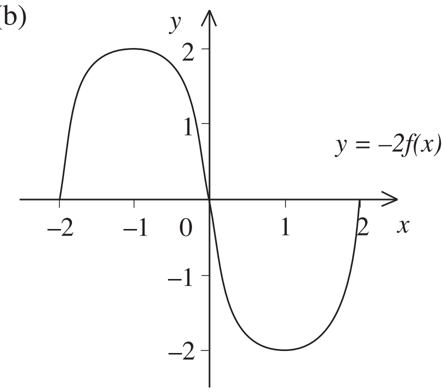
Sketch graphs of $y = g(-x)$, $y = g(-2x)$, $y = g\left(\frac{1}{2}x\right)$ and $y = g\left(-\frac{1}{2}x\right)$.

10. Use each graph below to sketch a graph of $y = f(x)$.

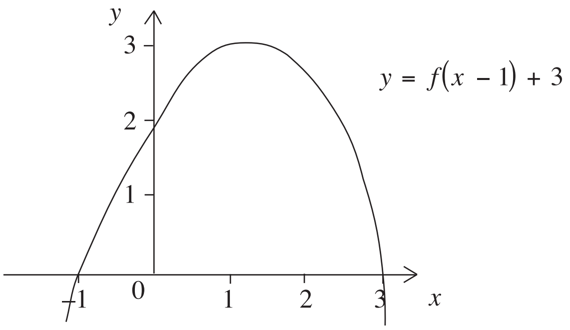
(a)



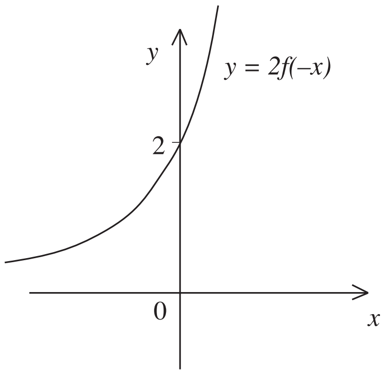
(b)



(c)

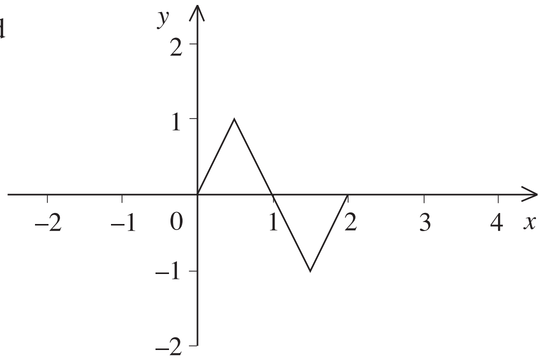


(d)

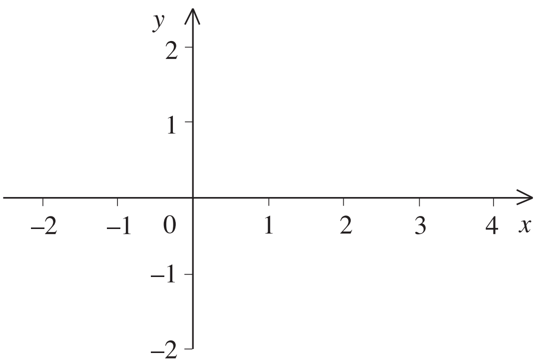


11. The function $y = f(x)$ is defined for $0 \le x \le 2$.

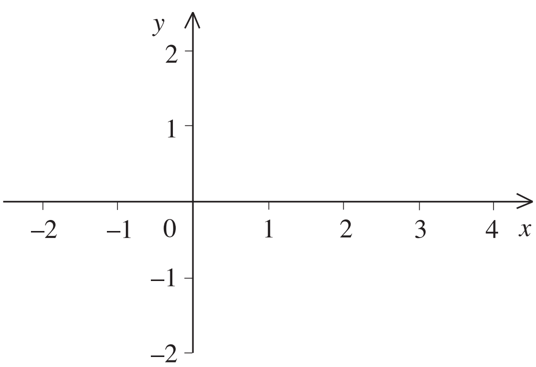
The function is sketched opposite.



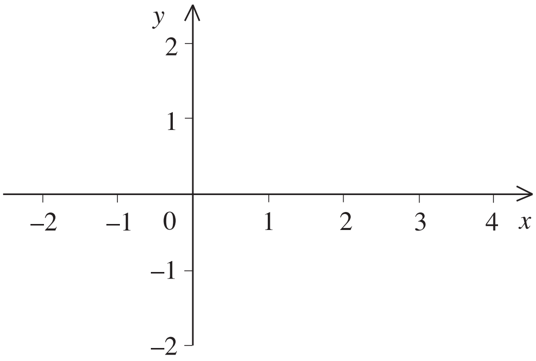
(a) Sketch $y = f(x) + 1$ on axes like the ones below.



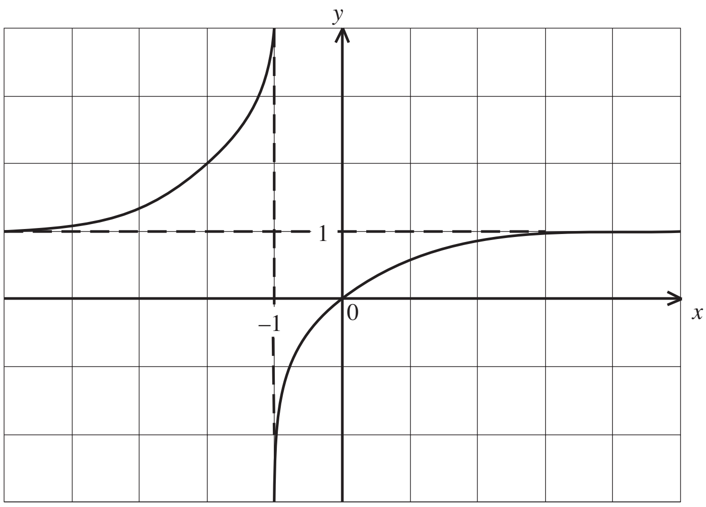
(b) Sketch $y = f(x - 1)$ on axes like the ones below.



(c) Sketch $y = f\left(\frac{x}{2}\right)$ on axes like the ones below.

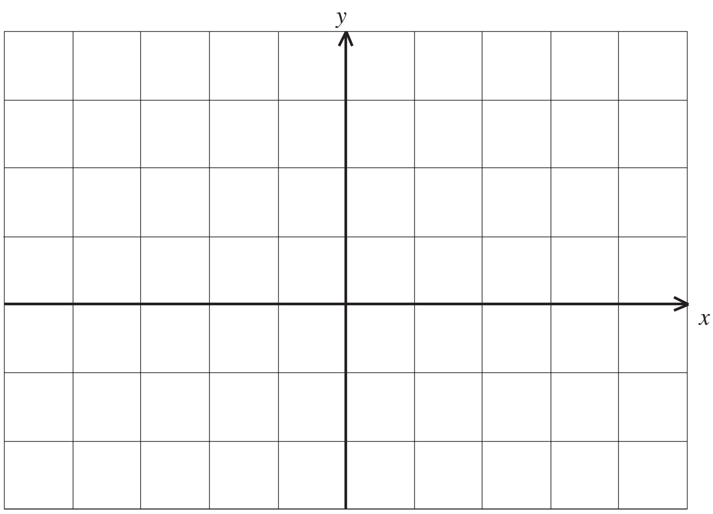


12. The graph of $y = f(x)$ where $f(x) = \dfrac{x}{x+1}$ is sketched below.



Hence, or otherwise, sketch on an axis like the one below

(a) $y = f(x - 1)$



(b) $y = f(2x)$