Investigation of Pre-Service Teachers' Pedagogical Content Knowledge Related to Division by Zero

Fatih Karakus
Sivas Cumhuriyet University
fkarakus@cumhuriyet.edu.tr

Although the topic of division by zero has been widely discussed in the literature, this subject is still confusing for students, pre-service teachers and teachers. Because of this, teachers at various grade levels may encounter difficulties in conveying the concept to their students. Therefore, in order to provide students with a strong conceptual understanding of division by zero, it is important to examine the ways that teachers and pre-service teachers structure their instructional explanations about division by zero. The aim of this study was to explore the instructional explanations given by pre-service teachers concerning division by zero, as well as the effects of teacher training programs on these explanations. The study consisted of a cross-sectional design and was carried out with 197 pre-service teachers of elementary mathematics. To determine the pre-service teachers' instructional explanations given at different grade levels, a written questionnaire was used. Analysis of the results revealed that although most of the pre-service teachers gave correct answers related to division by zero, few of them provided conceptual-based explanations. Rather, those who gave correct explanations mainly responded with rule-based statements.

Introduction

Zero is a complex and interesting concept in the field of mathematics that has been shown in prior studies (e.g., Ball, 1990; Crespo & Nicol, 2006; Wheeler & Feghali, 1983) to cause difficulties not only for students, but also for pre-service and in-service teachers of mathematics. The concept of dividing by zero, in particular, plays a crucial role in developing an understanding of certain mathematical notions, such as rational numbers and the relationship between multiplication and division (Quin, Lamberg, & Perrin, 2008). However, as with the overall concept of zero, division by zero is often unclear and confusing, and teachers at various grade levels may encounter difficulties in conveying this concept to their students. Therefore, in order to provide students with a strong conceptual understanding about division by zero, it is important to examine the ways that pre-service and in-service teachers structure the instructional explanations they provide. Additionally, identifying the effects of teacher training programs on pre-service teachers' instructional explanations may result in useful information about these programs' contributions to their pedagogical content knowledge. To address these concerns, this study provides data on pre-service teachers' pedagogical content knowledge related to division by zero, as well as the effects of their teacher training program on their content knowledge.

Pedagogical Content Knowledge

Teachers' knowledge is one of the most important factors relating to the quality of teaching. Therefore, over the last three decades, numerous investigations have been carried out to explore the nature and components of mathematics teachers' knowledge. According to the literature, the two major components of teacher knowledge are content knowledge and pedagogical content knowledge (Fennema & Franke, 1992; Shulman, 1986). Shulman (1986) describes content knowledge as the amount and organization of knowledge in the mind of a teacher. In this sense, a teacher must not only know the facts or concepts of a particular domain, but also have the ability to explain the structure of the concepts within that domain. He further divides a teacher's content knowledge into three categories: subject matter knowledge, pedagogical content knowledge and curricular knowledge. The subject matter knowledge is a critical component of what teachers need to know (Ball, 1990). There has been an interest in defining and analysing what subject matter knowledge is for teaching (Even, 1990). Even and Tirosh (1995) defined the subject matter knowledge as "knowing that" and "knowing why". Knowing that includes teacher knowledge about rules, procedures, algorithms and concepts that are related to a mathematical topic. Knowing why involves teacher knowledge about the underlying meaning of a mathematical procedure.

The quality of a teacher's or pre-service teacher's pedagogical content knowledge has been especially emphasized as relating directly to effective mathematics instruction (Baki, 2013; Bütün, 2012; Cankoy, 2010; Sanders & Morris, 2000; Toluk Uçar, 2011). According to Shulman (1986), pedagogical content knowledge is the knowledge of how to teach by transforming information into a form that is easy for students to understand. In other words, transformation is a process whereby pre-service teachers' subject matter knowledge is converted into a form appropriate for teaching, a form he calls pedagogical content knowledge (Kinach, 2002). This occurs when a teacher is able to interpret a topic and then apply appropriate teaching methods and to render the topic in a manner that allows students to internalize it (Baki, 2013).

Shulman (1986) proposed several key elements of pedagogical content knowledge, including knowledge of representations of subject matter on the one hand, and students' conceptions on the other. He emphasized the importance of using representations, analogies, illustrations, examples, explanations and demonstrations in order to make a subject comprehensible to students. Moreover, he suggested that the better a teacher's understanding of students' learning processes in a given domain, the more effectively he/she can teach in that domain. As such, a teacher should be aware of the most effective presentation methods, representations, analogies, examples and explanations (Shulman, 1986).

Among the most important indicators of both pre-service and in-service teachers' pedagogical content knowledge are the instructional explanations they can provide about mathematical concepts and rules. Leinhardt (1990, 2010) defines instructional explanation as an activity in which teachers explain subject matter content to students. The activity contains not only verbal explanations, but also teachers' arrangements, demonstrations or illustrations. Thus, instructional explanations are more than simple descriptions of the content to students rather they help students to construct a meaningful understanding for a concept.

Teachers provide instructional explanations for various purposes such as introducing a new content to their students, to answer students' questions, to help students move from what they already know to what they have to learn, take students' difficulties and misconceptions into account or clarify what is to be learned (Charalambos, Hill & Ball, 2011). Constructing meaningful understanding, good instructional explanations are important. Good instructional explanations provide precise, meaningful, correct information and they clarify the ideas to be learned. Inconsistent, incomplete or unclear instructional explanations can negatively affect students' learning (Charalambos, et. al. 2011). Some studies (e.g. Ball, 1990; Charalambos, et. al. 2011; Kinach, 2002; Leinhardt, 2001; Quin et. al, 2008) stated that especially pre-service and novice teachers provide incomplete, error-prone and unrelated explanations. Although Leinhardt (1990) did not directly compare expert and novice teachers, she stated that expert teachers provide better instructional explanations than novice teachers. Many of the studies noted above focused on the mistakes or inaccuracy of the explanation provided by pre-service or novice teachers. However, little research (e.g. Charalambos, et. al, 2011; Kinach, 2002) has been conducted to describe the development and transformation process of the pre-service teachers' instructional explanation. For example Kinach (2002) reports on the difficulties she encountered in a mathematics method course to help pre-service teachers shift from providing "telling-math" explanations (i.e. explanations that define the procedure) to "teaching-for-understanding" explanations (i.e. explanations that highlight the meaning of the procedure). Kinach (2002) reports that the transformation process is difficult because pre-service teachers enter teacher education programs with a constant belief of what mathematics is and what its teaching entails. For this reason she suggests that pre-service teachers explain not only the hows, but also the whys.

In studies on instructional explanations of pre-service teachers, the inadequacy of instructional explanations and lack of subject matter knowledge are often emphasized. On the other hand, little research has been conducted to determine the factors affecting the transformation and development process of the instructional explanations of pre-service teachers. It is increasingly reported in research outputs that the procedural/rule-based understanding of mathematics that pre-service teachers indicate in mathematics content courses and mathematics method courses is not adequate to teach the concept at the elementary or secondary school mathematics levels (Ball, 1990; Ma, 1999; Tirosh, 2000; Toluk Uçar, 2009, 2011). While pre-service teachers generally know the rules and methods related to a topic and how to apply them, they typically do not have enough knowledge about giving proper explanations for specific mathematical situations. Thus, according to the research, pre-service teachers' instructional explanations tend to be primarily rule-bound, rather than conceptual-based (Baki, 2013; Kinach, 2002; Toluk Uçar, 2011). It is important for a pre-service teacher to know what makes it easier and what makes it more difficult to teach a mathematical concept. That is, pre-service teacher's pedagogical content knowledge includes making a good instructional explanation for mathematical rules and concepts. For this reason, it can be said that instructional explanations are at the centre of pedagogical content knowledge (Charalambos, et al, 2011). For example, in the division of fractions, given only the "invert and multiply" algorithm without making any explanation is incomprehensible for students. Why does the "invert and multiply" algorithm work? How is it that we can convert a division expression into a multiplication expression by multiplying by the reciprocal of the divisor? Making appropriate explanations for these questions will help students to make more meaningful understanding in the division of fractions. Such instructional explanations are also closely related to the understanding of pre-service teachers for mathematical concepts (Kinach, 2002). Skemp (1976) classifies the understanding as instrumental and relational. Instrumental understanding includes knowing a rule with little need for explanations and being able to use it. Relational understanding is about knowing what to do and why. Teaching of a mathematical topic is also different for teachers who adopt these two views of mathematical understanding. For example, teaching division of fraction from the perspective of the instrumentalist would mean teaching the "invert-and-multiply" algorithm without attention to its underlying concepts and principles. In contrast, teachers adopting a relational view would focus on meaning and justifying why the invert-and-multiply algorithm works. In the instrumentalist view, remembering and memorizing rules, facts and procedures are essential for student achievement. The instrumentalist teacher has focused on the content in which emphasis is on students' performance and learning is seen passive acceptance of knowledge. However, the teacher who adopts a relational view of mathematics, has focused on student achievement that is much broader than remembering. This kind of teacher gives his/her students opportunities to derive rules, make conjectures and determine patterns.

The majority of the existing studies that have focused on instructional explanations have examined pre-service teachers' pedagogical content knowledge about integers (Baki, 2013; Kinach, 2002; Thanhieser, 2009) as well as both pre-service teachers' (Ball, 1990; Toluk Uçar, 2011) and in-service teachers' (Cankoy, 2010; Quinn, et al., 2008) pedagogical content knowledge related to fractions. However, few studies have explored pre-service teachers' pedagogical content knowledge related to division by zero.

Studies Related to Division by Zero

Although the topic of division by zero has been widely discussed in the literature (e.g., Ball, 1990; Cankoy, 2010; Crespo & Nicol, 2006; Çelik & Akşan, 2013; Tsamir & Sheffer, 2000; Tsamir & Tirosh, 2002; Quinn, et al., 2008; Watson, 1991), the subject is still unclear and confusing for students, pre-service teachers and in-service teachers alike. It is therefore reasonable that the understanding and misconceptions of students (Reys & Grouws, 1975; Tsamir & Sheffer, 2000; Tsamir, Sheffer & Tirosh, 2000; Tsamir & Tirosh, 2002) and pre-service teachers (e.g. Ball, 1990; Crespo & Nicol, 2006; Çelik & Akşan, 2013;Wheeler & Feghali, 1983) related to this topic have been examined by numerous scholars.

For example, Even and Tirosh (1995) asked pre-service teachers to explain why 4 divided by 0 is undefined. While most of their respondents answered that the result of this operation is undefined, they could not supply an appropriate explanation as to why this is true. Likewise, Ball (1990) investigated elementary and secondary pre-service teachers' understanding of division and found that their knowledge on the topic was based on memorization, more than on conceptual understanding. While the secondary pre-service teachers in Ball's study were more successful than the elementary pre-service teachers, most of the secondary pre-service teachers who gave correct answers related to division by zero justified their answers by stating that "it is undefined because this is a rule, and you cannot divide by zero." Wheeler and Feghali (1983) reached a similar conclusion, reporting that pre-service elementary school teachers did not have an adequate understanding of the number zero or the division of items with zero as a dividend or a divisor. Crespo and Nicol (2006) also examined the effects of two teacher education tasks on pre-service teachers' understanding of division by zero. According to their findings, the participants made progress during these instructional experiences; as such, their initial ideas were extended, and their later explanations became more conceptually based than rule-bound. Moreover, Çelik and Akşan's (2013) study of the perceptions of pre-service teachers on the concepts of infinity, indeterminate and undefined, yielded similar results to Ball (1990), who found that pre-service teachers who gave correct answers to problems involving division by zero typically justified their answers by giving a rule.

Teachers' perceptions and topic-specific pedagogical content knowledge have also been discussed in the literature (Cankoy, 2010; Quinn et al, 2008). For instance, Quinn et al (2008) investigated the perceptions of fourth through eighth grade teachers concerning division by zero and reported that, although a few teachers had a conceptual understanding, most did not. In addition, Cankoy (2010) investigated high-school mathematics teachers' topic-specific pedagogical content knowledge about "$a^0 = 1, 0! = 1$ and $a \div 0$, where $a \neq 0$." He found that experienced teachers proposed conceptual explanations more frequently than novice teachers, but all the participants' explanations were procedural and fostered memorization.

In the studies described here, the researchers focused mainly on pre-service and in-service teachers' knowledge and understanding of division by zero; furthermore, a small number of studies on teachers' perceptions (e.g., Quinn et al., 2008) and topic-specific pedagogical content knowledge (Cankoy, 2010) were explored. There are not many studies determining instructional explanations of pre-service teachers about division by zero, and also focusing on the transformation process in these instructional explanations in a teacher education program. Kinach (2002) mentions the importance of transforming instructional explanations from "telling math" explanations to "teaching-for-understanding" explanations and gives clues on what will be effective in this transformation process.

Explanations Demonstrating the Impossibility of Division by Zero

Past experiences with mathematical operations can cause the perception that every mathematical operation must result in a numerical answer (Tsamir & Sheffer, 2000). However, division by zero does not result in a number, as it is the only real number that does not have a multiplicative inverse. That is, every nonzero real number c has a multiplicative inverse $\frac{1}{c}$, and when we multiply c by $\frac{1}{c}$, the result is 1. However, if we let zero have a multiplicative inverse of 1 divided by zero, we can write the product of the multiplication of zero and this inverse as 1. That is,

$$\frac{1}{0}.0 = 1.$$

If we accept this definition as correct, we can state that zero divided by zero is equal to one, and 0+0 divided by zero is equal to 2. These results are meaningless, and thus, zero has no multiplicative inverse. Because zero has no multiplicative inverse, there is no number that, when multiplied by zero, results in 1. For this reason, the division problem $\frac{c}{0}$ is meaningless for any number c.

Additional explanations about why zero is undefined are also available. For example, Henry (1969) proposed three types of approaches based on the meaning of division to show the impossibility of division by zero: the standard partitive approach of division, the repeated subtraction concept of division, and the inverse of multiplication. Similarly, Watson (1991) suggested four types of models for demonstrating the impossibility of division by zero: the measurement division, the partitive division, the relationships to fractions, and the formal definition. Crespo and Nicol (2006) further defined three types of explanations regarding division by zero: explaining with examples, explaining with deductive logic, and explaining with analogy.

Quinn et al. (2008) similarly categorized teachers' correct explanations into three groups: dividing by successively smaller fractions, relating multiplication and division, and the partitioning and grouping approach. A review of these studies reveals that two types of explanations are commonly used to show the impossibility of division by zero: concrete (real-world) explanations and formal mathematical explanations (Tsamir & Sheffer, 2000). A concrete explanation can be defined as using real-word situations to give meaning to the mathematical statement; a formal explanation involves using only mathematical definitions and theorems (Tsamir & Sheffer, 2000). For students at the elementary level, it is expected that concrete explanations should be used; while formal explanations should be offered at the secondary and upper levels. The following concrete explanation, which refers to division as sharing or dividing evenly, is taken from Watson (1991, p. 375):

> Given six apples to be divided evenly among zero children (i.e., no children arrive to collect them), how many apples will each child get? It is clear that there are no sets into which [the] partition can take place. Hence the operation is impossible to perform. The impossibility of distributing the apples allows us to be justified in declaring the operation undefined.

In contrast, the following formal explanation relies on the definition of division as the inverse of multiplication, as taken from Tsamir and Sheffer (2000, p. 94).

> If 4÷0=c then c×0=4; but c×0=0 for every number c, therefore 4÷0 cannot be an ordinary number.

Division by Zero in the Turkish Educational Setting

In Turkey, students begin learning about division in the second grade. However, neither the mathematics teaching program nor the authorized textbooks offer much explanation about the fact that a number cannot be divided by zero. Students first begin to learn about rational numbers in the 7th grade. At this level, textbooks such as Bağcı's (2015) define rational numbers as those that "can be written as a/b where a and b (b≠0) are integers" (p. 37). According to this definition, numbers such as $-\frac{5}{2}, \frac{0}{5}, \frac{3}{8}$ are considered as rational numbers, since their denominators and dividends are integers. On the other hand, although their denominators and dividends are also integers, numbers such as $\frac{4}{0}, \frac{7}{0}$ are not considered as rational numbers, because their denominators are zero (Bağcı, 2015). This textbook, as well as others, presents $\frac{0}{0}$ as undetermined, but there is no explanation as to why this is the case. Then, in the 9th through 12th grades, students frequently encounter such undefined and undetermined cases while solving equations, as well as in defining functions and finding limits of functions. As with the 7th grade, rather than offering conceptual explanations related to division by zero, the focus on solving this undefined case is on rules and procedural operations.

With respect to preparing teachers to work with students on mathematical concepts, including division by zero, pre-service mathematics teachers in Turkey are required to take a course called "General Mathematics" in the first year of their teacher education program. During this course, the topics of numbers, equations, functions and graphs of functions are discussed in detail to address any gaps in their understanding of basic mathematics. In particular, undefined cases in numbers and in definitions of functions are emphasized. Furthermore, in subsequent years of their training, these uncertainties and operations for solving them are encountered again within the scope of a calculus course that deals with subjects such as limit, derivative and integral. Additionally, from the beginning of the third year of their program, pre-service teachers take courses in "special teaching methods" and "materials development" courses. In these courses, pre-service teachers prepare for the teaching of elementary school mathematics subjects, as well as examining the grade 5-8 mathematics curricula and textbooks.

Purpose of the Study and Research Questions

The aim of this study was to explore the instructional explanations given by pre-service teachers concerning division by zero, as well as the effects of teacher training programs on these explanations. Therefore, the research problems of this study were as follows:

- What are the instructional explanations proposed by pre-service teachers of elementary mathematics for the teaching of division by zero?

- Does the year of study in their degree program have any effect on the explanations of pre-service teachers of elementary mathematics concerning division by zero?

Methodology

A cross-sectional study design was used to investigate different groups of subjects at the same time; this design was chosen because it allows for simultaneously collecting data from larger groups at minimal cost (McMillan & Schumacher, 2006).

Participants

The research sample for this study was selected via convenience sampling; volunteers were requested from the body of students who were enrolled in the elementary mathematics teaching program at the researcher's university, as these students were easily accessible for administration of the data collection instrument. The resulting research group consisted of 197 pre-service elementary mathematics teachers. Of these, thirty-five were freshman (first year students), fifty-nine were sophomores (second year students), sixty-one were juniors (third year students) and forty-two were seniors (fourth year students). The aim in selecting students from four different levels was to determine the effects of the teacher education program on the students' pedagogical content knowledge regarding division by zero. At the time of the study, the freshman students had completed the general mathematics course previously described; the sophomores had completed calculus I, calculus II and mathematical problem solving courses; the juniors had completed courses in calculus III, and teaching mathematics I; and the seniors had completed courses in common misconceptions in mathematics education, instructional techniques and materials development, and school experiences in teaching mathematics.

Pre-service teachers encounter division by zero in topics such as finding the domain and codomain of a function, drawing its graph, limit, and derivative in their study of general mathematics, calculus I and calculus II courses. Moreover, they examine the elementary mathematics curriculum for 5-8th grades and prepare mathematics activities in their courses in teaching mathematics, instructional techniques and materials development. In addition, they are asked to apply their own activities concerning division, fractions and rational numbers in a real classroom environment in their mathematics teaching practicum.

As division by zero is covered in the subject of rational numbers in the 7th grade in Turkish schools, the practicum course in teaching mathematics is particularly useful in providing pre-service teachers with an authentic experience in teaching this subject. In this respect, because of their more extensive exposure, the senior students in this study were expected to give more rigorous explanations demonstrating the impossibility of division by zero.

Instrument and Data Collection Procedure

To determine the pre-service teachers' pedagogical content knowledge at each grade level, a written questionnaire was developed. The first questionnaire item read, "Suppose that you have an elementary school student who asks you what is 6 divided by 0. How would you respond?" The second questionnaire item read, "Suppose that you have an elementary school student who asks you what is 0 divided by 0. How would you respond?" The aim of these questions was to reveal the pre-service teachers' pedagogical content knowledge concerning division by zero. The questions were adapted from the literature relating to teachers' knowledge and perceptions about division by zero (Cankoy, 2010; Ma, 1999; Quinn et. al, 2008). The questionnaire was administered to the participants in the fall term of the 2015-2016 academic year. The pre-service teachers who completed the questionnaire were not allowed to compare their answers, and they were given sufficient time to complete their responses. Before administering the final form of the questionnaire, two mathematics educators and a mathematician checked the face validity of the test questions and agreed that they were valid and appropriate for measuring pre-service teachers' pedagogical content knowledge. After the instrument was administered to 197 pre-service teachers, the researcher and two mathematics educators chose the explanations of 25 of the participants at random and used inductive analysis (McMillan & Schumacher, 2006) to determine the categories and themes in the written responses.

Afterward, the researcher and the mathematics educators separately examined the full set of responses to the first item of the questionnaire and used previously determined categories and themes to code the explanations of the pre-service teachers in a deductive way. In this process, interrater reliability, which is defined by Creswell (2012) as consistency between scores that are assigned by two or more independent raters, was established as follows: first, if at least two of the mathematics educators and the researcher identified a theme in common, that theme was taken into consideration and coded; and second, if no common theme was found, the researcher and the mathematics educators examined the data again and arrived at a consensus on the conflicting themes.

One of the most popular methods for computing interrater reliability is percent agreement, which measures the percentage of agreement among raters. In this study, the percent agreement between the researcher and the mathematics educators was 91.8%, which indicates a good level of consistency (Miles & Huberman, 1994). The general themes and examples of the pre-service teachers' explanations can be seen in Table 1 and Table 2.

Table 1.
Pre-service teachers' themes and sample explanations related to 6÷0 and 0÷0.

Category	Sub-category	Theme	Sample of pre-service teacher explanation related to $6 \div 0$	Sample of pre-service teacher explanation related to $0 \div 0$
Incorrect explanations		No answer	-	-
		Explain as an infinite	Since there is an infinite number of zero in 6, the result is infinite. As a rule, $number \div 0$ equals infinite. You will learn why the result is infinite in high school.	The result is infinite. You will learn the reason in the future. Having zero in the denominator makes the result infinite.
		Use definition of rational number	$6 \div x, x \neq 0$ is a rational number. However, dividing 6 by 0 does not conform to the definition of rational numbers, so it is accepted as undefined.	According to the definition of a rational number, the denominator must be different from zero. For that reason, $0 \div 0$ is not a rational number. It is undetermined.
		Use the meaning of zero	Zero means nothing, and a number	

			cannot be divided by nothing.	Zero has [the] meaning of nothing. So, we cannot divide something that does not exist into something that does not exist.
		Write numerical answer	If we are on the set of real numbers, the result of a division operation is a real number. So, $6 \div 0 = 0$.	Since dividing zero by a number is zero, dividing zero by zero should be zero.

Table 2.
Pre-service teachers' themes and sample explanations related to 6÷0 and 0÷0.

Category	**Sub-category**	**Theme**	**Sample of pre-service teacher explanation related to 6 ÷ 0**	**Sample of pre-service teacher explanation related to 0 ÷ 0**
Correct explanations	Rule-based explanations	Write only "undefined/undetermined"	6 ÷ 0 is undefined	0 ÷ 0 is undetermined
		Write "undefined/undetermined" and explain as a rule	6 ÷ 0 is undefined, because this is a rule in mathematics	0 ÷ 0 is undetermined, because it was also taught to me as a rule.
	Conceptual-based explanations	Inverse of multiplication	Suppose that $\frac{6}{0} = x$. In this case 6=0.x. If we multiply a number by zero, the result is always zero. Clearly, there is no number that can be placed in x to make the equation true.	If $\frac{0}{0}$ serves as an answer, we can write $\frac{0}{0} = x$, so it would satisfy the equation 0=0x. In this equation, we can write any number as x; that is to say, we cannot find a particular x. For that reason, the answer is undetermined
		Sharing/dividing evenly (concrete)	If we share 6 pencils with 2 children evenly, each has 3 pencils. If we share 6 pencils with 3 children equally, each has 2 pencils. If there are no children to share pencils, how can we divide 6 pencils between zero children? Of course, not having any children makes division impossible.	If you have 0 balls to share equally among 0 children, how can we divide zero balls among zero children? How many balls does each child have? It is meaningless, since there is no ball and no children. Hence, the operation is impossible to perform.
		Sharing/dividing evenly (formal)	-	We are trying to divide nothing among nobody. It is meaningless; we can't do this.
		Repeated subtraction (formal)	Dividing 6 by 3 means finding out how many times 3 should be subtracted from 6 to reach 0. Similarly, how many times should 0 be subtracted from 6 to reach 0?	According to the definition of division, we should subtract 0 from 0; The remainder will always be 0, so there is no exact number that

	No matter how many times one subtracts 0 from 6, the remainder will always be 6. Thus, the process of subtracting 0 from 6 is infinite. We cannot find an exact number.	gives as an answer.
Intuitive notion of limit	$6 \div 6 = 1, 6 \div 3 = 2, 6 \div 2 = 3, 6 \div 1 = 6, 6 \div 3 = 2, 6 \div 0.5 = 12, \ldots$ If we decrease the dominator, the result increases. If the denominator gets closer to zero result will be very huge. There is no number that represent this situation.	-
Deductive reasoning	-	$\frac{0}{0}$ will be 1, because any number divided by itself is 1, and $\frac{0}{0}$ will be 0, because 0 divided by any number is 0. In this case, what would be answer? That is to say, I emphasize that it is undetermined.

Data Analysis

Qualitative and quantitative analysis methods were both used in analyzing the data in this study. Namely, to evaluate the pre-service teachers' instructional explanations related to division by zero, both inductive and deductive content analyses were used. The qualitative analysis of the data led to the development of two categories of explanation. The first group included pre-service teachers who did not give any explanation or gave incorrect explanation. Explanation of pre-service teachers who provided incorrect answers were separated into four sub-categories in the form of "explain as an infinite", "use definition of rational number", "use the meaning of zero", and "write numerical answer". Anthony and Walshaw (2004) and Russell and Chernoff's (2011) models to explain the understanding of zero guided the above system of coding. The second group included pre-service teachers who knew that the operation is undefined/undetermined, but could not provide any explanation as to why this is the case or those who were able to provide at a correct conceptual explanation of dividing by zero. Explanations of pre-service teachers who gave correct conceptual explanation separated into six sub-categories in the form of "inverse of multiplication", "sharing/dividing evenly (concrete)", "sharing/dividing evenly (formal)" "repeated subtraction", "intuitive notion of limit" and "deductive reasoning". Henry (1969), Reys and Grouws (1975), Tsamir and Sheffer (2000) and Watson (1991) models to show the impossibility of division by zero guided the above system of coding. Thus, participants' responses to each of two questions were analyzed and coded according to the model given in Table 1 and Table 2. Then, in determining whether there were significant differences in the percentages of the themes of the participants' responses in consideration of their class levels, a chi-square test was used. The chi-square test is based on the simple idea of comparing the frequencies observed in certain categories to the frequencies that might be expected to arise in these categories by chance (Field, 2009). The variable may have two or more categories. The chi-square test is more likely to yield significance if the sample portions for categories differ greatly from the hypothesized proportions and if the sample size is large (Green & Salkind, 2005). Moreover, when using the chi-square test, the expected frequencies should be greater than 5. Therefore, before using the chi-square test, contingency tables were created, and the expected frequencies were determined for each explanation. Although it is acceptable in larger contingency tables to have expected frequencies below 5 up to 20% of the time, the result is a loss of statistical power (Field, 2009). For this reason, these tables were not included in the data analysis. In the interpretation of the results of the chi-square test, the within-group percentage values were considered, and the findings were presented in a table. Additionally, to determine the strength of the relationship between the variables, Cramer's V was used. Cramer's V is the measure of association reflecting the strength of the relationship between two or more variables based on the chi-square distribution. Rea and Parker (2014) suggest the following guidelines (see Table 3) for interpretations of the Cramer's V effect size measures.

Table 3.
A guideline for interpreting effect size of the data

Measure	**Interpretation**
.00 and under .10	Negligible association
.10 and under .20	Weak association
.20 and under .40	Moderate association
.40 and under .60	Relatively strong association
.60 and under .80	Strong association
.80 to 1.00	Very strong association

Findings

Pre-service Teachers' Instructional Explanations Related to 6÷0

The frequencies and percentages of the pre-service teachers' explanations about 6÷0 are given in Table 4.

Table 4.
Frequencies and percentages of pre-service teachers' explanations about 6÷0

Category	**Sub-category**	**Theme**		**Freshman pre-service teachers**	**Sophomore pre-service teachers**	**Junior pre-service teachers**	**Senior pre-service teachers**	**Total**
Incorrect explanations		Explain as an infinite	f	5	18	18	5	46
			%	14	31	30	12	23
		Use definition of rational number	f	3	8	7	2	20
			%	9	14	12	5	10
		Use the meaning of zero	f	1	2	2	2	7
			%	3	3	3	5	4
		Write numerical answer	f	-	-	-	1	1

			%	-	-	-	2.4	1
		No answer	f	-	1	-	-	1
			%	-	2	-	-	1
Correct explanations	Rule-based explanations	Write “undefined” and explain as a rule	f	11	11	17	12	51
			%	31	19	28	29	26
		Write only “undefined”	f	10	6	2	4	22
			%	29	10	3	10	11
	Conceptual-based explanations	Inverse of multiplication	f	5	11	9	8	33
			%	14	19	15	19	17
		Sharing/dividing evenly (concrete)	f	-	1	5	1	7
			%	-	2	8	2	4
		Intuitive notion of limit	f	-	1	-	6	7
			%	-	2	-	14	4
		Repeated subtraction (formal)	f	-	-	1	1	2
			%	-	-	2	2	1

Table 4 shows that nearly 62% of the pre-service teachers gave correct explanations about 6÷0; while 38% of them produced incorrect explanations. Most of the participants knew that the result of 6÷0 was undefined, but they primarily gave explanations such as “[it is] undefined because this is a rule”. This shows that the participants’ correct explanations were based mainly on a procedural/memorization approach. In addition, seventeen percent of the pre-service teachers used the inverse of multiplication approach for explaining division by zero. A minority (fewer than 9%) of the participants included correct explanations such as sharing/dividing evenly or the intuitive notion of limit or repeated subtraction. This finding indicates that few pre-service teachers offer more conceptual-based explanations about division by zero. The incorrect explanation that was frequently written was the reference to zero as an infinite. Moreover, eleven percent of the respondents used the definition of a rational number, and 4% cited the meaning of zero as “nothing”.

According to Table 4, the freshman pre-service teachers primarily gave explanations such as “explain as a rule” and “write only ‘undefined’” with respect to division by zero. Very few of the freshman gave explanations that were classified as conceptual-based. On the other hand, when the sophomore and junior pre-service teachers’ explanations are considered, it can be seen that they mainly involved the categories of “explain as an infinite” and “write only ‘undefined’ and explain as a rule”. Furthermore, it was determined that both the sophomore and junior pre-service teachers gave explanations in the category of “use the definition of rational number” more often than both the freshmen and the seniors. The senior pre-service teachers gave more conceptual-based explanations than the freshman, yet these explanations were still not adequate. The most frequently given conceptual-based explanation was the category of “inverse of multiplication”. It was interesting to note that the seniors generally did not favour explanations in the category of “explain as an infinite,” which was the most frequent explanation among the sophomores and juniors. At this level, the pre-service teachers’ explanations were mostly correct and, in comparison with the other classes, it was clear that they gave more frequent and varied conceptual-based explanations.

The distribution of the pre-service teachers’ explanations concerning 6÷0 is shown in Figure 1.

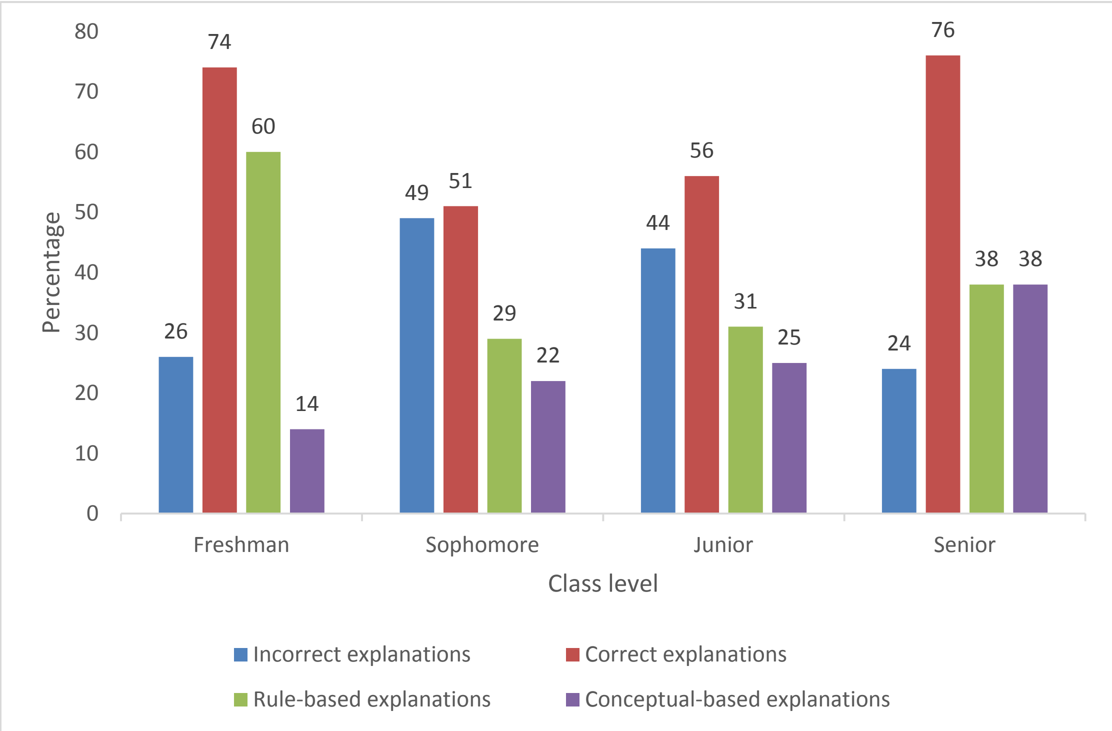

Figure 1. The distribution of the pre-service teachers’ explanations about 6 ÷ 0 with respect to class level.

As shown in Figure 1, the frequency of correct explanations was high for freshman and senior pre-service teachers, but there was a decrease among the sophomores and juniors. On the other hand, the correct explanations given by the freshmen were more frequently rule-based than those of the upper classes. Also, while there was a decrease in the number of rule-based explanations from the freshman to the junior participants, there was a slight increase in the number of senior pre-service teachers’ explanations. In addition, there was an increase in conceptual-based explanations from the freshmen to the seniors. Moreover, the number of incorrect explanations was low among the freshmen and seniors, while the sophomores and juniors gave incorrect explanations more frequently.

In order to determine whether there was a meaningful statistical difference between the incorrect and correct explanations according to class level, the chi-square test was used; the results are presented in Table 5.

Table 5.
Chi-square test results for pre-service teachers' incorrect and correct explanations that were statistically significant.

Comparison	Category				Pearson chi-square	p	Cramer's V
	Incorrect explanations		Correct explanations				
	f	%	f	%			
Freshman	9	26	26	74	5.011	.025	.231
Sophomore	29	49	30	51			
Sophomore	29	49	30	51	6.648	.010	.257
Senior	10	24	32	76			
Junior	27	44	34	56	4.520	.033	.209
Senior	10	24	32	76			

According to Table 5, there was a statistically meaningful difference between the freshman and sophomore pre-service teachers' explanations about division by zero. The difference was in favour of the freshmen, and the effect size of the difference was at moderate level. There was also a statistically meaningful difference between the sophomore and senior pre-service teachers' explanations about division by zero. The difference was in favour of the seniors, and the effect size of the difference was at a moderate level. In addition, there was a statistically meaningful difference between the junior and senior pre-service teachers' explanations in favour of the seniors; the effect size of the difference was at moderate level. There were also differences in the other class levels, but these were not statistically significant.

A chi-square test was also used to determine whether there was a meaningful statistical difference between the rule-based and conceptual-based explanations according to class level, the chi-square test was used. The results are presented in Table 6.

Table 6.
Chi-square test results for pre-service teachers' conceptual-based and rule-based explanations that were statistically significant.

Comparison	Category				Pearson chi-square	p	Cramer's V
	Rule-based explanations		Conceptual-based explanations				
	f	%	f	%			
Freshman	21	81	5	19	4.106	.043	.261
Junior	19	56	15	44			
Freshman	21	81	5	19	5.880	.015	.318
Senior	16	50	16	50			

As illustrated in Table 6, there was a statistically meaningful difference between the freshman and junior pre-service teachers' rule-based and conceptual-based explanations about division by zero. The difference was in favour of the juniors, and the effect size of the difference was at moderate level. There was also a statistically meaningful difference between the freshmen's and senior's explanations. The difference was in favour of the seniors, with an effect size at a moderate level. There were also differences in the other class levels, but these were not statistically significant.

Pre-service Teachers' Instructional Explanations Related to 0÷0

The frequencies and percentages of the pre-service teachers' explanations about 0÷0 are provided in Table 7.

Table 7.
Frequencies and percentages of pre-service teachers' explanations about 0÷0.

Category	Sub-category	Themes		Freshman pre-service teachers	Sophomore pre-service teachers	Junior pre-service teachers	Senior pre-service teachers	Total
Incorrect explanations		Explain as an infinite	f	1	-	3	1	5
			%	3	-	5	2	3
		Use definition of rational number	f	2	3	4	1	10
			%	6	5	7	2	5
		Use the meaning of zero	f	2	2	13	4	21
			%	6	3	21	10	11
		Write numerical answer	f	-	2	2	2	6
			%	-	3	3	5	3
		No answer	f	-	-	-	2	2
			%	-	-	-	5	1
Correct explanations	Rule-based explanations	Write only "undetermined"	f	14	29	13	7	63
			%	40	49	21	17	32
		Write "undetermined" and explain as a rule	f	11	13	17	17	58
			%	31	22	28	41	30
	Conceptual-based explanations	Inverse of multiplication	f	5	10	5	4	24
			%	14	17	8	10	12
		Repeated subtraction (formal)	f	-	-	-	2	2
			%	-	-	-	5	1
		Sharing/dividing evenly (formal)	f	-	-	1	-	1
			%	-	-	2	-	1
		Sharing/dividing evenly (concrete)	f			3	-	3
			%			5	-	2
		Deductive reasoning	f	-	-	-	2	2
			%	-	-	-	5	1

Table 7 indicates that nearly 78% of the pre-service teachers gave correct explanations, and few of them produced incorrect explanations relating to the concept of 0÷0. In this respect, explanations in the "write only undetermined" category were given most frequently. Another explanation that was frequently given was "write 'undetermined' and explain as a rule". In addition, twelve percent of the participants gave explanations relating to the "inverse of multiplication" approach, while 11% provided explanations in the "use the meaning of zero" category. Further, just 5% of the pre-service teachers provided explanations in the "use definition of rational number" category, while 3% gave explanations in the "write numerical answer" category and 3% gave explanations in the "explain as an infinite" category. Moreover, a very few gave explanations in the "sharing/dividing evenly (2%)", "repeated subtraction (1%)" and "deductive reasoning (1%)" categories.

In Table 7, it can be seen that, similar to their explanations of 6÷0, the pre-service teachers most often gave correct explanations for 0÷0; these were given at relatively similar percentages at each class level. Here, the freshman pre-service teachers primarily gave explanations in the category of "write only undetermined" (40%) and "write undetermined and explain as a rule" (31%). Moreover, some of the freshmen gave explanations in the category of "inverse of multiplication" (14%). A very few of the participants offered explanations in the categories of "use definition of rational number" (6%), "use the meaning of zero" (6%) or "explain as infinitive" (3%), Further, the sophomores gave explanations mainly in the category of "write only undetermined" (49%). Moreover, the sophomore students frequently gave explanations in the "write undetermined and explain as a rule" (22%) category. As with the freshmen, some of the sophomores gave explanations in the in the category of "inverse of multiplication" (17%), while Very few offered explanations in the categories of "use definition of rational number" (5%), "use the meaning of zero" (3%) or "write numerical answer" (3%).

Apart from the freshman and sophomore pre-service teachers, the other participants showed some diversification in their correct and incorrect explanations. For instance, of the juniors, twenty-one percent gave explanations in the categories of both "write only undetermined" and "use the meaning of zero". It can also be seen that some of the participants gave explanations in the "use the definition of rational number" (7%), "explain as a rule" (5%) and "write numerical answer" (3%) categories. Thus, while the number of correct explanations decreased, they did become more varied. Among the correct explanations, a decline was found in the category of "write only undetermined" and "inverse of multiplication"; however, explanations in the categories of "sharing/dividing evenly (concrete)" and "sharing/dividing evenly (formal)" were seen for the first time. The senior pre-service teachers primarily gave explanations in the "write undetermined and explain as a rule" (41%) category; furthermore, the number of their correct explanations increased and were varied similarly to those of the juniors.

The distribution of the pre-service teachers' explanations concerning the concept of 0÷0 is shown in Figure 2.

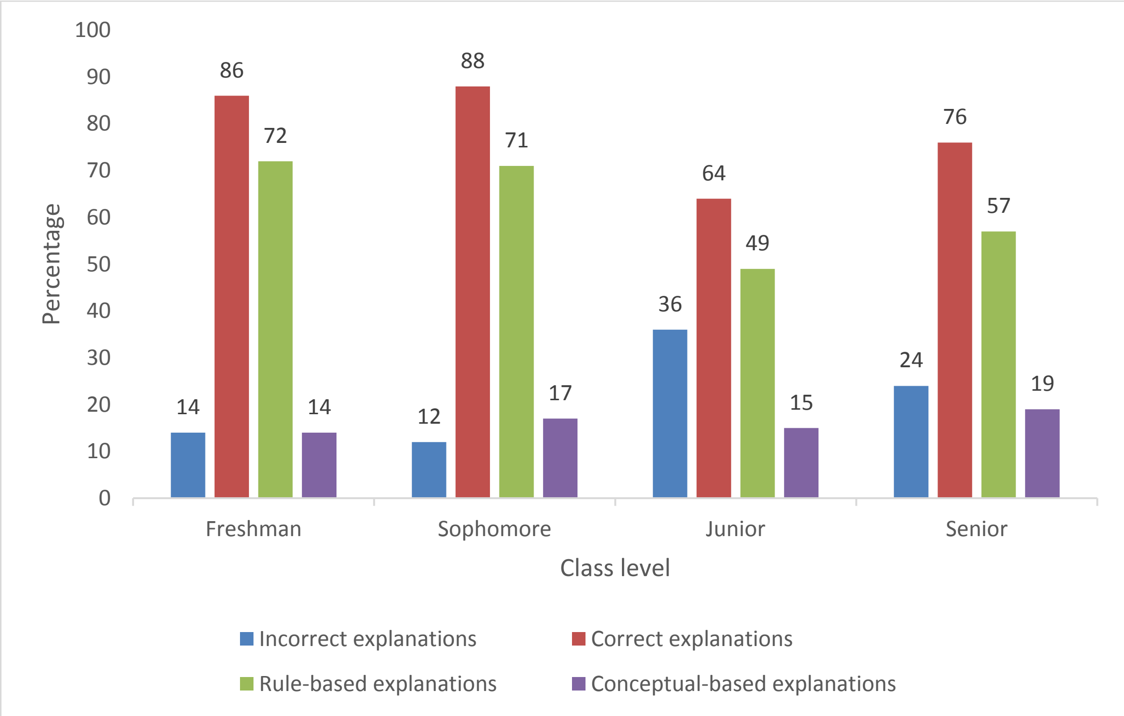

Figure 2. Distribution of pre-service teachers' explanations about 0 ÷ 0 with respect to class level.

According to Figure 2, the number of correct explanations was high for both the freshmen and sophomores, but there was a decrease for juniors and seniors. Both the freshmen's and sophomore's correct explanations were more frequently rule-based than those of the other students. In contrast to the decrease in the rule-based explanations among the juniors and seniors, there was a slight increase in the number of conceptual-based explanations given by the senior pre-service teachers. Further, while the number of incorrect explanations was low for the freshmen and sophomores, there were an increase for the junior and senior pre-service teachers.

To determine whether there was a meaningful statistical difference between the pre-service teachers' incorrect and correct explanations according to class level, the chi-square test was applied.

Table 8.
Chi-square test results about pre-service teachers' incorrect and correct explanations which were statistically significant.

Comparison	Category				Pearson chi-square	p	Cramer's V
	Incorrect explanations		Correct explanations				
	f	%	f	%			
Freshman	5	14	30	86	5.219	.022	.233
Junior	22	36	39	64			
Sophomore	7	12	52	88	9.585	.002	.283
Junior	22	36	39	64			

According to Table 8, there was a statistically meaningful difference between the freshman and junior pre-service teachers' incorrect and correct explanations relating to 0÷0. The difference was in favour of the freshman, and the effect size of the difference was at moderate level. There was also a statistically meaningful difference between the sophomore and junior pre-service teachers' incorrect and correct explanations about 0÷0. The difference was in favour of the sophomores, and the effect size of the difference was at moderate level.

In order to determine whether there was a meaningful statistical difference between the rule-based and conceptual-based explanations according to class level, the chi-square test was used. The chi-square test results show that there was no statistically meaningful difference between the pre-service teachers' rule-based and conceptual-based explanations relating to 0÷0.

Discussion and Implications

The results of this study demonstrate that the majority of the pre-service teachers could give correct answers for both 6÷0 and 0÷0. Although most of the pre-service teachers gave correct answers related to division by zero, only some of them were able to give conceptual-based explanations. This indicates that they mostly gave procedural/rule based explanations. Parallels to these results can be found in studies reporting that both teachers and pre-service teachers primarily gave rule-based explanations such as "it is undefined because this is a rule, and you cannot divide by zero" to justify their answers (Ball, 1990; Cankoy, 2010; Crespo & Nicol, 2006; Quinn et al, 2008). The ability to give instructional explanations for mathematical problems and concepts is one of the important dimensions of pedagogical content knowledge for mathematics teachers. In this respect, instructional explanations that support student understanding of the underlying mathematical concepts may be more effective, yet studies have shown that both in-service teachers and pre-service teachers generally use rule-based explanations (Ma, 1999; Kinach, 2002; Toluk Uçar, 2011) instead. This issue has been ascribed to various factors, including teachers' previous mathematical experiences, their beliefs in the nature of mathematics, and inadequate mathematical content knowledge (Dede & Karakuş, 2014; Richardson, 1996, Thompson, 1992). In comparing the participants' explanations for 6÷0 and 0÷0, it was clear that they gave more conceptual-based explanations for 6÷0 than for 0÷0. This may be explained by the epistemological obstacles of the concept, as well as the pre-service teachers' past experiences and their beliefs concerning mathematics (Cornu, 1991; Dede & Karakuş, 2014).

In comparing the responses according to class level, the freshman pre-service teachers' answers were mostly correct but rule-based for both 6÷0 and 0÷0. Such explanations given by pre-service teachers at the beginning of their teacher training program is an indication that they had received inadequate instruction in this subject. Likewise, previous studies have reported that pre-service teachers at the beginning of their university term generally have rule-bound mathematical understanding, and they have difficulty in giving appropriate mathematical explanations for a given mathematical situation (Ma, 1999; Toluk Uçar, 2011). Moreover, the available textbook explanations about division by zero are also relatively rule-bound. For instance, in Turkey, 7th-grade textbooks (e.g. Bağcı, 2015) typically give explanations such as "since its denominator is zero, 6÷0 is not a rational number, and for this reason, it should be considered as undefined." Such textbooks may affect pre-service teachers' instructional explanations. An interesting case relating to this issue was observed in the sophomore and junior pre-service teachers' explanations about division by zero. In those classes, the correct explanations decreased, but incorrect explanations increased. In addition, the rule-based explanations decreased, but conceptual based explanations increased. Among the sophomore and junior pre-service teachers, the most frequently given explanation was “as a rule, number÷0 equals infinite” for 6÷0. One reason for this may be that, at the sophomore and junior levels, the participants had taken courses in Calculus I and Calculus II, where they had frequently encountered the concept of undefined in the subjects of limit and derivatives. In this respect, Özmantar (2008) points out that, starting in high school, students encounter limits such as $\lim_{x\to 0^-}\frac{1}{x} = -\infty$ and $\lim_{x\to 0^+}\frac{1}{x} = +\infty$. For this reason, pre-service teachers who give explanations as “infinite” often have the tendency to think of 6÷0 as $\lim_{x\to 0}\frac{6}{x} = \mp\infty$. Furthermore, it was observed in the present study that some pre-service teachers gave incorrect explanations such as $\frac{number}{0} = \infty$. These participants considered the symbol ∞ to be a number. Özmantar (2008) also asserts that in equations such as $\lim_{x\to 0^-}\frac{1}{x} = -\infty$ and $\lim_{x\to 0^+}\frac{1}{x} = +\infty$, students may perceive the symbol of ∞ as a number. Tsamir, Ruth and Tirosh (2000) note that students may have an intuitive belief that all mathematical operations should have a numerical result; it is possible that the participants in this study who gave the explanation that $\frac{number}{0} = \infty$ may have a similar belief. An interesting point here was that in the case of 6÷0, the sophomores and juniors mainly cited as “infinite”, but for 0÷0, they rarely gave explanations as “infinite”. While many of them knew that 0÷0 is undetermined, they nearly always explained this as a rule, also noting that the reason that 0÷0 is undetermined would be taught in the subjects of limit and derivatives in high school. It can be inferred that pre-service teachers who consider 0÷0 as a type of approaching limit operation preferred giving explanations as only “undetermined” because they assume that elementary school students have no knowledge of these subjects. In the senior pre-service teachers, the rule-based explanations slightly increased compared to juniors and their conceptual-based explanations increased, too. In addition, in the case of 0÷0, freshman and senior pre-service teachers’ percentage of correct, but rule-based responses were similar and higher than sophomore and junior pre-service teachers. The reason may be the national exams. Before attending a teacher training program in Turkey, high school students must enter national exams. For that reason the education system is exam oriented, having negative effect on students’ mathematical understanding. Teachers focus much more on remembering and memorizing rules, facts and procedures. Similarly, when pre-service teachers graduate from a teacher training program, they must enter an exam for being a permanent teacher. Therefore, in the last year of the teacher training program, pre-service teachers focus much more on remembering and memorizing rules and procedures for being successful in the exam.

Although correct explanations were given at a high rate for all class levels, the participants’ conceptual-based explanations increased and become varied, but their rule-based explanations decreased at the higher class levels. This indicates that the teacher training program did impact the pre-service teachers’ explanations concerning division by zero. Similarly, it has been emphasized in the literature (e.g. Bütün, 2012; Gürbüz, Erdem & Gülburnu, 2013) that teacher training programs have a positive effect on pre-service teachers’ pedagogical content knowledge. This finding may be related to certain courses in the teacher education program, such as school experience and the teaching of elementary mathematics. In these courses, pre-service teachers have a chance to interact with students and to practice elementary school mathematics topics that they have previously encountered mainly on a theoretical basis. Therefore, it can be said that these courses were effective in improving the pre-service teachers’ pedagogical content knowledge; and thus, the decrease in the seniors’ rule-based explanations and the increase and variety in their conceptual-based explanations may be the result of these courses. Studies by Cankoy (2010) and Gürbüz et al. (2013) likewise support the notion that professional experience has a deep impact on teachers’ proficiency. In this sense, for the underclassmen, when the class level increased, the participants’ incorrect explanations decreased, and their correct explanations increased; but these changes were not statistically meaningful. It can be inferred from this that the teacher training program was beginning to impact the pre-service teachers’ instructional explanations; however, the change was gradual. These findings are supported by studies reporting that pre-service teachers do not have sufficient pedagogical content knowledge; that in-service and pre-service teachers generally know the rules and methods and how to use them, but they are not always able to provide complete explanations for given mathematical situations (e.g., Ball, 1990; Even, 1993; Işıksal, 2006; Ma, 1999; Tirosh, 2000; Toluk Uçar, 2011). Depending on their class level, the pre-service teachers’ most frequently-cited explanations with respect to conceptual-based explanation was “inverse of multiplication”. Crespo and Nicol (2006) and Quinn et al. (2008) likewise reported in-service and pre-service teachers’ use of the inverse of multiplication approach for explaining why dividing by zero is undefined. However, comparing the participants’ conceptual-based explanations for 6÷0 and 0÷0, it was clear that they gave more conceptual-based explanations for 6÷0 than for 0÷0. Thus, while positive effects of the teacher training program were observed concerning the participants’ explanations of 6÷0, such effects were not seen for 0÷0. The inherent difficulties of these concepts make them difficult to understand, as well as teach. In fact, Brousseau (2002) points out that, from the primary level to higher education, there are numerous mathematical concepts that have epistemological obstacles; and because of historical development and difficulties in the expression of these concepts, zero and division by zero can be considered as among these.

Overall, the results of this study showed that a significant proportion of the pre-service teachers were unable to give robust conceptual explanations regarding division by zero; furthermore, it was revealed that the teacher training program had little effect on their instructional explanations. Ball (1990) and Toluk Uçar (2011) found that pre-service teachers had more rule bound and superficial understandings when they started their teacher education program and for this reason they had more rule-based explanations. The results of this study are similar to the studies of Ball (1990) and Toluk Uçar (2011). The fact that pre-service teachers come to the teacher training program with such operational understanding and explanations requires examining the quality of the education they have received in the past. There is much interest in public exams held in senior classes of elementary and secondary schools in Turkey. The aim of these exams is to get the student a high score and to place him/her in a successful school. This leads to the emergence of operational understanding instead of conceptual understanding. In this context, the identification of the instructional explanations of elementary and secondary school teachers and investigation of the reasons of these explanations can lead to further research.

The results indicate that as class level increases, there is a decrease in rule-based explanations while a rise in conceptual-based explanations. As class level increases, the number of mathematics method courses increases and the number of mathematics content courses decreases in the teacher education program. In the mathematics method courses, pre-service teachers are getting more information about improving their pedagogical content knowledge. This suggests that the method courses in the teacher education program have positive effects on the instructional explanations of the pre-service teachers. Similarly, it can be said that mathematics content courses are effective in changing the pre-service teachers' instructional explanations since they take mathematics content courses in the sophomore and junior levels. In these levels, the reason that sophomore and junior pre-service teachers answer the result of $6 \div 0$ as "infinite" may be the calculus or algebra courses. Charalambos, et. al. (2011) and Kinach (2002) state how pre-service teachers' instructional explanations can be transformed from operational to conceptual and they also provide some suggestion in this transformation process. In this study despite not being supported strongly it is clear that the mathematics teacher education program is effective in the transformation of pre-service teachers' instructional explanations from rule-based explanations to conceptual-based explanations. In this context, the effects of the content and method courses that pre-service teachers have on this transformation process need to be examined in depth. In addition, the effects of other factors, such as subject matter knowledge and beliefs which influence the formation and transformation of instructional explanations of pre-service teachers should be examined. The present study also raises the need for further research, including an examination of in-service teachers' choices of examples or explanations relating to the same topic. In addition, it would be interesting to identify how pre-service teachers are being trained with respect to the choice and use of examples or explanations for a mathematics topic in various teacher education programs.

References

Anthony, W., & Walshaw, M. A. (2004). Zero. *Teaching Children Mathematics, 11*(1), 38.

Bağcı, O. (2015). *Ortaokul matematik 7 ders kitabı.* Ankara: Tutku Yayıncılık.

Baki, M. (2013). Pre-service classroom teachers' mathematical knowledge and instructional explanations associated with division. *Egitim ve Bilim-Education and Science, 38(167),* 300-311.

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education, 21(2),* 132-144.

Brousseau, G. (2002). Epistemological obstacles, problems, and didactical engineering. *Theory of Didactical Situations in Mathematics: Didactique des Mathématiques, 1970–1990,* 79-117.

Bütün, M. (2012). İlköğretim matematik öğretmeni adaylarının uygulanan zenginleştirilmiş program sürecinde matematik öğretme bilgilerinin gelişimi. *Unpublished doctoral dissertation, Karadeniz Technical University, Institute of Educational Sciences. Trabzon.*

Cankoy, O. (2010). Mathematics Teachers' Topic-Specific Pedagogical Content Knowledge in the Context of Teaching a0, 0! and a÷0. *Educational Sciences: Theory and Practice, 10(2),* 749-769.

Charalambous, C. Y., Hill, H. C., & Ball, D. L. (2011). Prospective teachers' learning to provide instructional explanations: how does it look and what might it take?. *Journal of Mathematics Teacher Education, 14(6),* 441-463.

Cornu, B. (1991). Limits. In Tall, D. *Advanced mathematical thinking* (pp. 153-166). Dordrecht: Springer.

Crespo, S., & Nicol, C. (2006). Challenging preservice teachers' mathematical understanding: The case of division by zero. *School science and mathematics, 106(2),* 84-97.

Creswell, J. W. (2012). *Qualitative inquiry and research design: Choosing among five approaches (3rd ed.).* Thousand Oaks: Sage.

Çelik, D., & Akşan, E. (2013). Preservice mathematics teachers' perceptions of infinity, indeterminate and undefined. *Necatibey Eğitim Fakültesi Elektronik Fen ve Matematik Eğitimi Dergisi, 7(1),* 166-190.

Dede, Y., & Karakuş, F. (2014). The effect of teacher training programs on pre-service mathematics teachers' beliefs towards mathematics. *Educational Sciences: Theory & Practice, 14(2),* 804-809.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics, 21(6),* 521-544.

Even R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education, 24(2),* 94-116.

Even, R., & Tirosh, D. (1995). Subject-matter knowledge and knowledge about students as sources of teacher presentations of the subject-matter. *Educational Studies in Mathematics, 29(1),* 1-20.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws, (Ed), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 147-164). New York: Macmillan.

Field, A. (2009). *Discovering statistics using SPSS (third edition).* London: SAGE.

Fraenkel, J. R., Wallen, N. E., & Hyun, H. H. (2012). *How to design and evaluate research in education (8th ed.).* New York: McGram-Hill.

Green, S.B. & Salkind, N.J. (2005). *Using SPSS for Windows and Macintosh analyzing and understanding data (4th ed.).* Mahwah, NJ: Pearson Prentice Hall.

Gürbüz, R., Erdem, E., & Gülburnu, M. (2013). Sınıf öğretmenlerinin matematik yeterliklerini etkileyen faktörlerin incelenmesi. *Ahi Evran Üniversitesi Kırşehir Eğitim Fakültesi Dergisi, 14(2),* 255-272.

Henry, B. (1969). Zero, the troublemaker. *The Arithmetic Teacher, 16(5),* 365-367.

Işıksal, M. (2006). A study on pre-service elementary mathematics teachers' subject matter knowledge and pedagogical content knowledge regarding the multiplication and division of fractions. *Unpublished doctoral dissertation. Middle East Technical University, Ankara, Turkey.*

Kinach, B. M. (2002). A cognitive strategy for developing pedagogical content knowledge in the secondary mathematics methods course: toward a model of effective practice. *Teaching and Teacher Education, 18(1),* 51-71.

Leinhardt, G. (1990). Towards understanding instructional explanations. Retreived March, 10, 2017 from ERIC database.

Leinhardt, G. (2001). Instructional explanations: A commonplace for teaching and location for contrast. *Handbook of research on teaching, 4,* 333-357.

Leinhardt, G. (2010). Introduction: explaning instructional explanation. M.K. Stein and L. Kucan (Ed.) *Instructional explanations in the disciplines.* NY: Springer.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States.* Mahwah, NJ: Erlbaum.

McMillan, J. H., & Schumacher, S. (2006). *Research in education. Evidence-based inquiry (6th ed).* Boston: Pearson Education.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook (2nd ed.).* Thousand Oaks: Sage.

Özmantar, M. F. (2008). Sonsuzluk kavramı: Tarihsel gelişimi, öğrenci zorlukları ve çözüm önerileri. In MF Özmantar, E. Bingölbali ve H. Akkoç. *Matematiksel kavram yanılgıları ve çözüm önerileri,* (pp.151-180). Ankara: Pegem Akademi.

Quinn, R. J., Lamberg, T. D., & Perrin, J. R. (2008). Teacher perceptions of division by zero. *The Clearing House: A Journal of Educational Strategies, Issues and Ideas, 81(3),* 101-104.

Rea, L. M., & Parker, R. A. (2014). *Designing and conducting survey research: A comprehensive guide.* John Wiley & Sons.

Reys, R. E., & Grouws, D. A. (1975). Division involving zero: Some revealing thoughts from interviewing children. *School science and mathematics, 75(7),* 593-605.

Richardson, V. (1996). The role of attitudes and beliefs in learning to teach. In J. Sikula (Ed.), *Handbook of research on teacher education (2nd ed.),* (pp. 102-119). New York: Macmillan.

Russell, G., & Chernoff, E. J. (2011). Seeking more than nothing: Two elementary teachers conceptions of zero. *The Mathematics Enthusiast, 8(1),* 77-112.

Sanders, S. E., & Morris, H. J. (2000). Exposing student teachers'content knowledge: Empowerment or debilitation? *Educational Studies, 26(4),* 397-408.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational researcher, 15(2),* 4-14.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics teaching, 77(1),* 20-26.

Thanhieser, E. (2009). Preservice elementary school teachers' conceptions of multidigit whole numbers. *Journal for Research in mathematics Education, 40,* 252-281.

Thompson, A. G. (1992). *Teachers' beliefs and conceptions: A synthesis of the research.* Macmillan.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31(1),* 5-25.

Toluk Uçar, Z. (2011). Öğretmen adaylarının pedagojik içerik bilgisi: öğretimsel açıklamalar. *Turkish Journal of Computer and Mathematics Education, 2(2),* 87-102.

Tsamir, P., & Tirosh, D. (2002). Intuitive beliefs, formal definitions and undefined operations: Cases of division by zero. In G. C. Leder, E. Pehkonen, & G. Törner, G. (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 331-344). Dordrecht: Springer.

Tsamir, P., & Sheffer, R. (2000). Concrete and formal arguments: The case of division by zero. *Mathematics Education Research Journal*, *12(2)*, 92-106.

Tsamir, P., Sheffer, R., & Tirosh, D. (2000). Intuitions and undefined operations: The cases of division by zero. *Focus on Learning Problems in Mathematics*, *22(1)*, 1-16.

Watson, J. M. (1991). Models to show the impossibility of division by zero. *School Science and Mathematics, 91(8),* 373-376.

Wheeler, M. M., & Feghali, I. (1983). Much ado about nothing: Preservice elementary school teachers' concept of zero. *Journal for Research in Mathematics Education*, *14(3)*, 147-155.