

## Conceptions of Expressions and Equations in Early Algebra: A Learning Trajectory

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This article describes a learning trajectory on expressions and equations based on the sixth-grade Common Core State Standards for Mathematics (NGA/CCSSO, 2010). A classroom teaching experiment using design research (Lamberg & Middleton, 2009) was conducted. The data was analysed using Corbin and Strauss' (2014) Constant Comparative method and Creswell's (2007) data analysis spiral. The realized trajectory that emerged and the design decisions that were made were documented. The findings reveal that student understanding of the changing meaning of letters and variables within a problem context played a central role in student development of conceptual understanding of expressions and equations. Specifically, solving the different problem types involved the following types of thinking: *Label Thinker*, *Formulaic Thinker*, *Substituter*, *Solver*, and *Functional Thinker*. The letter was interpreted as a label representing a category, a known value, a variable as an unknown value, and a variable as a changing quantity depending on the problem context. Implications for classroom instruction and curriculum development are discussed.

In 2010, the Common Core State Standards for Mathematics (CCSSM) (National Governor's Association/Council of Chief State School Officers [NGA/CCSSO], 2010) were introduced to the American educational community. Educators began incorporating these standards into their teaching practices and it became apparent that the standards specify *what* students should learn, but do not address *how* these concepts should be taught. The CCSSM (NGA/CCSSO, 2010) are organized as learning progressions. This means that the standards are carefully sequenced to support the development of more sophisticated mathematical ideas. Materials and tools to teach these standards are currently being developed. Curriculum designers and teachers must decide on how to support student learning of the central mathematical ideas of the standards by organizing tasks. Figueiredo, van Galen, and Gravemeijer (2009) indicate that even though tasks are designed as intended by the designers and teachers, students might interpret a task "in a way that differs much from how teachers or instructional designers look at it" (p.1). Teachers and designers have to decide on how to organize and use tasks to best support student learning. Hypothetical learning trajectories are helpful for thinking about how to organize tasks to support student learning.

Hypothetical learning trajectories (Clements & Sarama, 2004; Daro, Mosher, & Corcoran, 2011; Wilson, Sztajn, Edington, & Meyers, 2015; Simon, 1995) are paths that students might take to learn mathematics. Tasks are developed and carefully designed based on conjectures of anticipated student thinking and learning. For our purposes, we conceptualize learning trajectories as possible paths for supporting student learning through a set of instructional tasks (Clements & Sarama, 2004) that can inform teachers and curriculum developers. The actual learning path of students might differ from the original hypothesized learning trajectory (Figueiredo, van Galen, & Gravemeijer, 2009). Wilson et al. (2015) specify that a teacher can support student learning by building on their pre-existing knowledge by using a learning trajectories approach to teaching. This process involves selecting and organizing tasks based on a conjectured learning trajectory and adapting the tasks to support student learning. Teachers find it helpful to have a framework of student

thinking to make instructional decisions as students engage in sense making (Wilson et al., 2015).

In this study, we explored how sixth-grade students made sense of expressions and equations during a whole class teaching experiment through design research (Lamberg & Middleton, 2009). We based the hypothetical learning trajectory on the sixth-grade CCSSM for expressions and equations: (a) applying and extending previous understandings of arithmetic to algebraic expressions, (b) reasoning about and solving one-variable equations and inequalities, and (c) representing and analyzing quantitative relationships between dependent and independent variables (NGA/CCSSO, 2010). A review of research on expressions, equations, and functions was conducted to layout a hypothetical trajectory based on the CCSSM (NGA/CCSSO, 2010).

Even though research exists on how to extend student understanding of arithmetic to algebraic expressions (Knuth, Stephens, McNeil, & Alibali, 2006), as well as variables (Blanton, Brizuela, Gardiner, Sawrey, & Newman-Owens, 2017; Hackenberg & Lee, 2015; Kaput, 1995; Küchemann, 1981), there is very little research that specifically addresses how to support sixth-grade students to develop understanding of these standards. Daro, Mosher, and Corcoran (2011) indicate that further research on learning trajectories in the content area of algebra is needed. The purpose of our study was to document the realized learning trajectory that emerged in order to inform teachers and curriculum designers on types of thinking that occurred and mechanisms for supporting shifts in thinking.

### Theoretical framework

Algebra is composed of its own standardized body of symbols, signs, and rules that govern the language of algebra. It has its own grammar and syntax that allows one to formulate algebraic ideas clearly and compactly (Drijvers, Goddijn, & Kindt, 2011). Although this symbolic language is very powerful, it is also detached and formal in relation to the actual context of a problem. The Common Core State Standards for Mathematical Practice state that mathematically proficient students should develop skills to reason abstractly and quantitatively (NGA/CCSSO, 2010). Furthermore, students need to not only have the ability to decontextualize a problem and illustrate it symbolically, but also contextualize a symbolic representation and understand its referents.

The National Council of Teachers of Mathematics (NCTM, 2000) recommended that students in grades 6–8 explore relationships among symbolic expressions and graphs, and also use symbolic algebra to represent situations and solve problems. The symbolic language of algebra is more than the memorization of rules; it involves the ability to model mathematical situations with symbols, understand the manipulation of these symbols, and have a fundamental understanding of the concept of variables and algebraic structures (Kieran, 1996).

Learning the meaning behind symbols and variables is essential for students to become proficient in algebra. Students cannot understand how to solve an algebra equation without knowing the meaning of the equal sign and variables (Knuth et al., 2006). Students must find meaning in algebra not only to understand why they are solving algebraic equations, but also what situations the algebra represents. Kaput (1995) found that many students view algebra as “little more than many different types of rules about how to write and rewrite strings of letters and numerals, rules that must be remembered for the next quiz or test” (p. 4). Research on algebra shows that students have difficulty using letters as variables and studies have focused on how students learn to represent unknowns using letters, including ignoring letters, substituting specific values for letters, treating letters as labels of objects, using letters

an alphabetical code, or treating each letter as having a value of 1 (Hackenberg & Lee, 2015; e.g., Booth, 1989; Küchemann, 1981; MacGregor & Stacey, 1993).

Once students learn to work with variables without thinking about the numbers that the variable might represent, they have achieved manipulation of “opaque formalisms” (Kaput, 1995, p. 8). Variables have many different possible definitions. In mathematics, a variable is a letter that represents an arbitrary, varying, number. May and Van Engen (1959) stated, “Roughly speaking, a variable is a symbol for which one substitutes names for some objects, usually a number in algebra” (p. 70). By varying a quantity in an arithmetic problem, students begin to learn the notion of “variable”. By the middle grades, students should be familiar with finding missing values and open mathematical sentences where a blank square or underline represents the unknown value, but they may question whether a letter is used to represent a numerical value or an abbreviation for a word (Booth, 1989), and thus, not connect the missing value to the word “variable”. Students can begin to learn about variables by using a variety of symbols and letters to represent unknown quantities.

Variables represent more than one value. However, students have difficulty with this notion because they have learned that a variable represents a particular number (Küchemann, 1981). Usiskin (1988) identified the differences between two conceptions of variables: (a) variables as unknowns or constants and (b) variables as varying quantities (p. 10). Letters that represent unknowns or constants are used in algebraic equations where the main goal is to simplify or solve. Variables as varying quantities are seen in equations where variables are arguments. For instance, in a linear equation  $y = 3x + 1$ , both  $x$  and  $y$  are the arguments and can represent any values that make the equality true. In other words, the ordered pair  $(x, y)$  can be any  $x$  and  $y$  values that lie on the line and, thus, are a solution to  $y = 3x + 1$ . Some mathematics educators think that algebra should be introduced using variables as changing quantities instead of letters that represent numbers (Fey & Good, 1985; Usiskin, 1988) and Carpenter, Franke, and Levi (2003) suggest that transitioning from natural language to symbolic notation is less challenging if students understand that the letter or variable represents a specific quantity. Translating word problems into algebra often involves creating equations and inequalities, and then finding the value of one or more variables. Thus, learning how to solve equations becomes an essential component of algebra.

Modeling real world situations allows students to make sense of algebra. Many view mathematical modeling as the main objective of algebra (Izsák, 2003; Kaput, 1999; Schoenfeld, 1992). Mathematical modeling is the process of taking a real world situation and attempting to represent it with mathematics or “mathematize it” (Kaput, 1999, p. 17). Mathematical modeling is not referring to using manipulatives as an example of the mathematics, but rather, simulating real phenomena with equations. Students learn to model real situations using mathematics by thinking about situations that contain multiple and connected representations. Placing an algebra problem in context helps students make sense of the mathematics and supports conceptual understanding of abstract representations (Earnest & Balti, 2008). Mathematical modeling is learned through the generalization of arithmetic to algebra, using symbols in a meaningful way, studying structure, and studying patterns and functions (Kaput, 1999).

Empirical approaches to algebra include relating the symbolic systems to real-world situations, graphs and tables, or arithmetic patterns (Kirshner, 2001). This approach to simplifying algebraic expressions and solving equations places the emphasis on learning the referential meanings of algebra, not merely on the manipulation of symbols with no purpose (Booth, 1989). Therefore, mathematical modeling involves the conversion of a word problem, written or verbal, to an equation, inequality, or system (NGA/CCSSO, 2010).

The research literature indicates that there is no clear and concise definition of algebraic thinking. Specifically, little is known about children's ability to advance from arithmetic to algebra and use algebraic notation (Carraher, Schliemann, Brizuela, & Earnest, 2006). Algebra can be integrated into arithmetic, and arithmetic can be modified to include algebra through generalizing the properties of real numbers (Moseley & Brenner, 2009). Researchers contend (Darley, 2009; Ketterlin-Geller, Jungjohann, Chard, & Baker, 2007; Usiskin & Bell, 1983) that it is impossible to learn arithmetic without dealing implicitly (e.g.  $\_\_ + 2 = 5$ ) or explicitly with variables.

### *Purpose of the Study*

This study reports on a study of how sixth-grade students came to understand expressions and equations through a whole class teaching experiment using design research (Lamberg & Middleton, 2009). We focused on sixth-grade students because expressions and equations are formally introduced in the CCSSM (2010) in the sixth grade. The purpose of this study was to understand the kinds of student thinking that emerged when presented with tasks and to document a learning trajectory as a means of organizing and supporting student learning. There were two main research questions that framed our study:

1. What learning trajectory emerged as students participated in the teaching experiment?
2. What sequence of mathematical tasks led to student understanding of expressions and equations?

### *A Teaching Experiment Using Design Research*

A whole class teaching experiment using design research (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Collins, Joseph, & Bielaczyc, 2004; Kelly, 2003) was conducted in a sixth-grade classroom. The teaching experiment centered on the development of the instructional sequence based on Kaput's (1999) framework for algebraic thinking, and our conjectures of student learning path based on a review of research. An initial instructional unit on algebra for 11–12 year old students was developed, revised, and modified during a whole class teaching experiment based on student thinking (Lamberg & Middleton, 2009). Design research involves engineering learning environments, systematically studying what takes place, and making adjustments to the curriculum (Cobb et al., 2003; Collins et al., 2004; Kelly, 2003). Design researchers develop theories about the learning process, as well as documenting mechanisms for supporting shifts in student thinking (Lamberg & Middleton, 2009).

### *Participants and Initial Instructional Unit*

American students from a sixth-grade classroom in an urban elementary school in a western state participated in the study. The sample included a total of 22 predominantly Latino (a) students, ages 11 to 12. There were 11 female students and 11 male students. The majority of the students were from lower to middle socioeconomic backgrounds. The teaching experiment was conducted over four weeks through a design research approach (Cobb et al., 2003; Collins et al., 2004; Kelly, 2003). The lesson times ranged from an hour to an hour and a half and took place during regularly scheduled math instruction time.

The teacher that participated in this study had a master's degree in education and had taught fifth and sixth graders for two years. She regularly participated in professional

development sessions in the school district. The teacher was chosen because she has a close working relationship with the lead researcher and wanted to learn new methods for teaching algebra.

We developed a hypothetical learning trajectory based on a review of research on how students learn algebra and mathematics to address the sixth-grade CCSSM for Expressions and Equations (NGA/CCSSO, 2010). This hypothetical learning trajectory was based on Kaput's (1999) framework for algebraic thinking. Some connections include (a) generalizing arithmetic to algebra, (b) using symbols in a meaningful way, and (c) mathematical modeling (Kaput, 1999) and also a review of research on how students learn algebra. Figure 1 shows the hypothetical learning trajectory and provides problem types and examples:

| Hypothetical Learning Trajectory                                                                                                                  | Problem Types                                                               | Examples                                                                                                                                                                                 |
|---------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p>[Connection to Kaput's (1999) framework for algebraic thinking]</p> <p>Alignment with the CCSSM (2010)</p>                                     |                                                                             |                                                                                                                                                                                          |
| <p>Expressions</p> <p>[generalizing arithmetic to algebra]</p> <p>CCSSM: Apply previous understandings of arithmetic to algebraic expressions</p> | Write, read and evaluate expressions                                        | Express the perimeter of a square as $2l + 2w$ where $l$ is length and $w$ is width.                                                                                                     |
|                                                                                                                                                   | Generate equivalent expressions                                             | A square has the same length and width, so the perimeter could be written as $2x + 2x$ where $x$ is the length of each side of the square. An equivalent equation to $2x + 2x$ is $4x$ . |
|                                                                                                                                                   | Identify when two expressions are equivalent                                | $2x + 2x$ and $4x$ are equivalent because they are the same quantity regardless of the value for $x$ .                                                                                   |
| <p>Equations</p> <p>[using symbols in a meaningful way and mathematical modeling]</p>                                                             | Use substitution to determine whether a given number makes an equation true | Each side of a square is 3 inches. The perimeter of a square is $P=4x$ , so $P=4(3)=12$ , using substitution of $x=3$ .                                                                  |
|                                                                                                                                                   | Use variables to represent numbers and write                                | If Ann has some money and Joe has \$8 more than Ann, how much money does Joe have? Joe has $8 +$                                                                                         |

|                                                                                                                                                                                                 |                                                                                                               |                                                                                                                                                                                                                                                                                                                                                        |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| CCSSM:<br>Reason about and solve one-variable equations                                                                                                                                         | expressions for solving a real-world problem                                                                  | $a$ , where $a$ is the amount of money that Ann has.                                                                                                                                                                                                                                                                                                   |
|                                                                                                                                                                                                 | Solve real-world problems using the addition property of equality and the multiplication property of equality | The total amount of money that Ann and Joe have is \$14. Joe has $8 + a$ and Ann has $a$ , so $14 = 8 + a + a$ or $14 = 8 + 2a$ . Using the addition property of equality, subtract 8 from both sides of the equation: $6 = 2a$ .<br>Using the multiplication property of equality, divide both sides by 2: $3 = a$ .<br>Ann has \$3 and Joe has \$11. |
| Early Functions<br><br>[using symbols in a meaningful way and mathematical modeling]<br><br>CCSSM: Represent and analyze quantitative relationships between dependent and independent variables | Use variables to represent two quantities in a real-world problem that change in relationship to one another  | $P$ is perimeter and $s$ is sides of a square.                                                                                                                                                                                                                                                                                                         |
|                                                                                                                                                                                                 | Write an equation to express the dependent variable in terms of the independent variable                      | $P = 4s$ , where $s$ is the independent variable and $P$ is the dependent variable.                                                                                                                                                                                                                                                                    |
|                                                                                                                                                                                                 | Use graphs and tables to analyze the relationship between the dependent and independent variables             | Using a table, choose different values for $s$ and find $P$ using the equation $P = 4s$ . Graph the points using $s$ as the independent variable and $P$ as the dependent variable to see that as the sides of a square increase, the perimeter increases.                                                                                             |

Figure 1. Hypothetical learning trajectory of instructional unit

The initial instructional unit constituted the hypothetical learning trajectory. The tasks in the instructional unit were sequenced to support students understanding of expressions and equations and then apply this knowledge to understand functions. The context of *soccer* was used as a real world context to help students think about the meaning of expressions and equations. Figure 2 shows the sequence of the initial instructional unit based on the hypothetical learning trajectory. This initial instructional unit was modified and refined continuously based on feedback from the classroom teacher and the second author and daily analysis of the data after each teaching episode.

| Lesson | Anticipated Number of Days | Title of Lesson                                      | Objective of Lesson                                                                                                                                                                                                                                               |
|--------|----------------------------|------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 1      | 1                          | Variable, Expression, and Equation                   | Students will explore meaning of variable expression and equation.                                                                                                                                                                                                |
| 2      | 2                          | Adding and Subtracting Like Terms                    | Students will develop an understanding of variables in mathematics and will learn that like terms can be added and subtracted. Students will also learn to model patterns with algebra.                                                                           |
| 3      | 2                          | Adding and Subtracting Like Terms                    | Students will develop an understanding of variables in mathematics and will learn that like terms can be added and subtracted. Students will also learn to model patterns with algebra.                                                                           |
| 4      | 2                          | Addition Property of Equality                        | This lesson will extend the students' previous understanding of the equal sign as an equal relationship between two sides of an equation. Students will learn how to isolate a variable on one side of an equation using the addition property of equality.       |
| 5      | 2                          | Multiplication Property of Equality                  | This lesson will extend the students' previous understanding of the equal sign as an equal relationship between two sides of an equation. Students will learn how to isolate a variable on one side of an equation using the multiplication property of equality. |
| 6      | 2                          | Equivalent Expressions and the Distributive Property | This lesson will extend the students' previous understanding of like terms as quantities that can be added and subtracted. Students will learn the distributive property.                                                                                         |
| 7      | 2                          | Structure of Algebraic Equations                     | This lesson will deepen students understanding of the meaning of the equal sign by looking at the systemic structure of algebraic equations.                                                                                                                      |

|    |   |                                                                                                                 |                                                                                                                                                                                                                                                                                                                                     |
|----|---|-----------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 8  | 2 | Solving Algebraic Equations using the Addition Property of Equality and the Multiplication Property of Equality | In this lesson, students will learn how to isolate the variable in an algebraic equation using the addition property of equality and the multiplication property of equality.                                                                                                                                                       |
| 9  | 2 | Learning Variables as Changing Quantities                                                                       | This lesson will introduce the students to the graph of a line and an equation of the line. Students will discover that a point on the line makes the quantities on each side of the equation equal. By finding different ordered pairs that work in the equation, the students will see that variables can be changing quantities. |
| 10 | 2 | Early Functions                                                                                                 | Students will learn that functions have three components: the domain, the range, and the rule. Students will learn that a function can be represented in different ways such as verbally, in an arrow diagram, algebraically, and graphically. Students will also solve the challenge problem from the first lesson.                |

Figure 2. Lesson sequence in expressions and equations instructional unit

### *Data Collection and Analysis*

Data sources included field notes (Maxwell, 2005), video recordings, and documentation of anything that occurred in the classroom such as students' work and researchers' reflections. Data was collected from each teaching episode. The lessons were video recorded and student work was documented and analyzed. Teacher interviews and student interviews were recorded, transcribed, and evaluated. We used an iterative design to test emerging theories and hypothesis of what we observed and how students were learning (Gravemeijer & Cobb, 2006; Lamberg & Middleton, 2009).

The teacher and the researchers had regular and ongoing discussions about the in-class interventions and interpretations of the data. These discussions were important because they enhanced the quality of the study and research process. This teaching experiment was an iterative process conducted in three phases. The first phase consisted of observation and pretest. The second phase consisted of teach/reflect/plan, interview, and prospective analysis. The third phase consisted of posttest and retrospective analysis. Data collection and analysis was a parallel process with prospective analysis occurring throughout the teaching experiment and retrospective analysis occurring after the teaching experiment (see Figure 3).

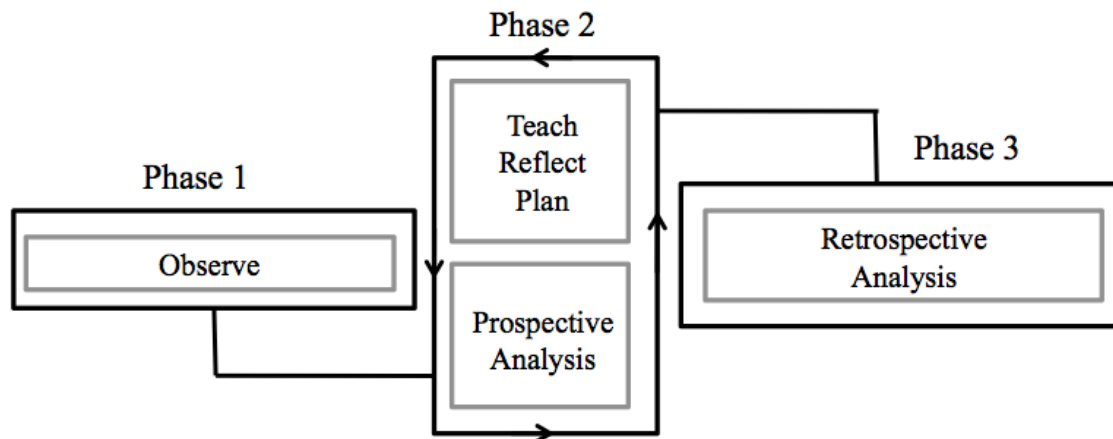


Figure 3. The iterative process of data collection and analysis in the teaching experiment

Phase 1 of the data collection and analysis consisted of observing the sixth-grade classroom in anticipation of the teaching experiment. The observations were intended to help us design the hypothetical learning trajectory, interventions, and instructional sequence. Exploratory observations and fieldnotes (Maxwell, 2005) occurred in the weeks prior to the teaching experiment.

Phase 2 of the data collection and analysis involved three parts: teach/reflect/plan, interviews/discussion, and prospective analysis. During phase 2, the actual teaching experiment took place for four weeks. The data collected in phase 2 included video of the teaching sessions, video of the student discussions and comments, and video and field notes of the teacher/researcher debriefing interviews, lesson plans, and student work.

*Teach/Reflect/Plan.* The teach/reflect/plan part of phase 2 included making decisions about the teaching approaches, the types of activities and tasks, and the order and ways that would address the objectives of the research (Molina, Castro, & Castro, 2007). The objective was to investigate the means of supporting and organizing student learning of algebra.

*Prospective analysis.* Data analysis was performed during two periods of the research process: throughout the study and after the teaching experiment had been completed. The early analyses of data were performed after each teaching episode. These prospective analyses informed the modifications of the interventions during future teaching episodes and facilitated creation and revision of hypotheses and conjectures (Molina et al., 2007).

Phase 3 of the data collection and analysis consisted of retrospective analysis.

*Retrospective analysis.* The final analysis was retrospective in that it included analysis of all the data collected during the teaching experiment. The product of this analysis was a historical explanation that detailed the pattern that emerged from the teaching experiment (Cobb et al., 2003).

The qualitative data generated by this study was voluminous (Patton, 1980) and the retrospective analysis consisted of formal analysis of the video recordings, student work, and field notes. According to Corbin and Strauss (2014), “concepts out of which the theory is constructed are derived from data collected during the research process” (p. 30). Therefore, the data was analyzed using a grounded theory approach to provide explanations about why events occurred within the teaching experiment and what these events mean in terms of student learning of algebra (Corbin & Strauss, 2014; Gravemeijer & Cobb, 2006). An adaptation of the Data Analysis Spiral (Creswell, 2007) was used to organize the analysis of the qualitative data. Creswell (2007) describes this process as “moving in analytic circles rather than using a fixed linear approach” (p. 150). Figure 4 shows an adapted version of

Creswell's (2007) Data Analysis Spiral for a teaching experiment using design research. After the instructional unit was complete, the data collection during phases 1, 2, and 3 was also complete.

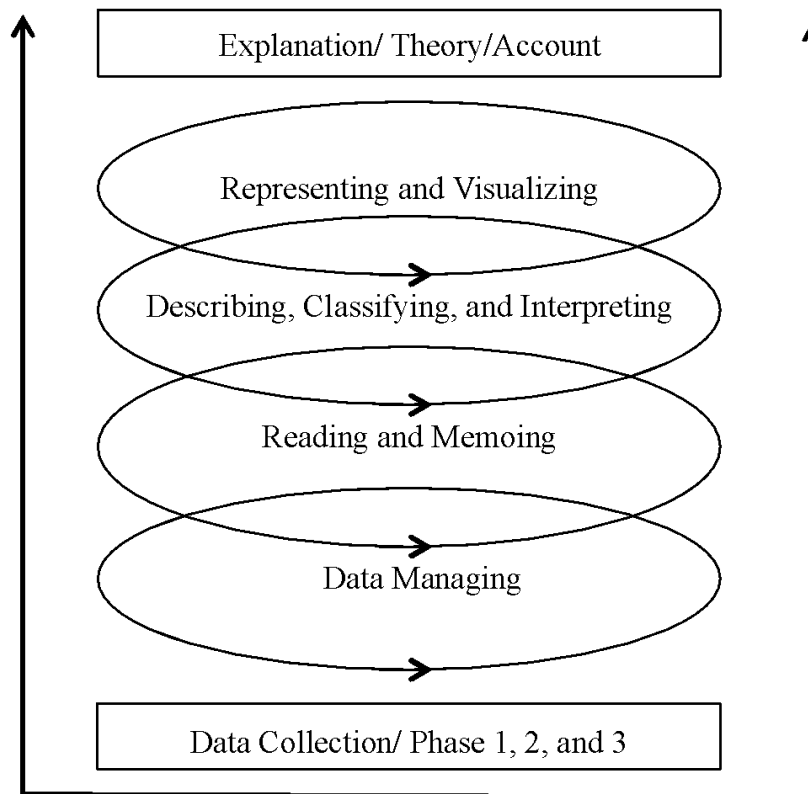


Figure 4. Data Analysis Loops (adapted from Creswell, 2007) for retrospective analysis of qualitative data from a teaching experiment using design research

## Results

The learning trajectory that emerged involved different types of thinking based on problem types. These types of thinking that emerged are shifts from: *Label Thinker to Formulaic Thinker, Substituter, to Solver and Functional Thinker*. These types of thinking corresponded to different problem types that increased in sophistication of difficulty. Students made sense of how letters and variables were used in various problem context in order to successfully solve problems. The understanding of the different meanings of letters and variables were critical for understanding expressions, equations and functions.

The meanings of letter or variable that emerged in the problem types are: *Letter as a Label, Letter as a Known Quantity, Variable as a Changing Quantity, Variable as an Unknown Quantity and as Independent and Dependent Variables*. The types of thinking emerged as the lessons unfolded. The types of thinking that emerged do not necessarily represent a linear progression; rather, they involve more sophisticated thinking. For example, understanding independent and dependent variables built on prior understanding of letters and variables. The following section elaborates on the kinds of thinking that emerged in the learning trajectory, the interpretation of letter or variable, problem types, errors and misconceptions, and mechanisms for shifting student thinking.

## Label Thinker

The *Label Thinkers* used letters to label a category or object and reasoned that a letter is a label for the name of a category, similar to Küchemann's (1981) findings of *letter used as an object* and Blanton's et al. (2017) *pre-variable* level of thinking in first-graders.

Figure 5 is an example of student work that demonstrates how a *Label Thinker* uses and interprets letters in an expression or equation.

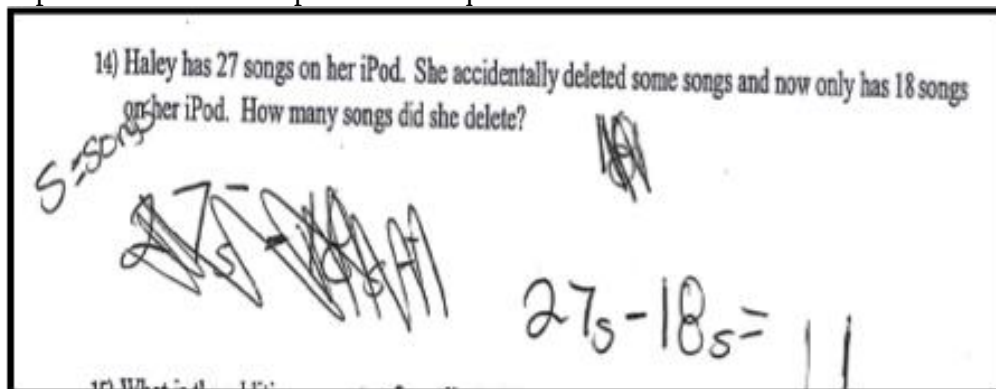


Figure 5. Example 1 of student work from a *Label Thinker*

In this example, the student labeled songs with the letter *s*. The student wrote “ $27s - 18s = 11$ ” and “ $s = \text{songs}$ ” where  $27s$  means 27 songs and  $18s$  means 18 songs. This finding is consistent with Küchemann's (1981) *Letter as Object*. The letter representing the object represents 1 unit.

In Figure 6, students were asked to find the total of two groups, boys and girls, by writing an algebraic expression or equation.

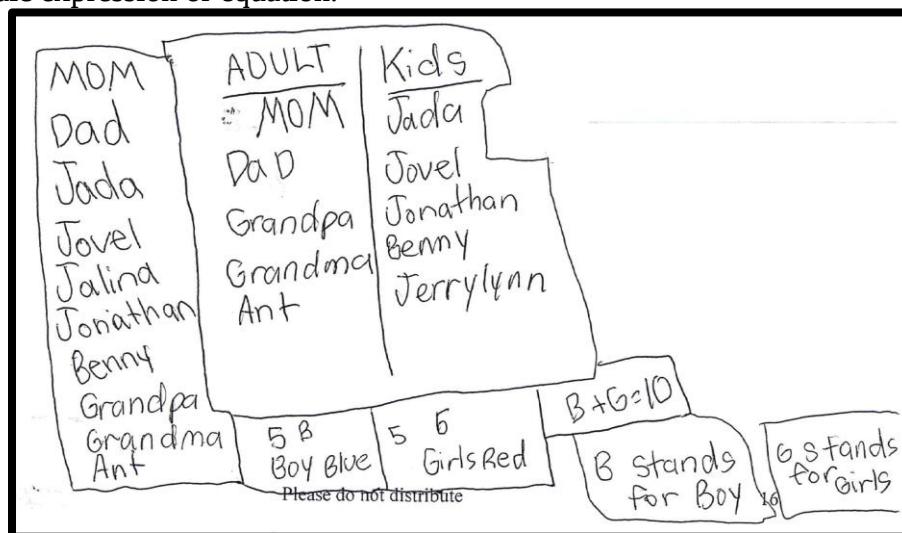


Figure 6. Example 2 of student work from a *Label Thinker*

In this example, the student listed family members and separated them into two categories: adults and kids. The student then counted the total number of boys ( $B=5$ ) and the total number of girls ( $G=5$ ) and wrote “ $B$  stands for Boy” and “ $G$  stands for Girls” and “ $B + G = 10$ ”. The letter represents a specific quantity within a category. This is similar to *Letter Evaluated* where the letter is assigned a numerical value (Küchemann, 1981). The student also wrote “ $5B$  Boy Blue” and “ $5G$  Girls Red” where  $5B$  means 5 boys and  $5G$  means 5 girls.

This is consistent with Küchemann's (1981) findings of *Letter Used as an Object* where a letter is shorthand for an object.

In another activity, students were asked to count the number of hexagons and pentagons in a soccer ball and represent the quantity as an equation. The letter represents one unit of the category. The students used letters to keep track of their counting. Our aim was for students to connect quantity with letter to shift their thinking from a letter representing a category such as a ball to thinking that it represents a quantity within a category.

- Ashley: I have 16 h's plus 4 h's and that equals 20.
- Teacher: 20 what?
- Ashley: 20 h's.
- Teacher: Which represents? What does h mean?
- Ashley: Hexagons.
- Teacher: Why did you write it 20 times?
- Ashley: There are 20 hexagons on the ball.
- Teacher: If I write a single h, what does that represent?
- Ashley: 1 hexagon.
- Teacher: So to show 20 hexagons we write?
- Ashley: 20h.

In this example, Ashley used the letter *h* to label hexagons. In the third to the last line of the transcript, Ashley might have made the connection that in  $20h$ , the *h* is a quantity of one hexagon. Figure 7 shows Ashley's work.

The image shows a piece of paper with handwritten work in blue ink. At the top, the equation  $4h + 16h = 20h$  is written. Below this, there are two rows of individual letters 'h' separated by plus signs. The first row contains 16 'h's, and the second row contains 4 'h's. The final part of the second row is followed by an equals sign and the expression  $20h$ , indicating the total count.

Figure 7. An image of Ashley's work as she represented hexagons with the letter *h*

*Mechanisms for shifting thinking.* To help shift students' thinking from *Label Thinker* to *Formulaic Thinker*, the teacher asked students questions such as, "What does the letter

mean?” and “How can we represent the number of hexagons and pentagons in an expression?” If students wrote an expression such as  $3h + 5h + 3p$ , then the teacher emphasized that the coefficient represents the number of units of a category, while recognizing that the letter is the category or a unit of one. On the other hand if students wrote an expression such as  $h + p$ , and the letters represented a specific category, then the teacher emphasized that each letter represents the number of items in each category or a known value (e.g.,  $h$  is the number of hexagons that a student can count on a soccer ball). In this instance, a distinction between a label representing 1 unit and multiple units was made.

### Formulaic Thinker

The *Formulaic Thinker* used letters to keep a record of a known value (e.g.,  $h + p$ ). Blanton et al. (2017) found similar results with first-graders and suggests that thinking of letters as representing variables with quantities reflects a fundamental shift in student thinking about variables. Küchemann (1981) noted this as the *Letter Evaluated* category of children’s interpretations of letters.

Figure 8 is an example of student work that demonstrates that the student is able to write a formula for the known quantities of different colors of candies. In this example, the student represented the number of red candies and the number of yellow candies with  $r + y$ . However, this same student represented blue with  $Bl$  and brown with  $Br$ , demonstrating that this student might be a *Label Thinker* that is transitioning to a *Formulaic Thinker*.

1. Write the number of each color below.

|          |           |
|----------|-----------|
| Red: 1   | Green: 7  |
| Blue: 4  | Orange: 4 |
| Brown: 1 | Yellow: 1 |

2. What **variable** should you use for each color?

|           |           |
|-----------|-----------|
| Red: R    | Green: G  |
| Blue: Bl  | Orange: O |
| Brown: Br | Yellow: Y |

3. Write an **expression** below for the number of red M&Ms and the number of yellow M&Ms in your bag.

~~scribbles~~ ~~scribbles~~  $r + y$

Figure 8. The student represented the number of red candies with  $r$  and the number of yellow candies with  $y$  and wrote an expression  $r + y$

Figure 9 shows another student’s work from the same activity in which the student, in the *Label Thinker* level, wrote  $1R$  for one red and  $2Y$  for two yellow.

1. Write the number of each color below.

|          |           |
|----------|-----------|
| Red: 1   | Green: 8  |
| Blue: 4  | Orange: 4 |
| Brown: 0 | Yellow: 2 |

2. What **variable** should you use for each color?

|          |           |
|----------|-----------|
| Red: R   | Green: G  |
| Blue: B  | Orange: O |
| Brown: b | Yellow: Y |

3. Write an **expression** below for the number of red M&Ms and the number of yellow M&Ms in your bag.

~~1R + 2Y~~  
~~Y + B~~

$1R + 2Y$

Figure 9. In this student work, the student used letters to label red and yellow

*Mechanisms for shifting thinking.* To help shift students' thinking from a *Formulaic Thinker* to a *Substituter*, the teacher asked students, "What is the value of the variable?" For instance, some students assumed that  $r = 3$  is the same as  $3r$ . In  $r = 3$ , the value of  $r$  is 3. In  $3r$ , the value of  $r$  is one unit.

### Substituter

A student at the *Substituter* level of thinking can make a one-to-one correspondence with a letter and a known quantity, similar to Küchemann's (1981) category of *Letter Evaluated*. Students represent situations with algebraic expressions, understand that a letter represents a known quantity, and can solve substitution problems such as in Figure 10.

~~4p + 1s~~

$4p + 1s$

$P = \$6.00$   
 $S = \$1.00$

$4(\$6.00) + \$1.00 = \$24.00 + \$1.00 = \$25.00$  At Walmart

$P = \$8.00$   
 $S = \$2.00$

$4(\$8.00) + \$2.00 = \$32.00 + \$2.00 = \$34.00$  At Safeway

$P = \$7.00$   
 $S = \$1.00$

$4(\$7.00) + \$1.00 = \$28.00 + \$1.00 = \$29.00$  At Baley's

$P = \$4.50$   
 $S = \$.75$

$4(\$4.50) + \$.75 = \$18.00 + \$.75 = \$18.75$  At Sak-n-Save

Figure 10. In this example, the student wrote a formula, replaced the letter with a given quantity, and simplified the expression

The problem context in Figure 10 was buying packs of cupcakes for different prices at different stores. This context related to the students' lives because they already understood that prices change at different stores. The teacher worked with the students through a whole class discussion to make sense of the example above.

- Leah: In the cupcake problem, the variable was the cost of the cupcakes.
- Teacher: Good. Anyone else want to add to that?
- Ian: A quantity was assigned to the variable.
- Teacher: We have a letter or symbol that represents a quantity.  
What about this quantity though? Remember when we are talking about the cupcakes?
- Class: The cost.
- Teacher: We had different stores. A quantity that can what?
- Class: Change.
- Teacher: Good. A quantity that can change.
- Jennifer: At every single store the cost changed.
- Teacher: So, can I use this formula at any store?
- Chris: Yes, as long as you are buying cupcakes. If you are buying cupcakes, then you need the price of the 6 pack of cupcakes and the price of a single cupcake.
- Teacher: Okay, anyone have anything to add to that? Or anything different?
- Mary: I disagree because some stores might not have packs of six.
- Teacher: Anything different?
- Gina: I say you can because you are going to have to buy the exact same thing. You are going to have to buy 1 cupcake and 4 packages of six but it will just be a different cost.

Students at the *Substituter* level of thinking realized that the letter was a placeholder for a specific quantity depending on the context. The students knew that there was a different, *known value*, for the prices of cupcakes and that these prices replaced the letter in the expression. Figure 11 provides another example of student work that shows a student at the *Substituter* level.

1.  $x + y$  where  $x = 35$  and  $y = 10$   
 $35 + 10 = 45$

2.  $2f$  where  $f = 7$   
 $2 \cdot 7 = 14$

3.  $5z - 3t$  where  $z = 4$  and  $t = 5$   
 $5 \cdot 4 = 20$   
 $3 \cdot 5 = 15$   
 $20 - 15 = 5$

Figure 11. This is an example of student work that shows that the student understood that the letter was a placeholder for a specific, known quantity

*Mechanisms for shifting thinking.* To shift students' thinking from a *Substituter* to a *Solver*, the teacher asked students to identify known values and substitute known values for the letters in a given expression. The teacher asked, "Do we know the value of  $x$  or do we need to find the value of  $x$ ?" In expressions that need to be evaluated, the letter is a specific, known value (e.g.,  $x + y$ , where  $x = 2$  and  $y = 3$ ); in equations that need to be solved, the variable is an unknown value (e.g.,  $x + 3 = 5$ ). The teacher also prompted students to identify whether they were given an expression to evaluate or an equation to solve.

### *Solver*

A student in the *Solver* level of thinking can solve an equation for an unknown value by balancing an equation. A *Solver* understands that the variable represents a quantity and knows how to find its value. Students transitioned from the *Substituter* level of thinking, where they were *given* the value for the letter, to the *Solver* level, where they had to *find* the value for the letter. A student in the *Solver* level of thinking has to realize that the value of the unknown is not provided and it is necessary to solve the equation to find the value, similar to Küchemann's *Letter Used as a Specific Unknown*. The following whole class discussion demonstrates student thinking in the *Solver Level*.

- Teacher: We have  $6 + x = 9$ . What do we have to do?
- Ian: Solve.
- Teacher: Solve for the unknown  $x$ . Do we know what  $x$  is?
- Joanna: Yes. You subtract 6 from 9 and get 3.

These students understood that they had to solve for the unknown value. This scenario is different than if the value for the unknown was provided. Figure 12 provides an example of student work where the students were in the *Solver Level* of thinking and the letter represents an unknown value.

|                    |                                                                                    |                                                                                                          |
|--------------------|------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------|
| 7.<br>$z + 5 = 17$ | $\begin{array}{r} z + 5 = 17 \\ -5 \quad -5 \\ \hline z = 12 \end{array}$ $z = 12$ | I wrote the problem down.<br>I did the opposite, adding<br>(subtraction) to both sides<br>I got $z = 12$ |
|--------------------|------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------|

Figure 12. This shows student work from two students that were in the *Solver Level* of thinking and thought about solving for an unknown

*Mechanisms for shifting thinking.* To shift students' thinking from *Solver* to *Functional Thinker*, the teacher provided a context that allowed the variables in an equation to be changing quantities (e.g. the formula for the perimeter of a square,  $P=4s$ ). The teacher said that there were different sized squares that had different side lengths, and thus, the perimeter for each square varied.

### *Functional Thinker*

A *Functional Thinker* understands the variable as a changing quantity and knows that a relationship exists between independent and dependent variables. In the four levels of thinking prior to *Functional Thinker*, the letter in an expression or an equation could be a label for a category, a placeholder in a formula, a known value to be substituted, or an unknown value to find. Students were introduced to independent and dependent variables by writing a formula for the perimeter of any square. This activity forced students to relate a letter as known value to variable as a changing quantity in one equation. Learning independent and dependent variables promoted relational thinking in that students had to see the relationship between these variables to learn functions. The following is a whole class discussion that occurred after students had time to work and think in small groups about writing a formula for the perimeter of a square.

Teacher: What is one of the equations you came up with?

Joanna:  $4s = p$

Chris:  $n$  times 4 equals  $p$

Eric:  $a + a + a + a$  equals  $p$

Mary: Also known as  $4a$

Ashley:  $4d$  equals  $p$

Leah:  $4x$  equals  $p$

Frank:  $4y$  equals  $p$

- Teacher: Are there any others that are not up here?
- David:  $b$  times 4 equals
- Teacher: Are there any others without just changing the variables?
- Ian:  $2x + 2x = p$
- Gina:  $3a + a = p$
- Teacher: So, what are all of these equations?
- Alex: Equivalent equations.
- Teacher: What does equivalent mean?
- Class: Equal.
- Teacher: Equal or the same.  $4s = p$  is called a function. It has input and output values. What that means is that I can input a number here and then get an output here. If I input, I can input any number and get an output.

Next, students made an arrow diagram (Figure 13) to show the inputs and outputs, varying the quantities of the sides and perimeters.

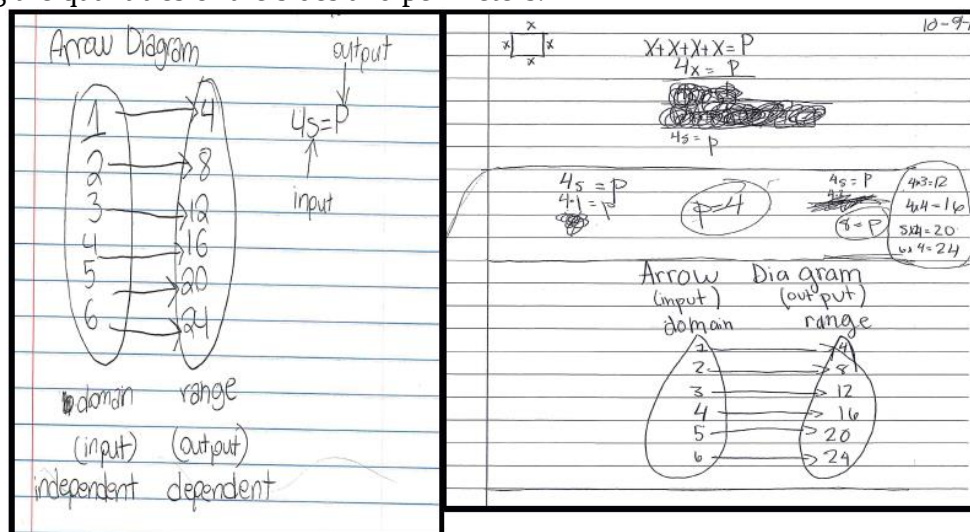


Figure 13. This shows the work of two students that were in the *Functional Thinker* level of thinking

For students to be *Functional Thinkers*, they had to also be *Formulaic Thinkers*, *Substituters*, and *Solvers*. *Formulaic Thinkers* use letters to keep a record of a known quantity (i.e.  $P$  for perimeter and  $s$  for sides). *Substituters* make a one-to-one correspondence with a letter and a known quantity (i.e. If  $s = 1$ , then  $P = 4$  in the equation  $4s = P$ ). *Solvers* can solve an equation for an unknown value (i.e. If  $P = 24$ , then  $4s = 24$ , so  $s = 6$ ). In the *Functional Thinker* level of thinking students had to combine the prior levels of thinking to understand that the variable is a changing quantity and know that a relationship exists between independent and dependent variables, similar to Küchemann's (1981) categories of *Letter Used as a Generalised Number* and *Letter Used as a Variable*. Moreover, students

used letters to represent variables as mathematical objects (Blanton et. al, 2017) and used variables to represent functional relationships (Blanton, Brizuela, et al., 2015).

## Discussion

The sixth-grade CCSSM (NGA/CCSSO, 2010) specifies that students must learn expressions and equations, but does not address how to present these concepts and research on how to specifically teach these sixth-grade concepts is lacking. Clements and Sarama (2004) state that the “overarching research goal the field of learning trajectories is to generate knowledge of learning and teaching. Therefore, scientific processes (e.g., documenting decisions, rationales, and conditions; hypothesizing mechanisms; predicting events and checking those predictions) must be carefully followed and recorded.” (p. 85). We discovered that the design decisions that we made to support student learning was influenced by how students interpreted the meaning of variables and problem types presented that were increasingly more sophisticated.

A learning trajectory (Clements & Sarama, 2004) based on students’ sense making of the meaning of letters that represent numbers in expressions and equations emerged as students deepened their mathematical understanding. The findings of how students interpreted with variable are consistent with Küchemann’s (1981) findings of how students interpreted letters and variables. Real world contexts supported students to use natural language and generalize the meaning of variable as indicated by prior research (Herscovics & Linchevski, 1994). The levels of thinking in the learning trajectory that emerged: *Label Thinker*, *Formulaic Thinker*, *Substituter*, *Solver*, and *Functional Thinker* (See Figure 14).

| Problem Types and Action Required                                                                                                                               | Letter or Variable in Problem Context                                                                                                                                                                                                                                                  |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p>EXPRESSIONS</p> <p><i>Label Thinker</i> – Find totals of various categories. Keep track of units within a category or categories</p>                         | <p><i>Letter as a label of a category</i> that represents 1 unit (e.g. 3 red and 2 green expression <math>3r+2g</math>)</p> <p><i>Number as a letter</i> where the letter represents a specific quantity within a category. (e.g., 5 boys and 3 girls expression <math>B+G</math>)</p> |
| <p>EXPRESSIONS AND EQUATIONS</p> <p><i>Formulaic Thinking</i> – Represent problem situation in an expression so that multiple situations can be figured out</p> | <p><i>Variable as a changing value</i> (e.g., Find price for 4 six pack cupcakes and a single cupcake expression <math>4c+S</math>)</p>                                                                                                                                                |
| <p>EXPRESSIONS AND EQUATIONS</p>                                                                                                                                | <p><i>Letter as a known quantity</i> (e.g., <math>c=2</math> and <math>S=3</math>. Substitute values to solve problem <math>4(2) +3=11</math>)</p>                                                                                                                                     |

|                                                                                                                                                                        |                                                                                                                                                         |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------|
| <i>Substituter</i> – Letter represents a known value that can be substituted into an equation to solve the problem.                                                    |                                                                                                                                                         |
| <b>EQUATIONS</b><br><i>Solver</i> – Variable is an unknown quantity and the equation must be balanced to solve. Then the variable becomes a letter as a known quantity | <i>Variable as unknown quantity</i> (e.g., $x + 5 = 11$ ). Balance the equation to solve for $x$ .<br><br><i>Letter as a known quantity</i> , $x = 6$ . |
| <b>EQUATIONS AND FUNCTIONS</b><br><i>Functional Thinker</i> – Able to understand dependent and independent variable to solve problems                                  | <i>Letter as known quantity and variable as a changing value</i><br>(e.g., $4s = \text{perimeter}$ , $s=1$ than $p=4$ , when $s=2$ , $p=8$ etc.)        |

Figure 14. Realized learning trajectory

Students in the *Label Thinker* level used letters to label a category or *Letter as Object* (Küchemann, 1981). The equations they generated involved finding a total. For example, they represented the total number of kids (the letter  $t$ ) as being made up of boys (the letter  $b$ ) and girls (the letter  $g$ ) or  $b + g = t$ . At this point, they were engaging with additive reasoning and thinking about the letter as a representation of a specific category. Students initially used the equal sign to represent the answer and learned like terms can be combined when they began to explore problems that required a generalization of a situation (Kaput, 2000; Moseley and Brenner, 2009; Russell, Schifter, & Bastable, 2011). The *Letter as Known Quantity* involved using a letter to represent a specific quantity in a category. This is similar to Küchemann's (1981) category, *Letter Evaluated*.

A student in the *Formulaic Thinker* level used letters to keep a record of a quantity that has the feel of a known quantity. The letter in a formula represented a quantity that had multiple values depending on the problem context. This level involved building on their prior understanding that the letter labelled a category. In this level, expressions represented generalized situations and led students to think about the meaning of the expression and also compare equivalent expressions, engaging them in relational thinking (Kieran & Chalouh, 1993; Molina, Castro, & Ambrose, 2005).

Once students understood that the letters could represent a known quantity, they entered the *Substituter* level where a letter could be substituted for a given value. However, we discovered that they were confused by the symbolic representation (e.g.,  $P = 5$ ). This notation did not require an action other than to indicate the value of  $P$ . As students began to understand this relation, they were able to use it to find the value of an expression by replacing that letter with the known value. Furthermore, once the numerical value was substituted into an expression, students needed to think about why a representation of the letter in the expression was no longer necessary. Instead, they were able to perform the arithmetic computation to simplify the expression.

Up until this point, students thought about the letter representing a known value. A student in the *Solver* level of thinking could solve an equation for an unknown value by balancing the equation. In this level of thinking, the variable represented an unknown quantity that could be found by isolating the unknown on one side of the equation. *Solvers*

identified the value of the unknown by performing computations using inverse operations. This process involved thinking about the meaning of the equal sign as a relationship or balance.

A student in the *Functional Thinker* level had to simultaneously attend to multiple meanings of letters that represent quantities to solve problems involving functional relationships. A *Functional Thinker* understands that a variable is a changing quantity. Similar to Blanton et al.'s (2017) study with first-graders, there appeared to be a progressively more complex understanding of the meaning of the variable as students advanced through these levels of thinking. The understanding of the different meanings of letters in expressions and equations was critical for students to successfully understand different problem types in various problem contexts. Furthermore, it appeared that the progression of this learning trajectory deepened student understanding of the meaning of variables.

Researchers have identified the importance of students learning about the different meanings of letters and variables (Blanton, 2008; Blanton et al., 2017; Carpenter, Franke, & Levi, 2003; Kaput, 1999; Küchemann, 1981). Our results of sixth-grade students' understandings of expressions, equations, and functional relationships was impacted by their thinking of the meaning of letters and is not unlike levels of thinking about variables observed in young children's thinking. Variables are commonly specified in algebra textbooks as unknown, changing, or constant. Developing problem types and carefully sequencing them based on how students understand letters in expressions and equations should be carefully considered to support students understanding of algebra. This research provides a starting point for teachers and curriculum to develop tasks to support student learning.

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