Early-Years Prospective Teachers' Definitions, Examples and Non-Examples of Cylinder and Prism

Fadime ULUSOY
Kastamonu University
fadimebayik@gmail.com

This study investigates early-years prospective teachers' definitions, examples, and non-examples of cylinders and prisms. The data were collected from 72 early-years prospective teachers through written tasks and classroom discussions. In written tasks, they were requested to define and then to give different examples and non-examples of cylinders and prisms. The results indicated that prospective teachers defined cylinders based on prototypical cylinder with circle bases without mentioning all critical attributes of cylinder. They mostly provided correct examples and non-examples for cylinders. They did not pay attention to critical and non-critical attributes of prism. Some prospective teachers drew cones and pyramids as the examples of prisms. Their daily-life examples had similarity with the examples presented in textbooks and other curriculum materials. This study revealed that there are relationships between prospective teachers' definitions, daily-life examples, drawings and non-examples. Possible reasons of these results are discussed and elaborated.

Keywords: definition; early-years teachers; definition, examples; non-examples; geometry

Geometry has an important role in the interpretation of physical environment and development of spatial visualization and problem-solving abilities (e.g. Jones, 2002). Children have more potential and capacity to learn and develop understanding about geometric concepts than traditionally recognized even in early-years (Clements, Swaminathan, Hannibal & Sarama, 1999; Sarama & Clements, 2009). Providing a strong base in early-years is necessary to support the development of children's geometric thinking (Tsamir, Tirosh, Levenson, Barkai & Tabach, 2015; van Hiele & van Hiele, 1958). In this sense, National Council of Teachers of Mathematics [NCTM] (2000) implies that "instructional programs should enable all students to recognize, name, build, draw, compare, and sort two- and three dimensional shapes; describe attributes and parts of two- and three-dimensional shapes (p. 96)" in K-2 geometry standards. In early-years, children begin to learn a shape as a whole if its appearance approximates that of the target shape (Aslan & Aktas-Arnas, 2007; Sarama & Clements, 2009; van Hiele, 1986). Furthermore, children aged 3-6 can describe and name geometric shapes and talk about which shapes are alike and what their differences are. Although young children begin to recognize attributes of geometric shapes they may not realize which attributes are critical or non-critical for identifying a shape (van Hiele & van Hiele, 1958) due to intuitive knowledge, which is self-evident and immediate, is often derived from experiences in everyday life (Fischbein, 1987; Vinner, 1991). However, since intuitions do not generally depend on logical and deductive reasoning about geometric concepts, learners can immediately generate invalid examples (non-examples) of geometric concepts that bear similarity with valid examples of the concept interpretation (Fischbein, 1993). Yet, geometric concepts are abstract ideas which stem from formal concept definitions (Tsamir, Tirosh & Levenson, 2008) and one of the aim of education in early-years is to promote children's concept images in line with the concept definition flexibly and correctly (Tsamir et al., 2008; Tsamir et al., 2015). In this regard, it is crucial to give suitable guidance and intervention before children develop inflexible and incorrect conceptions based on their intuitions. Some researchers found that there is a link between the quality of early childhood education and adolescents' academic achievements (Vandell et al., 2010).

The early-years teachers have crucial roles in fostering children's mathematical thinking (Tsamir et al., 2015). Teachers' subject matter knowledge is a key component of effective teaching that influences on student achievement (Ball, Thames & Phelps, 2008; Blömeke & Delaney, 2012; NCTM, 2000). In mathematics, teachers' knowledge of definition and concept images of geometric shapes influence their didactic approaches used in instructional process (Leikin & Zazkis, 2010), ways of representing concepts to their students and ways of orchestrating classroom discussions. Although there is an emphasis on the necessity of early-years teachers' correct use of mathematical language in order to prevent misconceptions (NCTM, 2000; Tsamir et al., 2015), early-years teachers may not always be ready to teach geometry without developing misconceptions in students' minds (Clements & Sarama 2011; Lee & Ginsburg, 2009). Considering the crucial role of learning geometry in early-years, several countries have prepared guidelines about how to teach geometric concepts to children aged 4–6 years. In these guidelines, they focused on definition and identification of two- and three-dimensional geometrical shapes. However, in Turkish National Mathematics Preschool Curriculum (TNMPC, 2013), there is only one objective about teaching geometry, i.e. students should "recognize geometric shapes (p. 22)". For the objective, following indicators are provided: Students are able to call the name of shapes and identify daily-life examples similar to the target shape. TNMPC (2013) provides some additional explanations such as "during the activities teachers should be careful when using the terms of triangle, circle, square, rectangle and ellipse correctly." There is no explanation about learning and teaching of three-dimensional geometric shapes in the curriculum.

In related literature, there are various studies on prospective mathematics teachers' knowledge of geometric solids (e.g. Bozkurt & Koç, 2012; Ertekin, Yazıcı & Delice, 2014; Gökkurt & Soylu, 2016; Koçak, Gökkurt-Özdemir & Soylu, 2014; Ubuz & Gökbulut, 2015; Unlu & Horzum, 2018). These studies present that prospective mathematics teachers have difficulties in defining geometric solids by using correct and precise mathematical language, identifying critical attributes and naming the shapes. Similar problems were also found in middle school students' understanding of prisms (Türnüklü & Ergin, 2016). However, the number of studies about early-years teachers' knowledge of geometric solids are limited in both national and international scale (Tsamir et al., 2015). That is, the details of early-years teachers' and young children's conceptualization of three-dimensional geometrical shapes is not known. Early-years education prepares a base to learn geometric shapes to children. At this point, teachers have a great role to help children use appropriate terminology in their descriptions of two- and three-dimensional shapes (NCTM, 2000). Early experience with geometrical shapes establishes a strong foundation for more-formal geometry in following years. For this reason, teachers need to know critical and non-critical attributes of geometric shapes. If early-years teachers do not learn geometric concepts perfectly, this situation may cause misconceptions, inflexible concept formation and prototypical images about geometric shapes. From this point of view, this study aims to investigate early-years prospective teachers' definitions, examples and non-examples regarding cylinders and prisms. The following research questions guide this study: (1) How do early-years prospective teachers define cylinders and prisms? What kinds of terms do they use when defining cylinders and prisms? (2) How do they exemplify cylinders and prisms? (3) What do prospective teachers offer as cylinder-like and prism-like daily-life examples? (4) What kinds of shapes do they draw as non-cylinders and non-prisms?

Theoretical Framework

Examples, Non-Examples and Prototypes

A geometric concept consists of a collection of examples. Examples have both critical and non-critical attributes (Hershkowitz, 1989). Researchers suggest that it is important to examine how an individual understands and identifies the critical elements of a concept in order to get information about the learner's personal example space (Watson & Mason, 2005). The critical attributes of a mathematical concept are connected to concept definition. However, non-critical attributes can be obtained by reasoning about the formal definition of a concept that includes both necessary and sufficient defining conditions (Tsamir et al., 2008). For instance, with regards to quadrilateral, critical attributes involve a planar shape, a closed shape, four sides, four vertices and four angles. However, the orientation and size of a shape are non-critical attributes of a quadrilateral (Levenson, Tirosh, & Tsamir, 2011). Although the entire set of critical attributes must be contained by all examples of a concept, non-critical attributes are included in only some typical examples of the concept. In another important study, Tsamir et al. (2008) established a relationship between children's intuitions, examples and non-examples by considering the self-evident, immediate, certain, global and coercive nature of intuitions (Fischbein, 1993). Tsamir et al. (2008) mentioned two types of non-examples such as *intuitive non-examples* and *non-intuitive non-examples*. Intuitive non-examples are accepted as non-examples intuitively and immediately by children without needing a justification, while non-intuitive non-examples are frequently identified as examples by children because they have significant similarities with valid examples of the concept. For this reason, identification of examples and non-examples considering the role of intuitions is the first step in establishing appropriate concept images (Levenson et al., 2011).

Concept image is the set of all the mental representations associated to the students' images that appear when they hear the concept name. The image might be nonverbal and implicit. On the other hand, *concept definition* constitutes a form of words which are used to specify the concept (Vinner, 1991). According to this framework, suitable and robust interactions between concept definition and concept image might guarantee the conceptual learning rather than instrumental ones. Unfortunately, learners do not make sense to link these elements appropriately because there might be irrelevant properties about the concept evoking in students' mind specifically. For instance, the results of some studies indicate that students at different grade levels have a concept image of equilateral triangle as a triangle with a right angle or slanted sides of equal length (Burger & Shaughnessy, 1986; Clements & Battista, 1992). According to the researchers, if students encounter limited examples that have common figural features of a geometric concept at school or in another context, these examples lead to a *prototype phenomenon* (e.g. Hershkowitz, 1990). The prototype examples are usually in "the subset of examples that has the longest list of attributes all the critical attributes of the concept and those specific (noncritical) attributes that had strong visual characteristics" (Hershkowitz, 1990, p. 82). The results of relevant studies indicate that students are not at the expected level of understanding of geometric concepts due to the overuse of prototypical examples (Hannibal, 1999; Ulusoy, 2015, 2016).

Development of Geometrical Thinking in Early-years

van Hiele (1986) proposed a framework to explain the development of geometrical thinking at five levels: visualization, analysis, abstraction, deduction and rigor. At the first level, children recognize shapes as a whole based on appearance and make a classification by comparing them with prototype examples (Burger & Shaughnessy, 1986; Clements, 1999; Clements et al., 1999; Hannibal & Clements, 2000; Jaime & Gutierrez, 1994; van Hiele, 1999). At this level, children do not pay attention to which attributes are critical or non-critical. At the second level, children define a geometrical shape considering its attributes (Burger & Shaughnessy, 1986; Clements et al., 1999; Hannibal & Clements, 2000; van Hiele, 1999). At the third level, learners perceive relationships between properties and shapes, try to give a verbal definition and simple arguments to justify their thinking. Based on van Hiele theory, most kindergarten children are commonly labelled at the visual level (Kellough, Carin, Seefeldt, Barbour & Souviney, 1996; Seefeldt & Barbour, 1998). However, recent studies concerning the development of geometrical thinking in early-years criticized to the van Hiele theory (Clements & Battista, 1992; Clements et al., 1999; Hannibal & Clements, 2000; Levenson et al., 2011; Tsamir et al., 2008). Clements et al. (1999) think that visual level in the van Hiele theory is not sufficient to explain and reflect pre-school children's geometric thinking since the theory focused on older children's development of geometrical thinking. They claimed that there is no clear-cut in the transition from visual level to analysis level as identified in the van Hiele theory. Similarly, Clements and Battista (1992) noted that there is not a clear transition from the first and the second level of geometrical thinking since children identify the shapes both by comparing them to the visual prototype and by focusing on critical attributes. As another idea, using the van Hiele levels of geometrical thought, Tsamir et al. (2008) grouped pre-school children's reasoning into visual reasoning and reasoning based on the shape's attributes. They considered Clements et al.'s (1999) suggestion that says naming figures as shapes or objects is also a type of visual reasoning. Tsamir et al. (2008) also categorized visual reasoning into two groups such as naming and purely visual reference to the whole figure. Furthermore, they divided the second level of van Hiele theory into two categories such as reference to critical attributes and reference to non-critical attributes. Therefore, researchers made an effort to elaborate ideas about pre-school children's development of geometrical thinking. From this point, it is important to investigate early-years teachers' knowledge of geometric solids in order to support the quality of their knowledge appropriately.

Studies on Teachers' Conceptions of Cylinders and Prisms

Prism is defined as a solid that is composed of two congruent closed curves in different parallel planes and all the line segments connecting these two curves (Clapham & Nicholson, 2009). Based on this definition, the base of the cylinder may be any curve (e.g. ellipse, polygon or circular region). However, in another source, a cylinder is defined as "a solid figure composed of two congruent circles lying on parallel planes and all of the line segments connecting these two circles" (Tsamir et al., 2015, p. 506). Although there is a strict distinction between cylinders and prisms in many sources, some sources emphasize the relationship between cylinders and prisms (Rutland & Hosier, 1961; Van de Walle, Karen & Bay-Williams, 2013) because they focus on the necessity of making more general definition of the cylinders to build a relationship between cylinders and prisms (Van de Walle et al., 2013). For example, Van de Walle et al. (2013) define prism as "a cylinder with polygons for bases and all prisms are special cases of cylinders" (p. 413). In Figure 1, there are some examples and non-examples of cylinders. In non-examples of cylinders while some shapes don't have two congruent bases, some shapes have only one base. Furthermore, some shapes have two bases in non-parallel planes. In addition, there are nonlinear segments connecting the points on the bases in some shapes.

Cylinders Prisms Non-cylinders

Figure 1. Examples and non-examples of cylinders and prisms (Van de Walle et al., 2013).

In recent years, there has been an increasing attention to learners' concept images and definitions of two- and three-dimensional geometric shapes. However, researchers mostly focused on prospective mathematics teachers' knowledge of geometric solids (Bozkurt & Koç, 2012; Ertekin et al., 2014; Gökkurt & Soylu, 2016; Horzum & Ertekin, 2018; Koçak, Gökkurt-Özdemir & Soylu, 2014; Ubuz & Gökbulut, 2015; Unlu & Horzum, 2018). Furthermore, some researchers focused on middle school students' concept images of geometric solids (Türnüklü & Ergin, 2016) and the presentation of geometric shapes in early-years books (Nurnberger-Haag, 2017) and mathematics textbooks (Bozkurt, 2018). However, there are a few studies that focused on how early-years teachers or students define and exemplify cylinders and prisms (Tsamir et al., 2015). Studies indicated that prospective middle school mathematics teachers have difficulties in distinguishing critical attributes (Ubuz & Gökbulut, 2015; Gökkurt & Soylu, 2016) and naming prisms (Horzum & Ertekin, 2018). For example, Bozkurt and Koç (2012) investigated 158 prospective mathematics teachers' knowledge of prisms. They found that prospective teachers had difficulties in defining prisms with inadequate use of mathematical language and focused on only three dimensionality and two bases for prisms. Similarly, Koçak et al. (2014) concluded that prospective middle school mathematics teachers drew only a right circular cylinder by defining limited number of critical attributes based on prototypical example (right circular cylinder). Additionally, their common daily-life examples consisted of pipe and pencil. In addition to the studies about prospective mathematics teachers' knowledge, Tsamir et al. (2015) found that early-years teachers had difficulties in both identifying examples and non-examples of cylinders and defining cylinders. Furthermore, Nurnberger-Haag (2017) found that cylinders and spheres were labelled circles incorrectly in the children's books. For example, coins (thin cylinders) were presented as the examples of circles. As a result of literature review, it can be concluded that studies on early-years prospective teachers' conceptions of geometric solids are very limited.

Methodology

In this study, qualitative research design was utilized in order to investigate early-years prospective teachers' conceptions of cylinders and prisms based on different sources and participants' interpretations (Creswell, 2007). More specifically, the case study method was preferred since it gives opportunity to examine a group or an event in depth (Merriam, 1998). In this study, an embedded single case study (Yin, 2013) was used to investigate early-years prospective teachers' conceptions of cylinder and prism. It included following multiple units of analysis based on the research questions: (i) definitions, (ii) mathematical examples, (iii) daily-life examples and (iv) non-examples of cylinders and prisms.

Context and Participants

This study was conducted with 72 early-years prospective teachers at a public university in Turkey. In the data collection process, they had completed the first year of 4-year programs at the faculty of education. In the first year of the program, they took common courses such as Introduction to Education, English for Academic Purposes, Turkish, History and Computer Applications. However, they did not take any course related to mathematics. In the second year of the program, they attended to Early Childhood Mathematics Education Course. In the current study, data was collected by the researcher before teaching of geometric concepts in the course.

Data Collection and Analysis

In data collection, participants provided a written definition of cylinder and prism in a mathematically acceptable way. Then, they were asked to draw at least four different examples of cylinders and prisms by explaining the reasons why they think their examples are different. Furthermore, they were asked to give at least four non-examples of cylinders and prisms and to explain the reason why they preferred them. They were asked to write their responses as follows: "I drew these examples/non-examples because …". Additionally, participants provided different daily-life examples that resemble to cylinders and prisms. They were asked to draw each daily-life example or to write the name of each daily-life example that they gave for cylinders/prisms in the paper. This enables the researcher how they visualize a daily-life example. There was no time limitation in the administration of the task, but it took approximately 60 minutes. After the participants completed their written responses, classroom discussions were held at the beginning of teaching of geometric concepts in Early Childhood Mathematics Education Course. In the discussion process, the researcher gave some definitions and examples/non-examples of cylinders and prisms that were selected from the participants' written responses. Participants criticized these definitions and examples/non-examples in terms of accuracy and visual, linguistics and pedagogical aspects. They reasoned about their responses and provided justifications and validations about how they were sure about their responses. Therefore, classroom discussions were used to validate and strengthen participants' written responses. Classroom discussions took about 45 minutes and were recorded through a high definition camera.

Data were analysed based on each research question according to content analysis. For the first research question, participants' personal concept definitions were examined in terms of critical attributes, terminology and mathematical language. After each term used in definition were listed, they were counted separately. This terminological analysis indicated which critical and non-critical attributes were mentioned by the participants to define cylinders and prisms. In addition to definitions, participants' drawings of cylinders/prisms and daily-life examples were examined. Participants' drawings were examined based on the criteria of orientation, aspect ratio and skewness. Furthermore, each participant's drawings were analysed in terms of how many criteria were included such as three criteria-based, two criteria-based, one criterion-based and no criterion drawings. For example, in the drawings, if the participant changed orientation and aspect ratio of cylinders, it was coded in two criteria-based drawing. Furthermore, drawings were grouped according to correctness and the number of dimensions in each drawings. For instance, if a participant drew a pyramid to exemplify a prism, this drawing was coded as incorrect and three-dimensional. On the other hand, participants' daily-life examples were counted and listed in a table. Thus, the most and least preferred daily-life examples were determined. Similarly, drawings of non-examples were grouped in terms of correctness and the number of dimensions in each drawing. In a study, researchers used similar palpable distractors and impalpable distractors to describe non-examples of shapes in the identification tasks (Aslan & Aktas-Arnas, 2007). In the current study, *non-example* was preferred instead of *distractor*. Therefore, participants' drawings were grouped as *palpable non-examples* and *impalpable non-examples*. Participants' explanations were used in the identification of palpability of a non-example. When drawing a non-example, if participants said the following types of explanations it was considered as an impalpable non-example: This drawing seems to cylinder/prism but it is not actually cylinder/prism", "students may think this drawing as a cylinder/prism", "this shape has two bases, but they are not congruent", "this shape has two bases, but they are not in parallel planes". In other words, participants' drawings which have significant similarities with valid examples of the concept were coded as impalpable non-examples. Two experts in teaching of early-years mathematics worked independently during the analysis. They examined and coded complete data. The inter-coder reliability was found .92 according to Miles and Huberman's (1994) formula. The items which caused disagreement were discussed until reaching an agreement.

Findings

The Nature of Definitions of Cylinder and Prism

According to Tsamir et al. (2015), there are six critical attributes of a cylinder: "a cylinder is (1) a solid figure composed of (2) two (3) congruent (4) bases on (5) parallel planes and (6) a casing (all of the line segments connecting the two bases) (p. 505)". The analysis indicated that none of the prospective teachers provided all critical attributes in their definition of cylinder. Instead, they wrote some of critical attributes of the concept. In order to examine the mathematical language of definitions, the terms that prospective teachers used in the definition of cylinder were presented in Table 1.

Table 1

Frequency (%) of Terms Used in Definitions of Cylinder (n=72)

Terms	Frequency (%)		Terms	Frequency (%)	
Two circles	55	(76)	Having volume	5	(7)
Shape	45	(63)	Expansion	4	(6)
Top-down/base-head	40	(56)	The opposed	4	(6)
Rectangle	27	(38)	Having height	3	(4)
3-Dimensional	26	(36)	Closed	3	(4)
Connecting	17	(24)	Prism-like	3	(4)
A shape that rolls	13	(18)	Solid	3	(4)
Thin/tall	10	(14)	Having no vertex	3	(4)
Parallel (two) lines	8	(11)	Having no edge	2	(3)
Oval/round	8	(11)	Mathematical unit	1	(1)
Congruent/equal	7	(10)	Mathematical product	1	(1)

According to Table 1, more than 60% of the prospective teachers used the terms *circle*(s) and *shape* in their definitions. Additionally, they used *oval-round* or *having no vertex/edge* to emphasize circular bases of cylinder. Some prospective teachers defined cylinder by using only the terms *circle* and *shape* as follows: “a shape having circle bases”. It can be drawn many shapes that fits this definition, but they cannot become a cylinder. In addition, 56% of the prospective teachers used the terms *top-down*/*head-base*. They focused on two bases by referencing top/down and head/base even though they did not write there are exactly two bases in a cylinder. For example, one prospective teacher wrote “a cylinder is a solid, its head has a circle and its base has a circle”. There were three critical attributes in this definition: solid, two and circle, but it is not enough to explain cylinder correctly and completely. Furthermore, top/down and head/base indicate the prototypical cylinder example. Furthermore, based on the definition including such terms, a cylinder in horizontal position becomes a non-cylinder since it has no bases in the top and down.

Another common term used in the definitions of cylinder was *rectangle*. Prospective teachers used this term as follows: “a cylinder is the combination of two circles and a rectangle” by referring right circular cylinder. This definition includes the critical property of two. Additionally, some prospective teachers who used the term rectangle have defined cylinder as the following: “a rectangle that rotates around itself creates a cylinder”. 38% of the prospective teachers wrote *three dimensional shape* in their definitions rather than writing *solid*. Some prospective teachers also tried to provide three-dimensionality by using terms such as a *shape having volume/height*. Moreover, some prospective teachers wrote *a rolled-up shape* instead of solid. They attempted to reference a critical attribute, but they could not use precise mathematical language. On the other hand, 14% of prospective teachers used the term *thin-tall* shape. This statement points to the concept images that are shaped by prototypical cylinders. However, if a cylinder is changed in terms of aspect ratio, the new cylinder does not seem tall and thin (e.g. from a pipe to a coin). In general, the most frequent used terms indicated that prospective teachers focused on the critical attributes of two bases and three-dimensionality of cylinder in their definitions, but other critical attributes were mostly neglected. For example, only 10% of prospective teachers emphasized that the bases should be congruent in their definition of cylinder. Similarly, a few participants attempted to emphasize parallelism of the bases in their definitions as follows: “a 3D shape that composed of two opposed circles”. Unfortunately, 11% of prospective teachers focused on parallelism in the definition differently. For example, one prospective teacher wrote “two circles and two parallel sides that connect the circles produces a cylinder”. Finally, in three definitions, prospective teachers mentioned there is a relationship between cylinders and prisms. They used a statement as *a prism-like shape* in the definition of cylinder. In the classroom discussion, when the researcher asked what they mean prism-like shape, they provided following statements. Group discussion showed that they constructed a relationship between cylinders and prisms without providing strong arguments. They focused on visual similarity of shapes instead of their critical attributes.

R	What do you mean prism-like shape?
PT23	Prism is a box-like shape such as matchbox and cylinder looks a rounded box. They are similar.
R	Do the other two friends think the same way?
PT49	I thought cylinder is also a prism. Any solid can be prism. The cylinder is rounded. The cylinder actually looks like a prism.
PT58	I thought the same way. I'm not sure, but cylinders can be grouped in prisms.

Terms that prospective teachers used in the definitions of prism were presented in Table 2. Similar to the terms used in the definitions of cylinders, prospective teachers mostly used the terms *shape* and *three-dimensional*. These prospective teachers mostly defined the prism as “*a three-dimensional shape*”. However, this definition is very broad and included only one critical property.

32% of prospective teachers focused on the shapes of bases in a prism such as triangles, rectangles and squares. For instance, a prospective teacher defined prism as “prism is a 3D shape with two bases that can be various shapes such as triangle, rectangle and square”. This definition included three critical properties: 3D, two and polygonal bases. They used various terms to express polygon bases. Likewise, some prospective teachers used terms *cornered* and *regular* in order to explain polygonal bases. As an interesting point, some definitions including the terms *triangle-like* and *apex* showed prospective teachers’’ incorrect concept images of prism since they defined pyramids rather than prisms (e.g. prism is a three-dimensional triangle-like shape with an apex).

Table 2

Frequency (%) of Terms Used in Definitions of Prism (n=72)

Terms	Frequency (%)	Terms	Frequency (%)
Shape	51 (71)	Cylinder-like	3 (4)
3-dimensional	40 (56)	Closed	3 (4)
Various bases (triangle/rectangle)	23 (32)	Solid	3 (4)
Connections of	18 (25)	Corresponding	3 (4)
Top-down/base-head	14 (19)	(4 or 6) faced	3 (4)
Two congruent	11 (15)	(6 or 12) sided	2 (3)
Triangle-like	8 (11)	Construction	2 (3)
Cornered	7 (10)	4-dimensional	1 (1)
Apex	4 (6)	Parallel	1 (1)
Regular	4 (6)	Not circle	1 (1)
Having volume	3 (4)	Rounded	1 (1)
Having height	3 (4)	Hollow	1 (1)

Examples of Cylinders and Prisms

Prospective teachers drew 195 shapes to exemplify different cylinders. Three incorrect drawings were found in the shapes. They drew truncated cone as an example of a cylinder. In the remaining 192 shapes, all prospective teachers preferred to draw cylinders with circles or oval bases. This situation was parallel to the structure of their definitions of cylinder since the vast majority of the prospective teachers used terms two circles and round/oval to emphasize circle bases. Table 3 presents which criteria prospective teachers used to provide different examples of cylinders.

As a general information, participants drew 192 correct examples. While they mostly drew cylinder examples in vertical (57%) and horizontal (31%) positions, they drew only 22 (12%) inclined circular cylinders. According to Table 3, 57% of prospective teachers used one criterion-based approach when drawing different examples of cylinders. In one criterion-based approach, they mostly focused on orientation of the shapes. Thus, they provided different examples by changing the orientation of a cylinder (see Figure 2).

Table 3

Criteria Used in Drawings of Cylinders

Focus	Criteria	Frequency	(%)	Total %
Three criteria-based	Orientation-aspect ratio-skewness	1	(1)	1
Two criteria-based	Orientation-aspect ratio	16	(22)	33
	Orientation-skewness	7	(10)	
	Aspect ratio-skewness	1	(1)	
One criterion-based	Orientation	35	(49)	57
	Aspect ratio	6	(8)	
	Skewness	0	0	
No criterion	A prototypical example	6	(8)	8
	Total	72	100	100

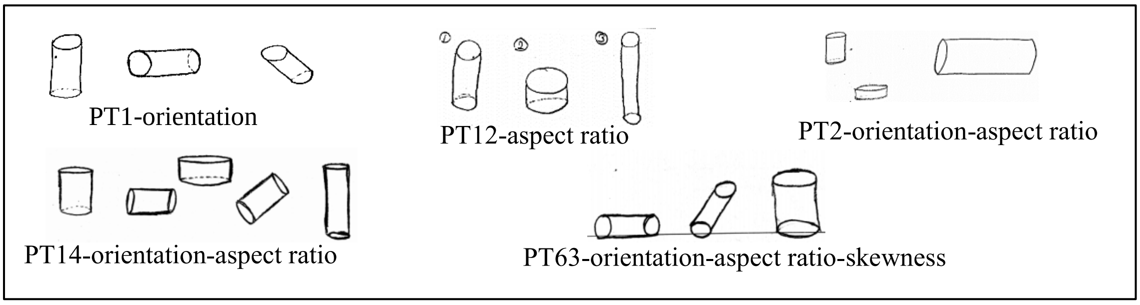

Figure 2. Prospective teachers' cylinder examples.

In one criterion-based drawings, the number of prospective teachers who drew various examples of cylinders considering aspect ratio was very few. Nobody provided different examples of cylinders in terms of only the criterion of skewness. This situation was mentioned in the classroom discussion.

R	Why do you prefer to change orientation of cylinders instead of skewness?
PT4	Slanted solids did not come to my mind.
PT24	I agree. We are accustomed to see cylinders in vertical or horizontal positions in our daily-life and textbooks. For this reason, I did not think to draw slanted cylinders.
R	Is there anyone who have different idea from your friends?
PT70	Drawing them is difficult. My ability of drawing is not good.
PT58	I think that slanted cylinders may be difficult for children.
PT22	I agree with you.

In the discussion, prospective teachers put different ideas about the reasons why they did not draw slanted cylinders. Based on the discussion, it seems that cylinders in horizontal and vertical positions were immediate intuitive responses. However, some prospective teachers found difficult to draw cylinders in slanted positions. Interestingly, some prospective teachers evaluated the issue in terms of a pedagogical perspective. On the other hand, 33% of prospective teachers tried to differentiate their examples of circular cylinders in terms of two criteria. In two criteria-based drawings, 16 prospective teachers drew cylinders considering orientation and aspect ratio together. At this point, they commonly drew three different examples of cylinders. In two examples, they changed the height and the radius of circle bases of cylinders. In another example, they changed the position of the cylinder. In terms of orientation and skewness, seven prospective teachers drew examples of cylinders. They commonly drew a cylinder in slanted position and right cylinders in vertical and horizontal positions. There was only one prospective teacher who drew different examples of cylinders considering skewness and aspect ratio. Unfortunately, only one prospective teacher used three criteria together to provide different examples of cylinders.

Table 4 shows that prospective teachers drew 185 shapes to exemplify prisms. While 75% of the drawings were correct, 25% of them were incorrect. In correct drawings, the most preferred shapes were triangle prisms and rectangle prisms. Correct examples indicated that prospective teachers paid attention to the critical attribute of polygonal bases. They generally did not provide different examples in terms of orientation, aspect ratio and skewness. Instead, they tried to provide different examples of prism by changing the shapes of bases in prisms (Figure 3-a-b) or size of prisms (Figure 3-c). In incorrect drawings, pyramids and cones were generally drawn as the example of prisms (Figure 3-d-e). Their incorrect drawings had similarities with their definitions of prism. Moreover, five prospective teachers drew truncated pyramid as an example of prism (Figure 3-f). In the classroom discussion three prospective teachers said “because there are two bases in this shape”. This showed the lack of knowledge of congruency of bases. Other two prospective teachers said that “there is no apex in this shape, but it becomes prism because it is very similar to prism”. It was obvious that they visualized a pyramid as a prism and a truncated pyramid as a pyramid.

Table 4

Correctness and Frequency of Examples of Prism

Prism examples given by prospective teachers		Frequency (%)		Total percentage
Correct examples	Rectangle prism	49	(26.5)	75.1
	Triangle prism	48	(25.9)	
	Square prism	22	(11.9)	
	Cube	18	(9.7)	
	Hexagon prism	2	(1.1)	
Incorrect examples	Rectangular pyramid	17	(9.2)	24.9
	Triangular pyramid	9	(4.9)	
	Cone	8	(4.3)	
	Cylinder	5	(2.7)	
	Truncated pyramid	5	(2.7)	
	Pentagon/Hexagon	2	(1.1)	
	Total	185	(100)	

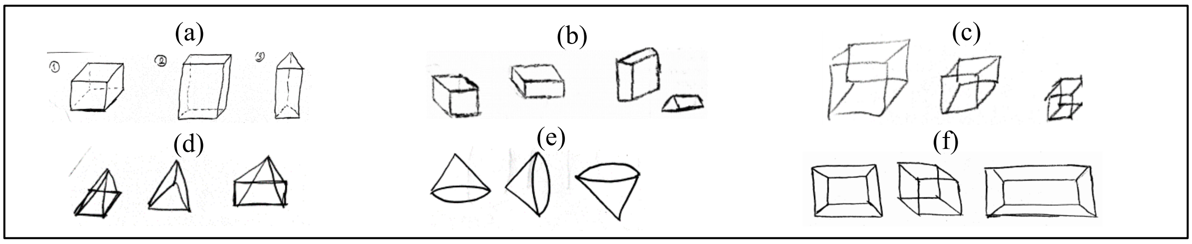

Figure 3. Prospective teachers’ examples of prisms

Daily-Life Examples of Cylinders and Prisms

Prospective teachers gave 31 different types of daily-life examples of cylinders, but they wrote 20 different types of examples of prism. Furthermore, the total numbers of daily-life examples were 184 and 148 for cylinders and prisms, respectively (see Table 5).

Table 5

Prospective Teachers' Daily-Life Examples of Cylinders and Prisms

Prospective teachers' daily-life examples			
Cylinders	Frequency	Prisms	Frequency
Box (pen rack, chip)	50	Box (matchbox)	35
Jar	13	White goods	14
Glass	13	*Roof*	14
Pipe (stove, water)	13	Cube toy	11
Roll	12	*Egypt pyramids*	10
Fluorescent lamb	12	Home/room	9
Plastic bottle	11	Eraser	7
Pencil	10	*Tent*	7
Barrel	8	*Cones* (hat, ice cream)	6
Road roller	5	Book/notebook	6
Coin bank	4	Wafer/chocolate	5
Battery	4	Lego	5
Salt cellar	3	Dice	*5*
Binoculars	3	Cube sugar	4
Rolling pin	3	Roll	2
Drum	2	*Diamond*	2
Cylindrical wood	2	Telephone/TV	2
Wheel	2	*Grater*	2
Vase	2	Ice cube	1
Flute	1	Beam splitter prism	1
Pipet	1		
Chalk	1		
Flower pot	1		
Flue	1		
Tunnel slide	1		
Pritt glue stick	1		
Lighter	1		
Hair roller	1		
Beaker	1		
Silicon	1		
Hourglass	1		
Number of different examples	31		20
Total frequency	184		148

Table 5 showed that the variety and the number of daily-life examples of prisms were less than that of cylinders. In all daily-life examples of cylinders, prospective teachers paid attention to select three-dimensional shapes with circle/oval bases. This situation was consistent with their concept definition of cylinder. The most preferred daily-life example of cylinders was box. Prospective teachers gave information about the shape of box such as circle/round/oval bases by indicating the type of box (e.g. pen rack and chip). Other daily-life examples of cylinders were jar, glass, pipe, roll, lamb, plastic bottle and pencil. Some prospective teachers added explanation or drawing in order to emphasize particular critical properties of cylinders. For example, three prospective teachers wrote “glass seems a cylinder if it has straight sides”. Another one wrote “vase would be a good example if it has regular and circular”. These explanations revealed that they considered some critical properties of cylinders. In the daily-life examples, they mostly gave cylinder-like materials with vertical position such as jar and glass. Furthermore, they gave some cylinder-like materials being mostly in horizontal position such as rolling pin, wheel and road roller. Orientations of examples given as daily-life examples also had consistency with the orientations of prospective teachers’ drawings of cylinders. Another interesting finding was the variety of examples. There were more examples in everyday-contexts (e.g. box and glass) and school-contexts (e.g. pencil and chalk), but there were a few examples of cylinders in different contexts like science (e.g. beaker) and cosmetics (e.g. hair roller).

As seen in Table 5, the most-preferred daily-life example of prisms was box (n = 35) again. They gave daily-life examples having polygon bases such as matchbox and shoe box. This supports most of prospective teachers knew the critical attribute polygonal bases of prism intuitively or conceptually. Their correct daily-life examples were commonly be evaluated as rectangular and square prism (e.g. lego, book and white goods). However, although most participants drew triangular prisms to exemplify prism mathematically, they did not prefer to give daily-life examples having similarity with triangular prisms. The researcher asked the reason why they did not prefer triangular prisms in your daily-life examples. One prospective teacher said “materials in our around mostly have a rectangular prism-shape. I think triangle-like examples is rare in daily-life”.

In the daily life examples of cylinders, hourglass was wrong. However, 22% of daily-life examples of prisms (n=32) were incorrect. The roof, tent, Egypt pyramids, ice cream cone, party hat, kitchen grater (Figure 4-a) and diamond (see Figure 4-b) create contradictions with prisms. For example, while the half of 14 prospective teachers who gave the roof as a daily-life example of prism drew a shape like in Figure 4-f, remaining prospective teachers drew the roof like Figure 4-g. Similarly, four prospective teachers visualized the tent as in Figure 4-h rather than Figure 4-i. Thus, concept images of prisms were constructed based on pyramids, which was consistent with prospective teachers’ incorrect concept definitions and images of prisms.

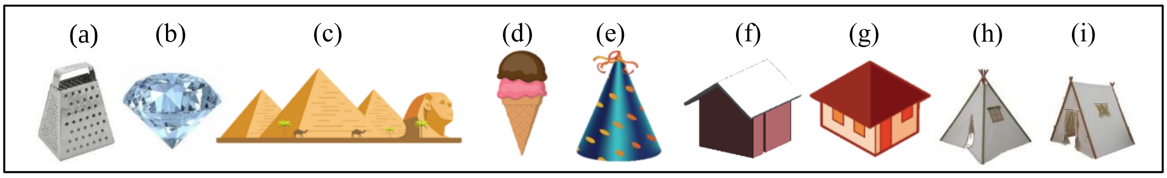

Figure 4. Prospective teachers’ problematic examples of prisms

On the other hand, ten prospective teachers immediately wrote or illustrated Egypt pyramids (Figure 4-c) as the examples of prisms. Egyptian pyramid would not be the example of daily-life, and it is not a prism. These prospective teachers even did not know whether there are two bases in a prism or not. Moreover, the example of grater showed that two prospective teachers did not know whether there are congruent bases in a prism. Finally, six prospective teachers expressed or illustrated the cone-like shapes as in Figure 3-d/e in their daily-life examples incorrectly. Thus, daily-life examples supported some prospective teachers' confusion between prisms and pyramids.

Non-Examples of Cylinders and Prisms

Prospective teachers drew 204 shapes as non-cylinders. They mostly drew two-dimensional shapes (71%) such as triangles, squares, circles and rectangles rather than three-dimensional shapes as non-cylinders (see Table 6). They drew cornered shapes grouped in two-dimensional non-cylinders. Many prospective teachers explained the reason: "I drew a square and a triangle since they have corners. Children can easily understand non-cylinders by examining these shapes". Another prospective teacher wrote: "square is a two-dimensional shape and has corners". A prospective teacher also provided following explanations: "I drew a star, a square and a triangle because they didn't look like a cylinder. The circle is different from the cylinder. It's simpler than cylinder". From the explanations, it can be inferred that they preferred two-dimensional shapes because they are easily distinguished them from cylinders. For this reason, these shapes were grouped in impalpable non-cylinders.

Prospective teachers frequently drew prisms as three-dimensional non-cylinders. Normally, prism is a special type of cylinder. However, since prospective teachers learned cylinders with circle bases, they considered prisms as non-cylinders. They wrote: "this shape (prism) has many corners, but cylinder is a rounded shape." Seven prospective teachers drew a cone as a non-cylinder. One wrote "this shape has an apex, but cylinder has not. We can easily understand that cone is not a cylinder". These explanations indicated that prospective teachers focused on the number of bases when drawing non-cylinders.

Table 6

Prospective Teachers' Non-Examples of Cylinders

Non-cylinder drawings			Frequency	Total frequency	(%)
2D	Palpable non-cylinders	Triangle	41		
		Square	34		
		Circle	28		
		Rectangle	18		
		Polygon (n>4)	7	141	(69)
		Oval	4		
		Trapezoid	3		
		Star	4		
		Parallelogram	2		
	Impalpable non-cylinders		4	4	(2)
3D	Palpable non-cylinders	*Prism*	36		
		Cone	7	46	(23)
		Pyramid	3		
	Impalpable non-cylinders		13	13	(6)
Total				204	(100)

8% of non-cylinders were grouped in palpable non-cylinders. Palpable non-cylinders either seems a cylinder visually or have many critical attributes of cylinders (Figure 5). In Figure 5-a-b-c, prospective teachers focused on similar appearance of cylinder surfaces. When the researcher asked why they drew such shapes, a dialogue between prospective teachers and the researcher happened as follows:

R	Why do you prefer to draw such non-cylinders?
PT7	For example, students may treat some shapes as cylinders if shapes have similarity with cylinders.
PT37	You are right. My sister says that pyramid is a triangle.
R	What do you want to emphasize in these shapes (Figure 5-a-b-c)
PT19	I focused on three-dimensionality.
PT44	I want to show the lack of two circular bases.

According to dialog, prospective teachers tried to focus on the critical properties of the cylinder in their drawings of palpable non-cylinders. Additionally, when they gave non-examples, they developed some pedagogical decisions by considering children's understanding of geometric shapes. There were some noteworthy points in prospective teachers' three dimensional palpable non-cylinders. Truncated cone (Figure 5-f) was mostly drawn as a 3D palpable non-cylinder. One prospective teacher wrote "this shape has not congruent bases." Another prospective teacher said "it seems cylinder, but not." Similar explanations were provided for Figure 5-i and Figure 5-h. They focused on the types and size of the bases. In addition, only a few prospective teacher focused on how the bases of the geometric shape converge (Figure 5-d-e-g). Their main concern was to reveal the necessity of linearity and parallelism of lines that converge the bases.

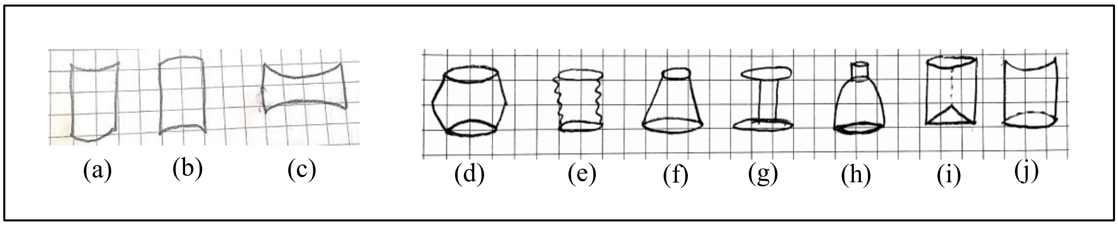

Figure 5. Prospective teachers' palpable non-cylinders

Table 7 indicated that the tendency towards two-dimensional non-prisms (81%) was more than the tendency in three-dimensional non-cylinders. Circle was mostly drawn as a two-dimensional non-prism. Interestingly, some prospective teachers drew heart, crescent, cloud and flower as non-prisms since they have no corner.

Written reasons revealed that the tendency in drawings of two-dimensional non-prisms was related to the lack of prospective teachers' knowledge of prism. One prospective teacher wrote "I'm not sure what prism really is, but I know that a heart and a square are not a prism." Some prospective teachers also wrote "I remember prism as a cornered shape. So, I drew crescent and circle as non-prisms." Written reasons confirmed that there was a confusion about the concept of prism, similar to the confusion in the definitions and daily-life examples. Therefore, they tended to focus on two-dimensional shapes. On the other hand, the most drawn shape (n=11) was cylinder as a three-dimensional non-prism. According to the written explanations, they preferred cylinders since it has no corner. Another interesting finding was that some prospective teachers presented prisms as non-prisms because they evaluated cones and pyramids as prisms. Only a few palpable non-prisms (e.g. truncated prism) were drawn by a limited number of prospective teachers.

Table 7

Prospective Teachers' Non-Examples of Prisms

Two-dimensional non-prisms		Three-dimensional non-prisms	
Shape	Frequency	Shape	Frequency
Circle	41	Cylinder	11
Rectangle	34	Cone	6
Triangle	30	Pyramid	6
Square	12	Prism	6
Star	9	Sphere	2
Oval	8	*Almost prisms*	3
Hexagon	6	*Truncated cone*	1
Parallelogram	6	*Truncated pyramid*	1
Heart-shape	4		
Crescent	4		
Cloud	2		
Flower	1		
Total	157 (81 %)		36 (19 %)

* *Note*. The shapes in italics are examples of palpable non-prisms.

Conclusion and Discussion

Early-years prospective teachers' definitions of cylinder and prism showed that they had difficulties in defining the concepts. They mostly used inappropriate terms that refer to more general attributes of geometrical solids such as three-dimensionality rather than critical attributes of the concepts. Analysis showed that they defined cylinders based on right circular cylinders. Furthermore, early-years prospective teachers also did not pay attention to all critical attributes of cylinders. As a result, their definitions of cylinder are insufficient to explain the concept. For example, some prospective teachers defined cylinder as "three-dimensional shape with two circular bases". However, they did not mention about parallelism and congruency of these circular bases. On the other hand, some prospective teachers' definitions of prism involved following terms: triangle-like, composition of triangles and a top point. Their drawings of examples and daily-life examples also supported this definition since they drew pyramids and gave pyramid roof and tent as daily-life examples. These results have similarities with the findings of some studies that were conducted with prospective mathematics teachers (e.g., Ertekin et al., 2014; Gökbulut & Ubuz, 2013). Tsamir et al. (2015) revealed that early-years teachers were able to define triangles, but they had difficulties in defining cylinders. Interestingly, prospective teachers did not even mention what kinds of surfaces cylinders and prisms have in their definitions. Instead, they gave more general definition for prism as "a three-dimensional shape". According to some prospective teachers, all geometric solids can be the examples of prisms. This might be related to their concept formation that depends on the number of bases or visual properties rather than cylindrical or prismatic surfaces. Classroom discussion supported this idea since they explained their difficulties in identification of prisms due to polygonal surfaces and bases. They compared a triangular prism and a rectangular prism, then they focused on triangular and rectangular surfaces in solids. As a result, they cannot choose necessary and sufficient characteristics of prisms when defining (Gökbulut & Ubuz, 2013). In defining process, they mostly used concept images rather than definitions of the concepts (Clements & Battista, 1992; Vinner & Hershkowitz, 1980).

Researchers argue that instruction of geometric shapes should include both prototypical and non-prototypical examples (Clements et al., 1999; Hershkowitz, 1989). However, in the current study, when early-years prospective teachers' examples of cylinders are examined, the results indicated that they mostly drew circular right cylinders to provide different examples of cylinders. Furthermore, most of them differentiated their drawings according to one criterion, especially in terms of orientation. Prospective teachers who drew different examples of cylinders based on two-criteria commonly preferred to change the orientation and aspect ratio of circular cylinders. The least used criteria in examples of cylinders was skewness (12.5%). These results showed that prospective teachers' awareness about possible effects of orientation, aspect ratio and skewness on children's development of geometric shapes is limited to the criterion of orientation. However, the influence of orientation on the children's classification of geometrical shapes is less than other criteria (Aslan & Aktas-Arnas, 2007). Nevertheless, it is important to consider orientation since some children could not identify the triangles in different orientations due to the position of top point (Hannibal & Clements, 2000). However, several studies revealed that skewness has a greater influence than orientation on the classification of triangles (Aslan & Aktas-Arnas, 2007; Clements, 1999; Hannibal & Clements, 2000). For example, many children do not identify triangles that have top points in skewed position. Another crucial point was that only 33% of the prospective teachers used aspect ratio when drawing different examples of cylinders although related literature says that aspect ratio has a greater effect on children's recognition of shapes (Aslan & Aktas-Arnas, 2007; Clements, 1999). However, in the current study, prospective teachers slightly changed this ratio. For example, there were a few examples of cylinders like a coin and a thin pipe. The reasons of such results may be related to pre-school educational materials in which the examples of geometric shapes are often presented with equal length of height and base (Bozkurt, 2018; Nurnberger-Haag, 2017) and in same orientation (Aslan & Aktas-Arnas, 2007; Hannibal, 1999). The results also revealed that some prospective teachers had problems about concept image and definition of prisms since they drew and defined pyramids and cones as the examples of prisms. Similar findings also found in middle school students' conceptions of triangular pyramids and prisms (Türnüklü & Ergin, 2016). On the other hand, drawings of prisms indicated that most of prospective teachers know that there are polygonal bases in prisms. As an interesting result, three prospective teachers established a relationship between cylinders and prisms although this relation is not mentioned in any Turkish mathematics textbooks (Bozkurt, 2018).

Use of daily-life examples in mathematics education is stressed (e.g. Van den Heuvel-Panhuizen & Drijvers, 2014) as an effective way to support children's conceptualization of geometric shapes and to make connections between mathematics and daily-life (TNMPC, 2013). The results indicated that prospective teachers' daily-life examples of cylinders were more diverse than the examples of prisms. Boxes were the most common daily-life example of cylinders and prisms. However, they give examples of rounded boxes for cylinders, but they provided examples of cornered boxes like matchbox and shoebox for prisms. This difference revealed that early-years prospective teachers know two polygonal bases of prisms as a critical property. Furthermore, prospective teachers' daily-life examples of cylinders have similarities with the examples presented in textbooks (Bozkurt, 2018; Koçak et al., 2014). The results also showed that daily-life examples of prisms included inappropriate examples such as cone-shaped party-hat and ice-cream and Egyptian pyramids. Furthermore, some problematic daily-life examples that can be used to illustrate both prisms and pyramids were detected such as roof and tent. Similarly, Sahin-Dogruer (2018) found that examples of roof and tent cause ambiguity in middle school students' understanding of cylinders and prisms if they are not represented appropriately.

In this study, in terms of the non-examples, prospective teachers mostly preferred to draw two-dimensional shapes for non-cylinders/prisms. The most common shapes were triangles, squares, circles for non-cylinders, and circles, rectangles, triangles were the most common non-prisms. These two-dimensional shapes are palpable non-examples for cylinders and prisms since they have no similarity with the concepts and lack of many critical defining attributes (Aslan & Aktas-Arnas, 2007; Tsamir et al., 2008). From this point, palpable non-examples can be used to emphasize critical properties of geometric solids. As another interesting result, prospective teachers preferred to draw hearts, crescents and clouds as non-prisms, they did not draw such shapes for non-cylinders. They visualized prisms as cornered shapes and cylinders as rounded shapes. The frequency of three-dimensional non-examples given for prisms (19%) was less than the frequency of three-dimensional non-examples given for cylinders (29%), which is related to prospective teachers' concept definition of prisms. Prospective teachers defined prism as a 3D shape, but they expressed they were not sure about the correctness of their definitions. As a result, they tended to draw two-dimensional shapes instead of three-dimensional shapes as non-prisms. Three-dimensional non-examples included some incorrect drawings. Some prospective teachers claimed prism as a non-example for both cylinders and prisms because they visualized prisms like a pyramid or a cone. On the other hand, the frequency of impalpable non-examples was very low for both concepts, especially for prisms. In this respect, Tsamir et al. (2008) argue that non-examples have a dual role in acquisition of geometrical shapes in terms of their intuitive nature. While intuitive non-examples support children's understanding of non-critical attributes of the concept, non-intuitive non-examples help children to develop reasoning on critical attributes. Similarly, Aslan and Aktas-Arnas (2007) found that children can successfully distinguish the palpable non-examples of triangles, but they were not successful in identification of impalpable non-examples.

As a final point, this study gives some suggestions and future research ideas. In this study, classroom discussions were used to support written data, but they also helped to elaborate prospective teachers' conceptions of geometric solids. In the future studies, researchers may use classroom discussions based on structured argumentation. In the current study, teachers were not aware of the potentials of using impalpable non-examples in learning and teaching of prisms and cylinders. From this point of view, teachers may be given opportunities to understand the potentials of using both palpable and impalpable non-examples of geometric shapes in group discussions. In this study, prospective teachers generally defined and drew cylinders based on a prototypical example. This situation may be related to the frequent presentation of prototypical examples of cylinders (right circular cylinder) in many textbooks instead of presenting inclusive definitions and non-prototypes examples (Bozkurt, 2018; Koçak et al., 2014). In this regard, textbook writers can change the presentation of definitions and examples in curriculum materials and children textbooks since it may have great influence on learners' reasoning about geometric shapes (Bozkurt, 2018; Nurnberger-Haag, 2017).

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