Secondary Mathematics Teachers’ Content Knowledge for Teaching the Concept of Function

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This study examined secondary mathematics teachers’ content knowledge for teaching the concept of function. Content knowledge, for the purpose of this study, includes the common content knowledge and specialized content knowledge domains of Ball and her colleagues’ Mathematical Knowledge for Teaching (MKT) framework. Data were collected through a 10-item questionnaire from 42 teachers practicing at fifteen different industrial vocational high schools in Ankara, Turkey. Findings revealed that most of the teachers placed a strong emphasis on the set correspondence and algebraic representations; found the uniqueness condition of functions very important for students to understand; and had a wide repertoire of examples of functions which constitute the uniqueness condition. The teachers’ choice of examples mostly included giving set correspondence relations and checking whether they define a function. Additionally, a strong focus on the computational aspects associated with functions was evident. The teachers’ understanding of the function concept might be influenced by the curricular materials, contextual factors, and high-stakes testing. The methodological and practical implications of these findings are discussed, and directions for future studies are offered.

To succeed in courses that build on quantitative thinking and relationships, students need to properly understand the concept of function and its essential features (Cooney, Beckmann, Lloyd, Wilson & Zbiek, 2010; Thompson & Carlson, 2017). While students experience difficulties in conceptualizing the concept of a function (e.g., Clement, 2001), research suggests that the learning opportunities teachers provide in the classroom do influence student learning outcomes (Baumert et al., 2010; Hatisaru & Erbas, 2017; Hill, Blunk et al., 2008; Hill, Rowan, & Ball, 2005). Heavy emphasis on algebraic and set correspondence representations, for instance, can lead to over-emphasis of algebraic functions and provide fewer explorations of different functions (Steele, Hillen & Smith, 2013). A rule-based view of function might also result in a situation where “students tend to view functions only in terms of symbolic manipulations and procedural techniques disassociated from an underlying interpretation of function as a more general mapping of a set of input values to a set of output values” (Carlson, Oehrtman, & Engelke, 2010, p. 115). An overemphasis on set correspondence would limit students’ conceptualization of functional relationship to correspondence approach (i.e., associating a unique *y*-value with an input *x*-value) when the same task should be approached via covariation (i.e., coordinating the two varying quantities while attending to the ways in which they change in relation to each other) (Ayalon, Watson, & Lerman, 2017; Carlson, Jacobs, Larsen, Coe, & Hsu, 2002; Thompson & Carlson, 2017). Furthermore, overemphasis on relations depicting functions as mappings between discrete datasets might lead to view that a function is a set of discrete data points and anything that can be mapped is a function, which would cause confusion between discrete and continuous data and confusing a point-wise relation with the underlying function (Spyrou & Zagorianakos, 2011; Watson & Harel, 2013).

To “prepare students well for their future understanding of functions and avoid typical problems” (Watson & Harel, 2013, p. 163), teachers’ knowledge about the function concept is critical, which might impede their instructional practices (Hill, Rowan, & Ball, 2005) and effective student learning (Baumert et al., 2010; Roux, Nieuwoudt & Nieuwoudt, 2015; Watson & Harel, 2013). It follows therefore that teachers need to have a deep knowledge of functions-related content (Hatisaru & Erbas, 2017; Nyikahadzoyi, 2015; Roux et al., 2015) and the knowledge and skills of this content unique to its teaching (Steele et al., 2013). These two domains of teacher knowledge, the former being referred to as *common content knowledge* and the latter as *specialized content knowledge* correspond with the Ball, Thames, and Phelps's (2008) *mathematical knowledge for teaching* (MKT) formulation, which is a powerful model that connects knowledge, teaching practice, and student learning (Speer, King, & Howell, 2015).

Previous studies revealed that both pre-service (e.g., Even, 1993) and in-service (e.g., Norman, 1992) teachers lack genuine understanding of the function concept, and recent evidence suggested the situation has not changed much (e.g., Hansson, 2006; Meel, 2000). Since then, despite there being some theoretical frameworks for unpacking what constitutes of teachers' MKT about the concept of function (e.g., Nyikahadzoyi, 2015) and how it influences student learning (Hatisaru & Erbas, 2017), no recent studies have focused on investigating in-service teachers' MKT about this concept or central issues inherent in their understanding and teaching of functions. This study aims to investigate Turkish secondary mathematics teachers' *content knowledge* (CK) for teaching the concept of function, the body of knowledge comprising common content knowledge and specialized content knowledge domains of MKT.

The study reports an aspect of a previous research examining relationships between teacher MKT about the function concept and student learning outcomes of this concept in industrial vocational high schools in Ankara, Turkey (Hatisaru, 2014). In the previous research, two teachers and their students were selected in a two-step sampling process. Firstly, 42 teachers were surveyed by a questionnaire measuring secondary mathematics teachers' CK for teaching the concept of function, and based on the survey results, one teacher from the strong and one from the weak MKT group were selected for case studies (for more details see Hatisaru & Erbas, 2017). In the current study, I use the data from those teachers who were surveyed, and drawing upon prominent literature relating to pre- and in-service teacher understanding of (Bolte, 1993; Cooney, 1999; Even, 1993; Hitt, 1998; Norman, 1992), and student difficulties in (Markovits, Eylon & Bruckheimer, 1986, 1988; Tall & Bakar, 1992; Vinner, 1983; Vinner & Dreyfus, 1989) the function concept, I measure teacher CK for teaching about this concept. The central research question that guides the study is: *What is the secondary mathematics teachers' content knowledge for teaching the concept of function?* The study not only complements but also extends the works that have capitalized on theoretical frameworks by focusing on in-service secondary mathematics teachers' CK about the concept of function, with a focus on the aspects presented in the study.

The related literature that informed the study is introduced in the following section, then the research design is presented in the second section. Results are reported in the third section and discussed in the fourth section followed by limitations and directions for future research.

Mathematical Knowledge for Teaching about the Concept of Function

Building on the Shulman's (1986) pedagogical content knowledge (PCK) notion, Ball and her colleagues developed a theoretical framework and set of measurement instruments for the assessment of elementary teachers' mathematical knowledge for teaching (MKT), the knowledge special to the teaching of mathematics (Ball, Lubienski, & Mewborn, 2001). MKT distinguishes between subject matter knowledge and PCK and refines both. It consists of three types of subject matter knowledge: common content knowledge, specialized content knowledge and horizon content knowledge. Common content knowledge is the mathematical knowledge and skill used in settings other than teaching (Ball et al., 2008). In this domain:

> Teachers need to know the material they teach; they must recognize when their students give wrong answers or when the textbook gives an inaccurate definition. In short, they must be able to do the work that they assign their students. (Ball et al., 2008, p. 399).

Contrary to common content knowledge, specialized content knowledge is the mathematical knowledge and skills unique to teaching. It enables teachers to "accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures and examine and understand unusual solution methods to problems." (Hill, Ball, & Schilling, 2008, p. 378). However, the separation between common content knowledge and specialized content knowledge is not very distinct (Hill, 2007; Hill, Schilling, & Ball, 2004; Howell, 2012; Kane, 2007; Ryan & McCrae, 2005/2006). In the secondary teacher knowledge context, the distinction between MKT domains in general (Buschang, Chung, Delacruz, & Baker, 2012), and between common content knowledge and specialized content knowledge in particular (Speer & King, 2009; Speer et al., 2015) becomes less clear. For instance, reviewing the correctness of mathematical solutions and formulating a response to students, which was defined as specialized content knowledge, also requires common content knowledge (Kane, 2007; Speer et al., 2015; Steele et al., 2013). In that case, the knowledge that was defined as specialized content knowledge and needed for the teaching overlaps with common content knowledge, which is common to a wide variety of settings (Speer et al., 2015).

Drawing on Speer et al. (2015), in this study, I operationalized the definition of CK (referring to the body of knowledge comprising Ball et al.'s (2008) common content knowledge and specialized content knowledge) as an understanding of the facts and concepts within the concept of function domain, making sense of what have been suggested by students, determining whether the responses of students are correct, and formulating correct responses to the students' solutions. Below, I present specific CK for teaching about the function concept based on this operational definition.

Content knowledge for teaching the function concept

A function is "a relation that uniquely associates members of one set with members of another set. More formally, a function from A to B is an object f such that every $a \in A$ is uniquely associated with an object $f(a) \in B$" (Stover & Weisstein, 2017, para. 1). The uniqueness, which is also known as "single-valuedness" (Cooney et al., 2010), emphasized in the definition is a main element in the treatment of the function concept in many secondary school mathematics curricula and mathematics textbooks. Yet, the following three essential understandings about the function should be explicitly addressed in the teaching and learning of functions:

- Functions are single-valued mappings from one set—the domain of the function—to another—its range.
- Functions apply to a wide range of situations. They do not have to be described by any specific expressions or follow a regular pattern. They apply to cases other than those of "continuous variation." For example, sequences are functions.
- The domain and range of functions do not have to be numbers. For example, 2-by-2 matrices can be viewed as representing functions whose domain and range are a two-dimensional vector space. (Cooney et al., 2010, p. 8)

Of these, the first refers to uniqueness and the latter two refer to arbitrariness. According to Even (1993):

> The arbitrary nature of functions refers to both the relationship between the two sets on which the function is defined and the sets themselves. The arbitrary nature of the relationship means that functions do not have to exhibit some regularity, be described by any specific expression or particular shaped graph. The arbitrary nature of the two sets means that functions do not have to be defined on any specific sets of objects; in particular, the sets do not have to be sets of numbers. (p. 96)

While the uniqueness is generally explicit in the function definitions, the arbitrariness is usually implicit (Even, 1993). According to Cooney et al. (2010), understanding uniqueness is part of developing an appreciation that the concept of function is sufficiently broad and flexible to encompass many "neoclassical" examples of functions (e.g., matrices, and arithmetic and geometric sequences) in addition to "classical" functions "whose domain and range consist of intervals within the real numbers and which are given by well-known formulas" (p. 18). As such, teachers are expected to know about the uniqueness and arbitrariness of functions (Cooney et al., 2010; Even, 1990; Nyikahadzoyi, 2015; Steele et al., 2013) and possess a wide-ranging repertoire of examples that best illustrate them (Nyikahadzoyi, 2015).

There is a fundamental need for students to develop a strong understanding of functions and functional perspective in different real-world contexts. Thompson and Carlson (2017) provided a definition of function from the covariation perspective, stressing the importance of providing students with opportunities to investigate the relationships between quantities and variables whose values vary or covary. They emphasized that concepts such as variable, function, and rate of change are essential in the understanding of calculus and the modeling of dynamically changing phenomena in science and engineering. They claimed that the conception of "continuous covariational reasoning, or reasoning about values of two or more quantities varying simultaneously" (p. 423) is central to understanding and using functions. Textbooks at different school levels, however, present different views of the concept of function such as an input-output process or assignment, a rule assigning x to $f(x)$, a correspondence from one set to another, or a conception of quantities varying simultaneously (Cooney et al., 2010; Thompson & Carlson, 2017). Teachers should have an understanding of different views on the meaning of function (Lloyd & Wilson, 1998; Nyikahadzoyi, 2015; Steele et al., 2013), and their knowledge should be sufficiently flexible to enable them to change the way they teach about functions and/or alter the textbook presentations to best suit the diversity of their students' needs.

As functions can be represented in multiple ways, including algebraic (symbolic), graphical, verbal, tabular (numerical) representations, representing and analyzing functions from different perspectives are critical for learning functions (Cooney et al., 2010). Understanding different representations for functions and the relationships among them would play an important role in developing a conceptually rich understanding of functions (Even, 1990; Lloyd & Wilson, 1998). Teachers should be familiar with various representations for functions and links among them (Even, 1993; Lloyd & Wilson, 1998). They "must make explicit to their students the connections among the various representations" (Van Dyke & Craine, 1997, p. 616).

The concept of function in the Turkish curriculum

In Turkey, secondary education lasts for four years (grades 9 to 12, ages 14 to 17) and is provided by general and vocational-technical education high schools. Vocational-technical education mainly aim to prepare students for higher education and post-secondary school employment in related branches. In the ninth grade, they offer the same common compulsory courses such as history, geography and mathematics, and implement the same curriculum as general secondary education schools. Mathematics teachers both in general and vocational-technical education receive the same type of training at the college level (UNESCO-UNEVOC, 2013). Teaching at schools is regulated by the national curriculum. Mathematics is taught as a mandatory and major subject during schooling, and it is part of university entrance exams, which students sit at the completion of the secondary school. Mathematical questions make up 33% of the questions for university entrance exams (European Schoolnet, 2018). Compared to their peers in general education, students in vocational-technical education have been less successful both in these national (Sever et al., 2016) and international (Alacacı & Erbaş, 2010) assessments.

In the Turkish secondary mathematics curriculum, the concept of function is first introduced formally to the students at 9^{th} grade (age 14-15 years). At the time of data collection for the current study, in the curriculum (Turkish Board of Education, 2011), the concept of function was primarily based on a set-theoretic perspective, as is the case for many school algebra or mathematics curricula and textbooks (Watson & Harel, 2013). In the curriculum guidebook (Turkish Board of Education, 2011), the concept of function was presented through an activity which corresponds to a group of children and the houses in a neighborhood, emphasizing that each child lives in a house and there might be houses that are not associated with children. The relation between the concept of Cartesian product, relation and function was emphasized. The learning objectives for each concept were itemized (e.g., "Explain what a relation is, represent it through a diagram, and draw its graph.") followed by hints and suggestions for explanations, including key examples or exercises. Basic function concepts were generally presented with functions defined in infinite sets, and then the focus solely shifted to real numbers or infinite intervals, which were the immediate subsets of real numbers. The solution methods for certain problem types were provided (e.g., the vertical line test, the horizontal line test). Various representations of a function were considered one by one, and different objectives were specified for each representation. However, no explanations about the purpose of transitions between different representations and what to focus on in these transitions were placed.

In the textbook published by the Ministry of National Education (2012) and officially required to be used throughout all schools in Ankara, two activities were utilized in the introduction to the function concept. The first included various visuals depicting that a function is a mechanism converting inputs to outputs (e.g., students start school, study and graduate). The second depicted various mappings between the set of customers at a restaurant and the list of dishes that they can order according to two conditions: all customers would have a meal and each customer could order only one dish. By means of this activity, it was highlighted that a function is a special relation which maps the elements of two sets. The necessary definitions, theorems, and explanations were given as endnotes after the exercises. Diagrams and algebraic representations were predominantly used in dealing with functions while there was little use of graphical representations, which were solely used for visual purposes rather than for the internalization of the concept. Activities that required converting among different representations of a function were included, but there was no trace of any information about what to take into consideration in these transitions, what would remain constant, and what would change. In a way, employing different representations in the internalization of the function was overlooked. The textbook did not involve any interdisciplinary exercises or activities, and no activities or questions involved real-world data or examined functional relations so as to highlight the uses of functions in different fields and thereby help students internalize functions.

The curriculum demanded that at the completion of instructional units on functions, students should be able to: define the function; draw its diagram; identify the domain, range, and image of a function; determine equality of functions; explain types of functions; explain the composition of functions through examples; find the inverse of a function; find the inverse from a graph; locate the images of any given pre-images on the axes in graphs and vice versa; interpret the behavior of a function in given intervals; and find $f+g$, $f-g$, $f \cdot g$, and f/g, where f and g are functions defined from R to R (Turkish Board of Education, 2011, pp. 67-68, translation by the author).

This study conceptualized teachers' knowledge regarding the function concept in terms of these expectations as required to teach about functions from such a perspective. The body of research summarized in the previous section also highlighted some components regarding teachers' CK for teaching the function. Table 1 shows the components have been considered as the focus of the study. However, they should not be regarded as a comprehensive list of what teachers need to know about functions in order to teach it effectively, but rather as a set of central issues inherent in teachers' understanding and teaching of the function concept.

Table 1

Components of secondary mathematics teachers' CK about the function concept and corresponded questionnaire items

Knowledge component	Questionnaire item
• Having different views about the concept of function	• Items 1, 2, and 3
• Knowing different representations of functions	• Item 4
• Having a range of concept images of functions	• Items 5 and 6
• Distinguishing between the range and image of a function	• Item 7
• Identifying equal functions	• Item 8
• Identifying pre-images, images, and (pre-image, image) pairs	• Item 9
• Finding the image of a given pre-image	• Item 10

The Study

The study was primarily qualitative in which a questionnaire was used to collect data about secondary mathematics teachers' CK for teaching the concept of function. While multiple-choice items may not provide insight as to the depth of teachers' knowledge, constructed responses could provide more information on teachers' cognitive types of content knowledge (e.g., Fauskanger, 2015). The questionnaire, therefore, included ten open-ended items adapted from the seminal works in teaching and learning of the function concept literature (e.g., Even, 1993; Hitt, 1998; Tall & Bakar, 1992; Vinner, 1983; Vinner & Dreyfus, 1989) and couched in the context of teaching. The items address the knowledge components of the function concept as summarized in Table 1. It is important to emphasize that the items require the teachers to identify students' errors, as well as provide appropriate explanations and correct responses to each item.

In Items 1 and 2, the teachers were asked to state how they think students should define and exemplify the function concept, and in Item 3, they were asked to give examples that could reveal students' understanding of it (Figure 1). As these items demanded teachers reflect on what students are ideally supposed to know, I expected their answers would reveal their own views about the function concept and what they value for students to know about it.

1. How do you think the students should define the concept of function? Please detail them.

2. What kind of examples do you think the students should give for functions? Please detail them.

3. What kind of questions do you think the students should be able to solve to show whether they understand the concept of function?

Figure 1. Questionnaire Items 1, 2, and 3 on 'Having different views about the concept of function'

The teachers were asked how they would respond to a student's inquiry about whether or not a function could be represented in different forms in Item 4 (Figure 2). The teachers' responses would be indicative of their knowledge of different representations of functions and the extent to which they place particular emphasis on certain representations, which might also provide clues about how functions are dealt with in the classroom.

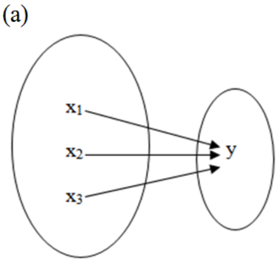
4. Assume one of your students inquires whether or not a function can be shown in different ways. What are some of the different ways that you could use to show functions to respond to this student?

Figure 2. Questionnaire Item 4 on 'Knowing different representations of functions'

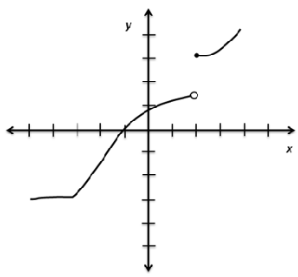
In Items 5 and 6 (Figure 3), the teachers were presented with a student's mistakes and were asked to decide whether the student was correct, and to provide their reasoning. The responses to the items were intended to reveal the teachers' concept images of functions in the context of evaluating the reasonableness of the student's answers.

5. Assume you ask your students to identify if the representations (a) through (f) define a function and one of the students marks them all as "Not a function." For each case, decide whether the student's response is correct or incorrect? Explain why.

(a)



(b)



(c)
$f: R \to R$

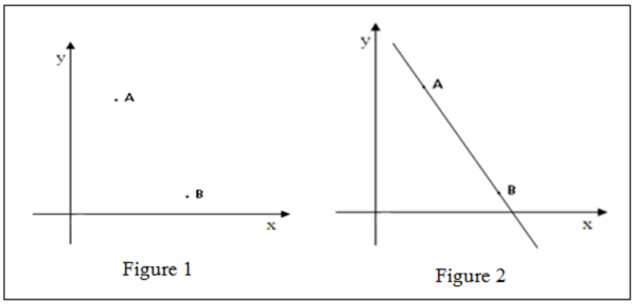
$$x \to \begin{cases} -3x^2+3, & x \geq 0 \\ 5, & x < 0 \end{cases}$$

(d) A correspondence that maps all positive numbers to 1, all negative numbers to -1 and 0 to 3.

(e) $y = 4$

(f) $\{(1,4),(2,5),(3,9)\}$

6. Assume you ask your students to give an example of a graph of a function that runs through points *A* and *B* (Figure 1). A student draws the graph in Figure 2. When you ask, "Is it possible to draw the graph of another function that passes through the points A and B?" the student responds, "No."

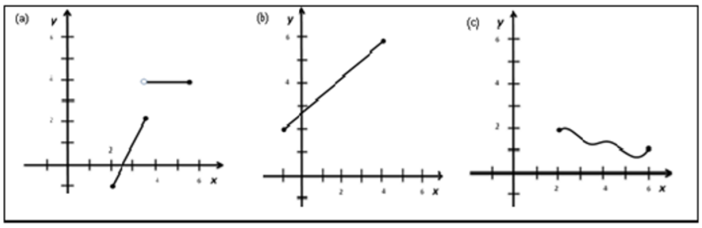


Do you think the student is correct?
If so, explain why.
If not, how do you think the student should have responded?

Figure 3. Questionnaire Items 5 and 6 on 'Having a range of concept images of functions'

The teachers were essentially asked to evaluate situations where a student was expected to relate the domain, range, graph and/or rule of a function in Items 7 and 8 (Figure 4). In the context of explaining and justifying the student's mathematical ideas, the teachers' responses to Item 7 were intended to reveal their knowledge of distinguishing among the terms: domain, range, and image and how they would be used to define a function and its graph within given constraints. The teachers' responses to Item 8 were expected to reveal their understanding of comparing two functions by attending to their domain, range, and applying rules.

7. Assume you ask your students to identify which of the graph(s) in (a) through (c) represent a function whose domain is $\{x : 2 \le x \le 6\}$ and whose range is $\{y : -1 \le y \le 4\}$.



One of the students marks them all. For each case, decide whether the student's response is correct or incorrect. Explain why.

8. As for $f : N \to N$, $f(x) = 4x + 6$, assume you ask your students to identify which item(s) in (a) through (d) "equals" to the function *f*. One of the students identifies all of them as equal to the function *f*. For each case, decide whether the student's response is correct or incorrect. Explain why.

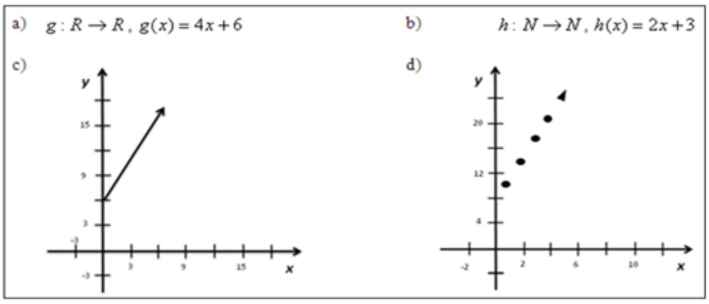
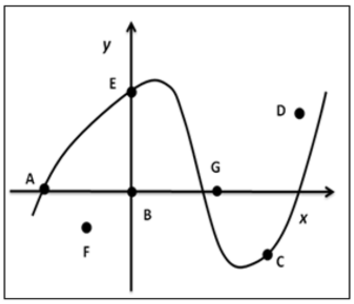


Figure 4. Questionnaire Items 7 and 8 on 'Distinguishing between the range and image of a function' and 'Identifying equal functions' respectively

In Items 9 and 10 (Figure 5), the teachers were required to decide whether the student's answer was correct, and to articulate how they think the student should have responded to the questions. The teachers' responses were intended to reveal the depth of their knowledge of identifying pre-images, images, and (pre-image, image) pairs on graphs through their justifications of the student's answers in this context.

9. Regarding the graph below, assume you ask the following questions to your students: (a) Which points represent an element of the domain? (b) Which points represent an element of the range? (c) Which points represent (pre-image, image) pairs? (d) Which points do not represent (pre-image, image) pairs?



One of the students comes up with the following responses:
(a) *A*, *E*, *C*
(b) *E*, *B*, *G*
(c) *B*, *G*
(d) *F*, *D*
Are the student's responses correct or incorrect? If incorrect, how do you think the student should have responded?

10. Assume you ask your students to locate the pre-images of the point *A* on the following graphs. One of the students responds as given below each graph. For each case, please check if the student's responses correct or incorrect. If incorrect, how do you think the student should have responded?

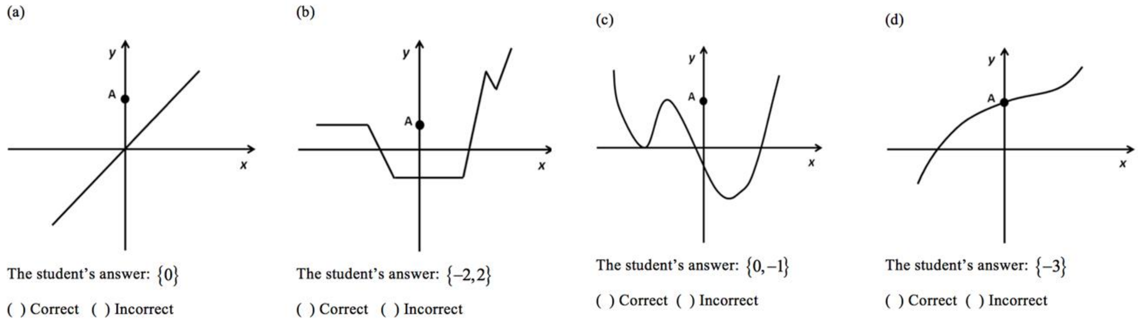


Figure 5. Questionnaire Items 9 and 10 on 'Identifying pre-images, images, and (pre-image, image) pairs' and 'Finding the image of a given pre-image' respectively

In addition, the questionnaire included a section related to teachers' demographic information, namely gender, the years of teaching and professional learning experience.

The content validity of the questionnaire was demonstrated through expert evaluation (Fraenkel, Wallen, & Hyun, 2012). One mathematician and one mathematics educator were given the questionnaire to make judgements about the degree to which the items match the knowledge components of the function concept shown in Table 1. Before using in the study, the questionnaire was piloted with three secondary mathematics teachers not participating the actual study to ensure the clarity of the items.

Participants and Data Collection

The participants in this study were secondary mathematics teachers who were practicing at industrial vocational high schools in Ankara, Turkey due to its convenience as the location where I used to live and work. To recruit the participants, all 20 industrial vocational high schools in the city and the mathematics teachers working there were considered as the target population. I first contacted with the principals at each school. With the help of the principals, I approached the mathematics teachers and briefed them about the questionnaire that they would be invited to complete. To increase the participation rate and allow participants to think and reflect about the items without any time restriction, I suggested the teachers complete the questionnaire during their available time. Each school employed between four to twelve full-time mathematics teachers. While none of the teachers in five schools chose to volunteer to take part in the study, one to six teachers (a total of 42 teachers, 31 female and 11 male) in each of the remaining 15 schools volunteered to participate in the study. I collected the questionnaires either during my personal visits or the teachers posted them in a sealed envelope to protect participant confidentiality. Most of the teachers reported that they had not participated in any in-service training about mathematics education in general and about the teaching of functions in particular. The teaching experience of all teachers ranged from two years to 25 years. The average teaching experience of all teachers was about 7 years in industrial vocational high schools and 13.5 years in general.

Data Analysis

The data were analyzed using content analysis (Creswell, 2007) by utilizing a "descriptive and interpretive" approach (O'Toole & Beckett, 2010, p. 43). After closely examining all answers, I developed a rubric (see Appendix A) to guide the data analysis process for each item. To ensure the validity of my interpretations and data analysis, a mathematician assisted me in analyzing the data. We implemented an inter-rater reliability process (McHugh, 2012). We coded ten teachers' responses to the questionnaire independently and agreed on 93% of the coding. To reach full consensus, we reviewed and discussed the coding for the discrepancies and then revised the rubric to code the rest of the questionnaires. Throughout the analysis, we consistently attempted to discuss and resolve issues that required further attention for consensus.

The teachers' definitions, exemplifications, and questions (Items 1 to 3 respectively, see Figure 1) were coded according to the definitions which consider a function as taking inputs to outputs, a rule assigning x to $f(x)$, a mapping from one set to another, or a set of ordered pairs such that no two pairs have the same first entry but different second entries. The responses that did not provide a clear view of the teachers' thinking about the function, or that showed just a basic understanding of the function were coded as "Other." The teachers' responses were further analyzed if they contained references to the uniqueness condition of functions and the analogies for the function (e.g., a machine transforming olives into olive oil). Their views identified in Items 1, 2 and 3 were compared in order to examine how many teachers presented different views when answering these three items. The teachers' explanations for alternative representational forms for functions (Item 4, see Figure 2) were classified into two general categories: different representations of a function, and function notations, and were also compared with their choices of representations in Item 3. To identify their concept images of functions (Item 5, see Figure 3), the teachers' justifications of whether a relation defines a function were examined. The arguments expressed in their answers were grouped into the several categories including the uniqueness, arbitrariness, split-domain, discontinuity, prototypical examples, and vertical line test. The teachers' concept images were further analyzed based on the number of functions they thought could go through two points and whether their explanations contained references to arbitrary functions or used specific examples or types of functions (Item 6, see Figure 3). Items 7 and 8 (see Figure 4) focus on being able to distinguish between the range and image of a function and identify equal functions, respectively. For both aspects, the teachers' evaluations of the student's responses were recorded if they found them as correct or incorrect. In each case, the teachers' relevant statements were quoted to illustrate their knowledge. In Items 9 and 10 (see Figure 5), the teachers' evaluations of the student's responses were recorded if they found them as correct or incorrect, and the teachers' own responses for were coded as correct, incorrect, or partially correct.

Results

The results were presented under the following knowledge components: having different views about the concept of function, knowing different representations of functions, having a range of concept images of functions, and understanding the core concepts related to functions as evidenced by the responses the participants gave to Items 1 through 10.

Having different views about the concept of function

The results revealed that 17 teachers defined a function (Item 1) as sets of ordered pairs such that no two pairs have the same first entry but different second entries (see Table 2). The other 21 teachers gave definitions that presented functions as taking elements of one set to another set (e.g., taking inputs to outputs, mapping from one set to another). The view of function in the responses of five teachers could not be verified and were categorized as "Other."

This category also included statements indicating that students did not need to define or cannot exactly define the function. Among the 38 valid definitions, 24 mentioned the uniqueness condition as exemplified in the following statements: “There must not be an element left in the domain that is not assigned to an element in the range,” and “Each element of the domain has exactly one image.”

Table 2

View of function presented in teachers’ responses on how students should define the concept of function

View of function	*n**	Illustrative example
Set of ordered pairs	17	“I want them to define relation first and then explain that function is a special relation.”
		“A relation from set *A* to set *B* is a function if there is no element of domain that is not paired with an element of range and an element of domain is paired with exactly one element of range.”
Mapping from one set to another	9	“Given two non-empty sets, every element of domain corresponds to the elements of range, but it can only correspond to exactly one element of range.”
		“A function is a mapping in which every element of domain is paired with no more than one value.”
Taking inputs to outputs	7	“We put in input and get output. For every piece of information, there is another one with which it is paired.”
		“Depending on the type of function, they can define it as a mechanism that transforms an element to another one.”
Rule assigning x to $f(x)$	5	“They can define it as a rule that transforms one element to another. For example, a rule which maps a natural number to its square.”
		“Given two sets that are domain and range, a (mathematical) rule that maps each element of domain to one and only one element of range is called function.”
Other	5	“It [the function] must have domain, range, and image sets.”
		“Students do not need to define function.”
		“Students cannot define function.”
No response	1	-

Note. This and subsequent tables show, *n* as the number of teachers whose answers were categorized in the respective row or column.

A total of 31 teachers’ exemplifications (Item 2) presented a function as a mapping from one set to another or as a rule assigning x to $f(x)$ (see Table 3). For some of these teachers, exemplifying a function as a rule would be simpler and more comprehensible for students. Eleven teachers provided examples that presented a function as taking inputs to outputs or as a set of ordered pairs. The exemplifications of six of the teachers seemed to reflect students’ views about the concept of function. Two of these referred to the basic meaning of “function” (e.g., functions of a phone). Of the 42 valid exemplifications, 12 referred to the uniqueness condition of functions.

While 32 teachers gave questions (Item 3) that represent a function as taking elements from one set to another, four teachers wrote questions that represent a function as a set of ordered pairs (see Table 4). The questions that represented a function as a rule assigning x to $f(x)$ mostly focused upon routines, such as evaluating an algebraic function at specific point(s) and, in a few cases, finding the inverse or composition of functions. Few teachers found these types of questions easy but level appropriate questions for their students. In 11 responses, the view of function could not be defined, and they were grouped as “Other.” In this grouping, four teachers devised questions that asked for the conditions for being a function. Two other teachers drew attention to real-life situations, but it was not clear what kind of function situations they had implied.

Table 3

View of function presented in teachers' responses on how students should exemplify the concept of function

View of function	*n*	Illustrative example
Mapping from one set to another	17	“They might present it by showing domain, range and co-domain on a Venn diagram.”
Rule assigning *x* to $f(x)$	14	“ $f(x)=2x-4,\ g(x)=5$ ” “They can give simple comprehensible examples like $f(x)=4x-3$ ” “I think the following is a simple but nice example: $f: A \to B,\ x \to f(x)=2x-1$ ” “Like substituting different numbers for *x* in a function such as $f(x)$ or $g(x)$ ”
Set of ordered pairs	6	“They should exemplify it not just as rules but also as diagrams [set correspondences] and sets of ordered pairs.”
Taking inputs to outputs	5	“Producing olive oil (output) by processing olives (input) in the factory.” “They can give examples like defining a function and seeing what some elements that they decide are transformed into.”
Other	6	“The examples related to the devices they are using, i.e., computers and cell phones.” “The functions of a (cell/smart) phone.” “One can lose his/her life functions. A tool may have various functions.” “1 more than two times 7 ($7 \cdot 2+1$).”
No response	3	-

Of the 47 questions devised by the teachers, 23 required students to decide if a given relation defines a function, and 27 referred to the uniqueness condition of functions. In questions, the form of function representation was mostly provided. In general, the teachers either provided a function in the form of a set correspondence (n = 16) or an algebraic expression (n = 14). Few teachers favored a set of ordered pairs representation (n = 4) and a graphical representation (n = 5). Even the 15 teachers who mentioned graphs, tables, and verbal statements as different representations of a function, preferred an algebraic expression and/or a set correspondence representation when directing a question to students. Among all of the questions, only the following setting was deemed realistic: “Every year a tree grows 5 cm more than twice its age. If its height was 10 cm when it was planted, what would its height be after three years?”

Table 4

View of function presented in teachers' questions to reveal students' understanding of the function concept

View of function	n	Illustrative example
Mapping from one set to another	17	"I would give the children and their mom an example and make them do the mapping." "I would give a set correspondence and ask if this correspondence defines a function." "[Gives a set correspondence and asks] $f(2)=?$"
Rule assigning x to $f(x)$	15	"$A=\{1,2,3\}$, $B=\{3,5,7\}$, if $f:A\rightarrow B$ and $f(x)=2x+1$, then $f(2)=?$" "$f:R\rightarrow R,\ f(x)=2x+3$, $f(1-2x)=?$ $f^{-1}(x)=?$"
Set of ordered pairs	4	"Which of the following are functions?" [S/he presents some sets of ordered pairs.] "What is a Cartesian product and a relation?"
Other	11	"I would give examples from daily life." "Like the most important functions we use in daily life." "What is the difference between domain and range?" "I would ask a question about whether or not the relations I present are functions." "I would give an element that has two images and have them checked."

Most of the teachers presented only one definition and example. Only eight teachers provided two different types of definitions and/or examples that present two different views and two teachers provided three that present three different views about the function concept. However, this might not mean that the teachers had only one view about the function. They might have had different views but might not have presented all of their possible views in the respective item. Therefore, I compared the views of function that they exhibited in Items 1 through to 3. This comparison revealed that among the teachers whose views could be defined in their responses (n = 41), eight teachers exhibited only one view, 25 teachers exhibited two views, and eight teachers exhibited three different views about the function concept. These findings suggest that 33 out of 41 teachers exhibited two or three different views about the function when defining or exemplifying it or generating a question about it. Interestingly, while most teachers provided definitions that favored the set of ordered pairs view, their examples and questions favored the mapping from one set to another and the rule assigning x to $f(x)$ views. A small number of responses favored the taking inputs to outputs view.

14 out of the 140 answers (two definitions, nine examples, and three questions) included analogies such as a mother-child relationship (n = 6) (e.g., "Set A is the set of children, and Set B is the set of mothers; then, (*i*) no element will be left out in Set A because every child has a mother, (*ii*) a child cannot have more than one mother.") or a machine (n = 5) transforming inputs (e.g., fruits, olives, or wheat) into an output (e.g., juice, olive oil, or flour). Among 42 valid exemplifications and 47 questions, only one setting was realistic; the rest were abstract. For many teachers the uniqueness was a central element in the treatment of a function. Of the 80 valid definitions and examples, 36 referred to this element. Among the 36 teacher-generated questions, 15 asked students to decide whether a given set correspondence defines a function by using the uniqueness condition (n = 23), or directly asked to explain necessary (and sufficient) conditions for a relation to be a function (n = 4).

Knowing different representations of functions

Explanations provided by 22 teachers addressed either the specific function names (e.g., *cos*, *log*) and notations (e.g., *f*, *g*), or did not reveal enough to make a judgement on their thinking about a query relating to whether a function can be shown in different ways (Item 4). As Table 5 captures, fifteen teachers pointed out different representations of a function in their explanations. They mostly mentioned set correspondences (n = 12), graphs (n = 9), algebraic expressions (n = 9), and sets of ordered pairs (n = 8) as alternative ways of expressing a function. Compared to these representations, they mentioned tables (n = 2) and verbal statements (n = 2) less frequently.

Table 5

Focus of teachers' responses to different representations of a function

Focus of the response	n	Illustrative example
Different representations of a function	15	"Yes, there are: Tables, graphs, and expressions." "We can represent functions through sets of ordered pairs, set correspondences, or by their rules."
Notations	17	"It can be expressed as $x \rightarrow y,\ f(x) = y,\ f : x \rightarrow y$" "It has a variety of representations: *f*, *p*, *cos*, *log*." "No different representations, but just *f* could be changed in $f(x)$."
Other	5	"I would tell them that I gave them all possible representations, and there is no other." "There are other representations, which I will show another time." "My students do not ask because they are vocational school students."
No Response	5	-

To further analyze this aspect, I compared the teachers' responses to different representations of functions with their responses to Item 3, where they gave questions by mostly providing the form of function representation. This comparison revealed that seven out of the 27 teachers who did not give a response to the question and whose answers were coded as "notations" and "other", presented no indication as to the different forms of a function, but the other 20 teachers stated questions that present a function in the form of a set correspondence (n = 14), an algebraic expression (n = 8), a set of ordered pairs (n = 2), and/or a graph (n = 1). As highlighted above, the idea of representations was not emphasized in the curriculum. Therefore, it is possible that these 20 teachers did not refer to different representations of a function in their explanations because they might not know what was meant by the word 'representation'.

Having a range of concept images of functions

The results showed that most of the teachers accurately identified the student's responses as incorrect in Items 5 and 6, with the exception of 5f which was (incorrectly) identified as correct by 12 teachers (Figure 6). The nature of a total of 252 arguments that were used by the teachers to justify the student's responses is presented below in Figure 7.

As shown in Figure 6, about one fourth of the explanations (64 of 252) were based on the uniqueness condition. In 5f, for example, four teachers stated that "The domain is not defined. The student cannot know whether any element of the domain is left out", which implies that they would need to check if the given relation satisfies the uniqueness condition. About one-third of the teachers' explanations (82 of 252) were based on prototypical examples. Some of the teachers' concept images were so strongly related with their prototypical examples that they did not accept some of the given representations as functions. To illustrate, in 5f, eleven teachers stated either that the relation does not define a function or that whether it defines a function could not be determined: "It is not a function. There is no domain and range." or "Since the domain is not known, saying that it is a function would not be correct. I am uncertain. It is much more like a relation." The reasons for not accepting some other representations or relations as functions also related to domain and range. Indeed, in all items, the domain and range of the functions were defined, but these teachers wanted the domain and range of a function represented explicitly.

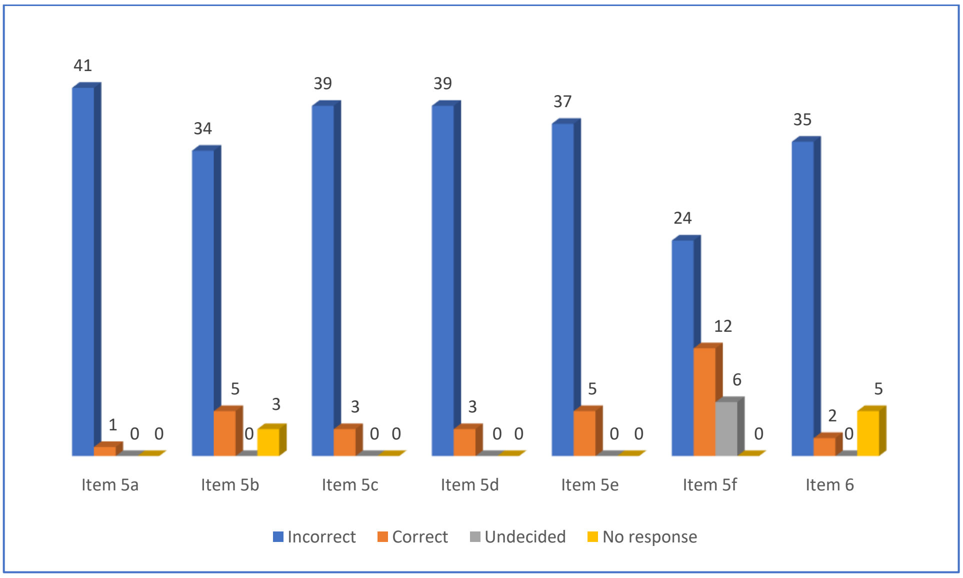


Figure 6. Distribution of teachers' evaluations for the student's responses in Items 5 and 6

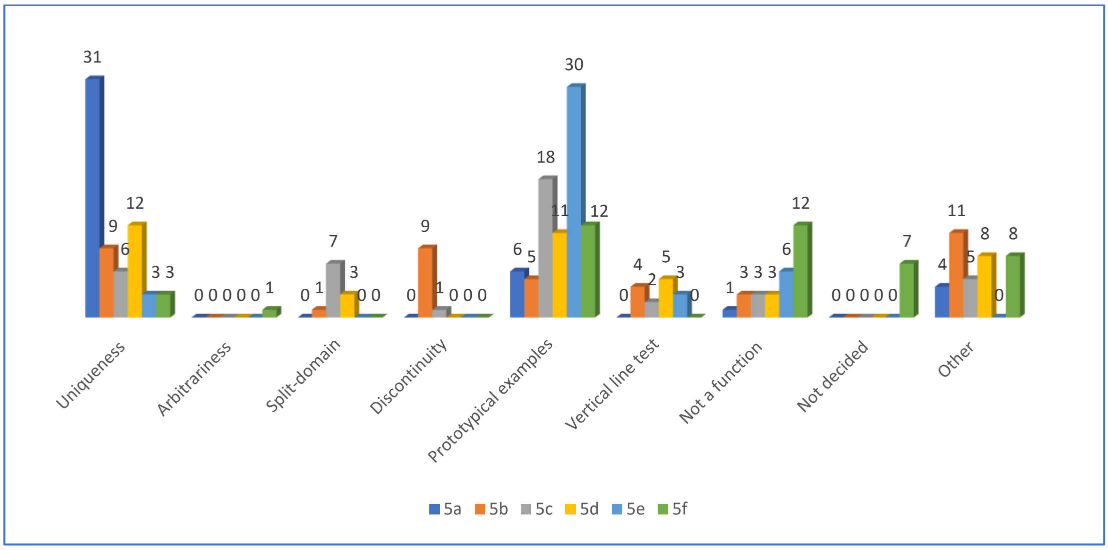


Figure 7. Arguments used by teachers' for justifying the student's responses in Item 5

Besides its domain and range, some teachers expected that a function should always be represented with the $f(x)$ notation. To illustrate, five teachers inaccurately identified the student's response to $y = 4$ (Item 5e), as correct stating that "It is not a function because of *A* = ?, *B* = ? and *f* = ?" or "It is the line $y = 4$. To be a function, it must be stated that *f* is from *R* to *R* and $x \to f(x) = 4$." For ten teachers, a function must map the elements of two sets by means of an arithmetical or algebraic rule. One of these teachers tried to find an equation or a rule to represent $\{(1,4),(2,5),(3,9)\}$ (Item 5f). As the teacher could not find one, the teacher stated that the student's response was correct. This might be an indication of having a concept image that a function must be comprehensively represented by an algebraic equation.

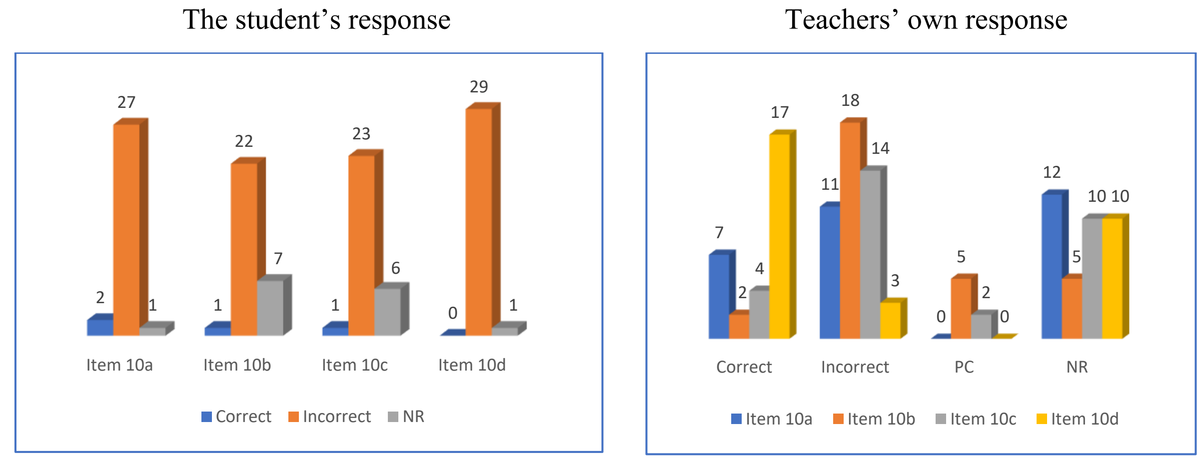
The arbitrariness condition of functions was used by very few teachers as an argument. Only one teacher referred to it in their argument for $\{(1,4),(2,5),(3,9)\}$, stating that: "The student might have thought that functions have to exhibit some regularity." This statement indicates that for this teacher a function does not need to follow a regular rule. In order to investigate the teachers' understanding of arbitrariness further, the questionnaire included Item 6. The results revealed that 35 of the teachers accurately identified that the student's answer was incorrect, five teachers did not give a response, and two teachers inaccurately identified that the student's response was correct. These two teachers might possess the (inadequate) concept image that only linear functions pass through two points.

Item 6 also asked the teachers how they think students should have responded to the question. The results revealed that of these 35 teachers, three did not give an explanation. 19 teachers stated that many or infinitely many functions could be drawn through the points *A* and *B*, without providing an indication of their characteristics, that is, either specific or arbitrary functions. Five teachers thought that a finite number of functions could pass through these two points. One of them indicated its nature (i.e. "an absolute value function") and the other four did not mention their nature but stated that "The student should have said 'yes' and drew a curve." Also, six teachers mentioned specific functions (e.g., parabolas) but did not indicate how many there would be. Only two teachers gave indications of the arbitrariness condition of functions. One teacher stated: "I can draw an infinite number of functions that do not contradict the rule of a function and use the vertical line test to justify them." Even though there was no word used to describe the arbitrariness in this statement, the response suggests that any shaped graph that does not violate the uniqueness condition, can pass through the points. As the teachers were not interviewed, it is not known whether these 30 teachers who did not refer the characteristics of functions, or who mentioned specific functions, were aware of the arbitrariness condition of functions. However, it seemed that some of the participating teachers showed limited understanding of this condition.

Understanding the core concepts related to functions

The findings indicated that while identifying two equal functions (Item 8) most of the teachers considered the domain and range, as well as whether the elements in the domain have identical images. However, in some cases the teachers' explanations were rather indicative of their limited understanding of concepts related to functions. For example, out of thirteen teachers who inaccurately identified the student's answer to 7a as incorrect, six teachers expressed that the graph embodies the numbers given as the domain and range, but there is no function defined or represented by this graph. While four of them thought that there is no indication of the sets from which the *x* and *y* values are drawn, three teachers thought that there are numbers in the domain that are not assigned (to a number in the range) as there is a discontinuity in the graph. One of the teachers who accurately identified the student's response to 7b as incorrect indicated that "It is incorrect. How the function was defined is not given; $y = f(x)$ is not given." This teacher apparently believed that to be deemed a function, a rule (defining the relation or association between *x* and *y*) must be given, regardless of its graph structure. The thirteen teachers who inaccurately identified that the student's answer to 7c is incorrect experienced difficulty in identifying the function graph whose image set is a proper subset of the range. According to these teachers, the domain is shown correctly in the graph but not the range. This indicates that they focused on the endpoints of the graph to determine the domain and range and ignored the fact that $f: A \to B$ means that $f(A) \subseteq B$. Indeed, among all teachers only four teachers contended that the function, the graph of which is given, lies within the given domain and range limits. The comments of two teachers to 7a and three teachers to 7c, who identified the student's answers to these items as correct, revealed that the teachers had attended to whether the graphs pass the vertical line test, rather than focusing on the domain and range that the function should have.

In Item 9, even though 30 teachers recognized the student's incorrect responses, about one-third of the teachers provided no explanation as to why the student was incorrect, particularly in 9c and 9d. Almost half of the teachers either provided incorrect or partially correct answers for 9a through 9d.



Note. NR: No Response; PC: Partially Correct

Figure 8. Distribution of teachers' evaluation of the student's responses and their own thinking for Item 10

With finding the pre-image of an image, in Item 10, while seven of the teachers identified the student's responses as incorrect and one teacher identified them as correct without further commenting on why, four of the teachers left the question unanswered (see Figure 8). Most of the other 30 teachers realized that the student's answers were indeed incorrect, but the majority gave incorrect reasons for why the answers were incorrect. From their statements it appeared that these teachers considered that the images should be points on the graph and these points should be given (or marked) explicitly. Furthermore, statements regarding 10b and 10c revealed that the majority of teachers considered that the graphs in 10b and 10c are not functions because they are not one-to-one. In 10b, only two teachers indicated that the response could well be correct and highlighted that still an indefinite number of real numbers with point *A* as their image existed. Most of the teachers gave correct responses for 10d in comparison to the other parts of Item 10.

Discussion and Concluding Comments

The present study investigated secondary mathematics teachers' content knowledge (CK) for teaching the concept of function in industrial vocational high schools in Ankara, Turkey. As a component of teachers' mathematical knowledge for teaching (MKT), CK was operationalized, for the purposes of this study, to include: an understanding of the facts and concepts within the concept of function domain, making sense of what have been suggested by students, determining whether the responses of students are correct, and formulating correct responses to the students' solutions. Despite there being some development of theoretical frameworks for unpacking teachers' MKT for the concept of function (e.g., Nyikahadzoyi, 2015), no recent studies have focused on investigating in-service teachers' MKT about this concept, or central issues inherent in their understanding and teaching of functions. This study contributes to the field of research into MKT by designing a test instrument and its associated rubric based on seminal research studies in the topic of functions and using them to examine secondary mathematics teachers' knowledge for teaching the concept of function, an approach that has been predominately used within the elementary teaching context.

The findings indicated that most of the teachers placed strong emphasis on the set correspondence and algebraic representations, and their limited focus on graphs, tables, and verbal statements, in relation to the function concept supports the findings of other studies (e.g., Cooney, 1999; Norman, 1992). The teachers found the uniqueness condition of functions very important for students to understand and had a wide repertoire of examples of functions which constitute this condition, and their choice of examples mostly included giving set correspondence relations and checking whether they define a function. As reported in other studies (e.g., Ayalon et al., 2017), most teachers privileged the computational aspects associated with functions.

The participants in the present study appeared to take an abstract approach to functions given that very few of their explanations presented functions in a real-life setting. Some teachers did, however, use analogies when defining and exemplifying the function, presumably with the aim of making the formal definition of a function less abstract. While a few of the analogies underlined the operation aspects of functions (e.g., a machine transforming olives into olive oil), most of them emphasized the uniqueness condition (e.g., the mother-child relationship, i.e. every child has a mother, but a child cannot have more than one mother) and placed little regard on how the values of the quantities or variables in these relationships vary or covary. Very few responses included references to functions as a tool for modeling real-world events (Even, 1993; Sánchez & Llinares, 2003).

The findings indicated that when justifying students' concept images of functions, the prototypical examples (e.g., $f: A \to B, x \to f(x) = 2x - 1$) for the function played a key role in most of the participant teachers' reasoning. They tended to emphasize the uniqueness condition but did not refer to arbitrariness property, which is less visible as a criterion in definitions of function compared to univalence condition, but nevertheless should be included any mathematically valid conception of function (Steele et al., 2013).

Various studies have revealed that textbooks and curricular materials have the potential to influence teacher knowledge, and how that knowledge is translated into specific classroom activities (Davis, 2009; Hill & Charalambous, 2012; Huang, Ozel, Li, & Osborne, 2014; Randahl, 2016; Van Zoest & Bohl, 2005). It is possible that the teachers in this study were strictly following the curriculum or textbook, which seemed to emphasize the set-theoretical conception of a function followed by symbolically oriented, rule-based practices. As such, the teachers' enacted CK may reflect the way in which mathematics curriculum and accompanying textbooks treat functions, an idea that is also highlighted by others (Mesa, 2004; Hill & Charalambous, 2012).

These findings may also be particularly relevant in Turkey where the transition from secondary education to the university context depends, to a large degree, on students' scores on a nationwide multiple-choice standardized examination with mathematics as a significant component. (European Schoolnet, 2018). Research suggests that community (parents, schools) expectations in relation to these kinds of examinations influence teachers' instructional practices (Altinyelken, 2011; Altinyelken & Sozeri, 2017) and subsequently their MKT. For example, if the examination focuses on symbolic manipulation and procedural techniques as opposed to concepts associated with functions, then teachers are likely to privilege the former.

The findings of this study suggested there were limitations in the participant teachers' CK in relation to the following core concepts of functions: the distinction between the range and image of a function; identifying pre-images, images, and (pre-image, image) pairs; and finding the pre-image of an image. This might be a contextual dilemma given that these teachers had been teaching in vocational-technical high schools for most of their careers. None of the teachers reported having participated in any professional development activities specific to mathematics education. Research suggests that students' mathematical knowledge and performance in vocational-technical education are usually very low (e.g., Hatisaru & Erbas, 2013, 2017; Alacacı & Erbaş, 2010; Lewis, 2000). In Turkey, although all mathematics teachers go into the same teacher education pathway, low expectations and beliefs like "Vocational students cannot learn (much)!" or "Vocational students do not understand mathematics" among mathematics teachers in vocational-technical high schools (Hatisaru, 2014) might not encourage them to seek strong CK. These perceptions may serve as a disincentive for mathematics teachers in these schools to seek to enhance their CK.

Limitations and Directions for Future Research

In this study, I surveyed teachers' CK for teaching in one area, the concept of function, rather than extending to mathematics topics. Although other theoretical perspectives might have been used to explore teacher knowledge, I chose to use the MKT framework (Ball et al., 2008) because of the way in which it conceptualizes teacher knowledge (Charalambous, 2016). Other approaches may generate different results from those obtained in this study and I hope that other researchers will investigate that possibility. The data in this study was collected from mathematics teachers from fifteen different industrial vocational high schools in a region within Turkey. As such, this sample might not be representative of mathematics teachers in general education high schools and of the entire population of secondary mathematics teachers within Turkey or in other countries. Notwithstanding these limitations, the study contains several implications and directions for future research.

From a theoretical standpoint, the study implies that CK is the common denominator in teacher knowledge conceptualizations (Charalambous, 2016; Garner, 2007). I considered that in the secondary teacher knowledge context, common content knowledge and specialized content knowledge were not distinct and therefore operationalized CK as including both common and specialized content knowledge. I believe, as Howell (2012) stated: "We should seriously consider the drawbacks of such a strong focus on the difference between SCK [specialized content knowledge] and CCK [common content knowledge] for secondary MKT, where the argument to be made is less that teachers need a lot of knowledge and more that they need to know how to think about mathematics as mathematicians." (pp. 238-239). In addition, I suggest measuring teacher knowledge by using open-ended items (Fauskanger, 2015) situated within the teaching context. To provide plausible explanations for the patterns that emerge, future research should include interviews with teachers for data triangulation to overcome the limitation of this study. Future studies could also include classroom observations to investigate how teachers' CK for particular mathematics content, is used in the act of teaching.

From a practical perspective, the study's findings lend credence to the idea that CK for teaching mathematics should not be assumed to develop in practice, and that continuous professional learning is crucial for teachers (Hill & Charalambous, 2012). In Turkey, the

Ministry of National Education provides professional development seminars to teachers of mathematics, but these seminars are usually limited to presentations or in the introduction of national school curricula when they are updated or revised (Hatisaru, 2018). The 2017-2023 Teacher Strategy Document points out the need to enhance the quality of teacher development activities and provide more and varied training for teachers based on their needs (Directorate of Teacher Education and Development, 2017). The errors made by participant teachers in this study could inform either personal development or collective understanding during such teacher professional learning activities. In relation to this, an interesting follow-up study would be to test the effectiveness of professional learning in teacher CK in specific mathematics content areas. The results also suggest the possible dampening effect of student academic profiles on teacher knowledge. Nevertheless, it must be acknowledged that this result is tentative. I believe that the ways in which student academic profiles contribute to teachers' CK for teaching mathematics content, should be verified through further studies.

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Appendix A: The function concept questionnaire rubric

Item	The knowledge component	Themes and Codes	Coding
1. How do you think the students should define the concept of function? Please detail them. 2. What kind of examples do you think the students should give for functions? Please detail them. 3. What kind of questions do you think the students should be able to solve to show whether they understand the concept of function? [Adapted from Cooney, 1999]	Having different views about the function concept	(i) Taking inputs to outputs (ii) A rule taking *x* to *f(x)* (iii) A mapping from one set to another (iv) A set of ordered pairs - Other - Not responded	The responses were coded as (i) if they contained references to an operation or manipulation, as (ii) if they contained references to a rule, as (iii) if they contained references to a correspondence or mapping between the elements of two sets, and as (iv) if they contained references to a relation and/or a Cartesian product. The responses that did not provide a clear view on teachers' thinking about the function or that define or exemplify it not in a mathematical sense but in a literal way (i.e., using the word "function" as synonymous with a duty or a service expected to be performed as in "function of a teacher" or "function of a machine or device,") were coded as "Other." The responses were further analyzed if they contained references to the uniqueness condition of functions and the analogies for the concept of function. The exemplifications and questions were analyzed whether settings for them were abstract or realistic. The responses to Item 3 were also analyzed in terms of the choice of representation of a function (e.g., an algebraic expression, a graph, a set of ordered pairs). The views were compared in order to examine how many teachers presented different views when answering these three questions. When making this comparison, "1" was assigned to each different view that was represented in the explanations and "0" to the same view. The total indicated the number of different views represented in each teacher's explanations across the three items.
4. Assume one of your students inquires whether or not a function can be shown in different ways. What are some of the different ways that you could use to show functions to respond to this student? [Adapted from Bolte, 1993]	Knowing different representations of functions	(i) Different representations of a function (ii) Function notations - Other - Not responded	The explanations were coded as (i) if they contained references to different representations of a function such as set correspondences, graphs, and algebraic expressions. They were coded as (ii) if they contained references to different function notations such as $x \to y$, $f(x) = y$, and $f: x \to y$, but not to different representations of a function. The responses were also compared with the choices of representations in Item 3.
5. Assume you ask your students to identify if the representations (a) through (f) define a function and one of the students marks them all as "Not a function." For each case, decide whether the student's response is correct or incorrect. Explain why. [Adapted from Even, 1993; Hitt, 1998; Markovits et al., 1988; Tall & Bakar, 1992]	Having a range of concept images of functions.	- Uniqueness - Arbitrariness - Split domain - Discontinuity - Prototypical examples - Vertical line test - Not a function	The student's responses to the item (a) through item (f) are incorrect. The teachers' evaluations of the student's responses were coded if they found his/her response as correct or incorrect. The arguments expressed in the teachers' answers were grouped into the eight categories. They were coded as uniqueness if they contained references to the uniqueness of the image of each element in the domain

		- Not decided - Other	and as arbitrariness if they contained references to the arbitrary nature of functions implying that functions do not have to exhibit any regularity (i.e., described by any specific expression or by a particular shaped graph). The explanations were coded as split-domain if they included arguments like "The domain of the correspondence splits into two subdomains, in each of which a different rule of correspondence holds. As a consequence, the graph may change its character from one subdomain to the other" (Vinner & Dreyfus, 1989, p. 361). If the statements included arguments like "The graph has a gap. The correspondence is discontinuous at one point in its domain" (Vinner & Dreyfus, 1989, p. 361), they were coded as discontinuity. In the explanations, if there was a tendency to transform the respective relation into graphs and to determine their functionality by using the vertical line test, they were coded as the vertical line test. The statements were coded as prototypical examples if they involved prototypes, typical examples or situations related to functions, as they appeared to be accepted with confidence and called upon without need for further justification. Examples of such statements are: "a function must be represented by $f(x)$", "a constant function and a piecewise function should be represented by a graph," "a function is usually described by a rule," "this is a constant function," and "this represents a piecewise function."
6. Assume you ask your students to give an example of a graph of a function that runs through points *A* and *B* (Figure 1). A student draws the graph in Figure 2. When you ask "Is it possible to draw graph of another function that passes through the points *A* and *B*", the student responds, 'No'. Do you think the student is correct? If so, explain why. If not, how do you think the student should have responded? [Adapted from Even, 1993; Markovits et al., 1988]	Having a range of concept images of functions	- Arbitrary functions - Specific functions - No indication	The student's responses are incorrect. Indeed, there are infinitely many arbitrary functions passing through two points. The functions do not need to be specific functions (e.g., parabolas). The teachers' evaluations of the student's responses were coded if they found his/her response as correct or incorrect. The teachers' concept images were further analyzed based on the number of functions they thought could go through two points and whether or not their explanations contained references to arbitrary functions or used specific examples or types of functions.
7. Assume you ask your students to identify which of the graph/s in (a) through (c) represent a function whose domain is $\{x : 2 \le x \le 6\}$ and whose range is $\{y : -1 \le y \le 4\}$. One of the students marks them all. For each case, decide whether the student's response is correct or incorrect. Explain why. [Adapted from Markovits et al., 1988]	Distinguishing between the range and image of a function		The student's responses to item (a) and (c) are correct; but, that to item (b) is incorrect. The teachers' evaluations of the student's responses were coded if they found his/her response as correct or incorrect. In each case, the arguments expressed in the teachers' answers were noted and relevant statements were quoted to illustrate their knowledge.
8. As for $f : \mathrm{N} \to \mathrm{N}$, $f(x) = 4x + 6$, assume you ask your students to identify which item/s in (a) through (d) "equals" to the function *f*. One of the students identifies all of them as equal to the function *f*. For each case, decide whether the student's response is correct or incorrect. Explain why. [Adapted from Markovits et al., 1988]	Identifying equal functions		The student's responses to item (a), (b), and (c) are incorrect; however, that to item (d) is correct. The teachers' evaluations of the student's responses were coded if they found his/her response as correct or incorrect. In each case, the arguments expressed in the teachers' answers were noted and relevant statements were quoted to illustrate their knowledge.

9. Regarding the graph below, assume you ask the following questions to your students: a. Which points represent an element of the domain? b. Which points represent an element of the range? c. Which points represent (pre-image, image) pairs? d. Which points do not represent (pre-image, image) pairs? One of the students comes up with the following responses: *a. A, E, C* b. *E, B, G* c. *B, G* d. *F, D* Are the student's responses correct or incorrect? If incorrect, how do you think the student should have responded? [Adapted from Markovits et al., 1988]	Identifying pre-images, images, and (pre-image, image) pairs	The teacher's response: - Correct - Partially correct - Incorrect	The points that represent an element of the domain are *A*, *B*, *G*.; the points that represent an element of the range are *B*, *E*; the points that represent (pre-image, image) pairs are *A*, *E*, *C*; and the points that do not represent (pre-image, image) pairs are *B*, *D*, *F*, *G*. If the teachers' responses comprehensively indicated these points, they were coded as correct. If not, then they were coded as incorrect. Responses that included some of the points or relevant explanations were coded as partially correct.
10. Assume you ask your students to locate the pre-images of the point *A* on the following graphs. One of the students responds as given below each graph. For each case, please check if the student's responses correct or incorrect. If incorrect, how do you think the student should have responded? [Adapted from Hitt, 1998]	Finding the image of a given pre-image	The teacher's response: - Correct - Partially correct - Incorrect	In 10a, the pre-image of *A* is a positive real number. In 10b, an indefinite number of real numbers exist, the image of which is point *A*. In 10c, there are four real numbers whose images can be *A*. In 10d, the pre-image of *A* is 0. The responses that comprehensively indicated these points were coded as correct and those that did not as incorrect. While 10a and 10d have one single answer, 10b has (infinitely) many and 10c has four. Responses that included one positive and one negative pre-image but did not include the possibility of more were coded as partially correct in 10b. Responses including at least two correct pre-images were coded as partially correct in 10c.