# Cognitive Processes in Problem Solving in a Dynamic Mathematics Environment

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While one branch of literature is replete with investigations of problem solving and another branch frequently investigates student use of dynamic mathematics environments (DMEs), most of the studies in both of these fields consider whether or not students can solve problems. Far fewer number of studies consider the cognitive processes associated during either problem-solving experiences or DME use and only a handful of studies consider cognitive processes associated with problem solving when working in a DME. This paper reports a novel approach to investigating, defining, and categorizing the cognitive processes used by students in mathematical problem-solving while working in a DME with examples found in student work. Using this approach, problem-solving is found to be nonlinear, iterative, and idiosyncratic. Insights gained by this analysis have both theoretical and practical applications in mathematics education.

Mathematical representations, and translations between representations, are central to student mathematical learning. A rich history exists of investigating student processes of translating from one static representation to another. The manner in which a student interprets, generates, uses, and translates between mathematical representations, provides insight into student understanding of the underlying mathematics.

With the advent of Dynamic Mathematics Environments (DMEs) (e.g., The Geometer's Sketchpad and GeoGebra) mathematics education is evolving from the still powerful era of the graphing calculator to one in which multiple mathematical representations are dynamically interconnected through a computer interface. DMEs allow users to transform mathematical relations and to explore mathematical properties; they provide both novel means through which users can interact with mathematical concepts and novel types of investigable mathematical tasks. The application of DMEs continues to influence mathematics instruction in the areas of concept and skill development, reasoning, communication, and problem solving (NCTM, 2000). However, there remains a dearth of research on the impact of DMEs in respect to problem solving.

While problem solving has been investigated for decades, it has often been examined through the lens of whether or not students can complete problem tasks rather than cognitive processes they use in solving the problem. There is a need to investigate problem solving in the context of DMEs beyond simply whether or not students can solve particular problems; ripe for investigation are the cognitive processes involved in DME problem solving. While examining problem solving in a DME may provide previously unrecognized insight into problem solving in general, this paper focuses on the nature of student cognitive processes when problem solving in a DME.

## Background Literature

The background literature for this study considers several dimensions. These include representations, transformations and translations, problem solving, and dynamic math environments and are now addressed.

### *Representations, Transformations, and Translations*

Mathematical representations (verbal, oral or written; numeric or tabular; symbolic or algebraic; and graphical or diagrammatic) are external objects which serve as a symbol scheme for encoding, communicating, and operating on or with mathematical relations, objects, or concepts (Cobb, Yackel & Wood, 1992; Kaput, 1987a; Zhang, 1997). Through somewhat parallel and complementary nomenclature, numerous authors explain each of the characters and syntactic rules associated with each representation as: a representation with a given format and operators (Ainsworth, 1999); a set of characters and rules for identifying and manipulating a symbol scheme (Kaput, 1987b); a symbolic encoding of mathematical relations, knowledge, and operations (Zhang, 1997); a collection of permitted configurations defined by primitive elements, characters, and signs (Goldin, 1987); and signs and their defined rules of association Duval (1999, 2006).

*Transformations* are the rewriting of a representation into another form of the same representation. For instance, $2x^2+10x+12$ can be transformed to $2(x+2).(x+3)$. Through performing transformations, students can better ascertain particular characteristics of the mathematical concepts contained in the representation (Edwards, 2003). While students employ different processes as they interact with mathematical representations, the entire translation process can be encapsulated through three interrelated activities: interpretation (the activity of interpreting either the source or target representation; implementation (the activity of translating the source representation to the target representation); and preservation (the activity of determining conceptual equivalence between the source and target representations) (Adu-Gyamfi, & Bossé, 2014; Adu-Gyamfi, Bossé, & Chandler, 2015; Adu-Gyamfi, Stiff, & Bossé, 2012).

While interacting with mathematical representations, students employ semantic elaboration or syntactic elaboration (Brown, Bossé, & Chandler, 2016; Kaput, 1987a, 1987b). Syntactic elaboration constitutes considering and acting *locally* (Leinhardt, Zaslavsky, & Stein, 1990) on a representation’s attributes and characteristics without *globally* (Duval, 2006) comprehending the mathematical meaning encoded in the representation. Semantic elaboration denotes *globally* interacting with a representation’s encoded conceptual information without being overly focused on the representation’s precise, *local* characteristics and attributes. When performing translations between representations, students are more error prone when they use local actions (syntactic elaboration) than when they use global actions (semantic elaboration) (Adu-Gyamfi, Stiff, & Bossé, 2012; Bossé, Adu-Gyamfi, & Cheetham, 2011a, 2011b; Dreyfus & Eisenberg, 1987; Dunham & Osborne, 1991; Gagatsis & Shiakalli, 2004).

*Translation* denotes the process through which well-formed concepts, ideas, and relations of one mathematical representation (the source) are successfully reformulated into a different type of representation (the target). Some describe the translation process as moving from one mode of representation to another (Janvier, 1987) requiring an interpretation of a source into a different representational modality (Roth & Bowen, 2001). Rather than the term *translation*, the perspective of registers of semiotic representations employs the term *conversion*, denoting the process used to change representation register without changing the objects being denoted (Duval, 2006).

Simultaneous characteristics and commonalities exist among numerous descriptors of the processes involved in representational translations including: reading and encoding (Kaput, 1987b); analysis and synthesis of the meaning of the source representation, target representation formulation, transfer of the meaning of the source representation into the target representation, and restructuring of target representation (Lesh, Post, & Behr, 1987); and selective encoding, combination, and comparison (Sternberg, 1984). Synthesizing these mechanisms, Bossé, Adu-Gyamfi, and Chandler (2014) define four processes that are key to representational translation: *unpacking the source*, *preliminary coordination*, *constructing the target*, and *determining equivalence*. These processes are defined later in respect to this study's initial conceptual framework.

## *Problem Solving*

Although problem solving continues to be a rich topic in the literature, much of today's work builds off seminal research on mathematical problem solving through the 1980's and 1990's. Through this era, problem solving was recognized and treated as independent content (e.g., Cobb, Wood, & Yackel, 1991; Evan & Lappin, 1994; Lester, Masingila, Mau, Lambdin, dos Santon, & Raymond, 1994; Olkin & Schoenfeld, 1994; Schifter & Fosnot, 1993; Schoenfeld, 1994; Stanic & Kilpatrick, 1989; Van Zoest, Jones, & Thornton, 1994). For instance, Schoenfeld (1994) investigated problem solving as a sense-making tool. Others investigated how student inspection of their own problem solving and that of others led them to better understand problem solving (Carpenter, 1989; Stacey & Groves, 1985; Thompson, 1985). Lester et al. (1994) investigated the effects of problem solving instructional experiences develop students' conceptual understanding. Others investigated how problem solving: develops students' mathematical self-confidence (Schifter & Fosnot, 1993); motivates student learning (Stanic & Kilpatrick, 1989); and affects task engagement, time on task, and overall enjoyment (Olkin & Schoenfeld, 1994). Pajares and Kranzler (1995) investigated the circular nature of problem solving: greater mathematical knowledge leading to better mathematical problem solvers and stronger problem-solving skills leading to more efficient mathematics learning. More modern research regarding mathematical problem solving considers problem solving heuristics, problem solving as an instructional method, teacher questioning techniques, interactions among teachers and students, student dialogue, and socially mediated understanding.

## *Dynamic Mathematical Environments*

Dynamic mathematics environments (DMEs) are computer environments that facilitate users being able to create and dynamically manipulate mathematical objects (e.g., parameters, functions, geometric objects, measurements, data elements) while the objects remain constrained by particular properties (Dick & Hollebrands 2011; Hollebrands, 2003; Hollebrands & Lee, 2012; Hölzl, 1996; Hoyles & Noss, 2003; Jones, 2000; Lopez-Real & Leung, 2006). These environments provide the opportunity for users to connect the world of mathematical theory to their experiences (Baccaglini-Frank & Mariotti, 2010) and to observe and experiment with logical dependencies among mathematical structures, properties, elements, and transformations (Hollebrands & Lee, 2012). Examples of DMEs include dynamic geometry environments (DGEs) (e.g., The Geometer's Sketchpad, GeoGebra, Cabri), dynamic statistical environments (e.g., Fathom, TinkerPlots), and Cinderella, among a growing number of others.

Dynamically interconnected representations allow concretizations of mathematical concepts through instantaneous feedback of mathematical ideas and novel forms (Arzarello, Micheletti, Olivero, Robutti, Paola, & Gallino, 1998; Goldenberg & Cuoco, 1998; Pea, 1985; Ruthven, Hennessy, & Deaney, 2008). Through this, DMEs have become instrumental in mathematical problem solving as students can create and manipulate constructions and investigate multiple dynamic representations simultaneously (Arzarello et al., 1998; Arzarello, Olivero, Paola, & Robutti, 2002; Baccaglini-Frank, & Mariotti, 2010, Garofalo, Drier, Harper, Timmerman, & Shockey, 2000; Lee, Harper, Driskell, Kersaint, & Leatham, 2012; Lee, & Hollebrands, 2006).

Considering student problem solving in DMEs, Brown, Bossé and Chandler (2016) catalog eighteen types of errors which student can perform. These errors consider domains (mathematics, technology, and problem solving), processes (interpretation, activity, and evaluation), and interactions (syntactic, and semantic). Additionally, the sequence of errors encountered in DME work is nonlinear and iterative. In another study of student problem solving in DMEs, Jobrack, Bossé, Chandler, and Adu-Gyamfi (2018) find that mathematical and technological understandings are both interdependent and independent. They show that while skill in using technologies does little to mitigate student mathematical weaknesses, stronger mathematical skills can overcome technological weaknesses when problem solving in a DME. When students have strengths in both mathematics and technology, the two fields become interconnected and problem solving is enhanced. When students demonstrate weaknesses in both mathematics and technology, both dimensions act independently and neither seems to have a significant effect on the other. At other times, each weakness seems to exacerbate the weaknesses of the other.

## Research Questions

As previously stated, problem solving has been investigated for decades. However, the majority of this research has centered on the question as to whether student could or could not solve particular problems and not what cognitive processes, they employ in doing so. Additionally, the role of DMEs as a tool for teaching and learning as well as to how it may facilitate problem solving have been investigated. However, investigating the cognitive processes associated with problem solving in a DME is relatively novel and a potentially fruitful realm of study. This study simply analyzes the work and articulations of two students' problem solving in a DME in order to discern some commonalities and differences in their cognitive problem solving processes.

## Initial Conceptual Framework

As will be seen later in the research methodology, the immediately following framework was the beginning foundation for this study. In employing this framework, initially developed to recognize cognitive processes associated with students translating from one mathematical representation to another Bossé, Adu-Gyamfi, and Chandler (2014), it was deemed insufficient to capture all of the cognitive processes students use when problem solving in a DME. As will be seen in the research methodology, this framework had to be modified to account for this more multidimensional investigation.

The cognitive processes associated with translating from one mathematical representation to another include the following (Bossé, Adu-Gyamfi, & Chandler, 2014):

### *Unpacking the Source*

Each mathematical representation contains a set of densely packed micro-concepts; these micro-concepts must be unpacked in order comprehend the mathematical concepts encoded in the representation (Bossé, Adu-Gyamfi, & Chandler, 2014; von Glasersfeld, 1987). Using different vocabulary, Lesh et al. (1987) discuss the analysis and the synthesis of meanings of the source representation; Kaput (1987b) defines reading the representation; and Duval (2006) and Sternberg (1984) (via selective encoding) consider the recognition of, and differentiating between, relevant information and irrelevant information.

As students unpack a representation, they may overly focus on select micro-concepts, inadvertently omit particular micro-concepts, and be drawn away from important micro-concepts (Bossé, Adu-Gyamfi, & Chandler, 2014). Lesh et al. (1987), Kaput (1987b), and Sternberg (1984) note that translator often unsatisfactorily attempts to differentiate valuable from irrelevant information. Some micro-concepts associated with a particular representation can cause some students to become more confused than informed. Micro-concepts that distract the translator toward irrelevant information are denoted *confounding micro-concepts* (Bossé, Adu-Gyamfi, & Chandler, 2014)

### *Preliminary Coordination*

Since source and target representations include different characters, configurations, and syntactic rules, translations require one to identify and reformulate mathematical concepts, relations, or objects in a manner that transcends the mere networking of facts between representations to making associations between representations (Duval, 2006). The initial steps of preliminary coordination entail the amorphous conceptualization of notions from the source representation into forms commensurate with the target representation and are articulated as: target representation formulation and source to target (Lesh et al., 1987); selective encoding, combination, and comparison (Sternberg, 1984); and reading and encoding (Kaput, 1987b).

Consistent with Kaput’s (1987b) syntactic and semantic elaborations and the findings of others (e.g., Clement, Lockhead, & Monk, 1981; Knuth, 2000; McGregor & Stacey, 1993), Bossé, Adu-Gyamfi, and Chandler (2014) find that students employ different techniques within the process of preliminary coordination. These include *fact-mapping* (a local, syntactic, mechanical, algorithmic heuristic focused on disconnected micro-concepts and lacking conceptual understanding) and *concept-mapping* (a global, sematic, conceptually interconnected, conceptually dense heuristic). Additionally, students employ heuristics such as formations of: the *meta-target* (an amorphous investigation of the source in respect to the target to attempt to ascertain relevant and irrelevant information); the *target skeleton* (an initial prepopulated form of the target representation, employing syntactic elaboration and the rudimentary attempts at networking (Duval, 2006) between the source and target representations); and the *transitional target* (a temporary, working model of the target proposed as incomplete, inadequately answering the investigation, and in need of revision).

*Constructing the Target*

After preliminary coordination, students construct the target representation employing techniques previously defined by: Lesh et al (1987) as latter aspect of target representation formulation, source-target content transference, restructuring of target representation; Sternberg (1984) as selective combination; Kaput (1987b) as encoding; and Duval (2006) as discerning associable concepts across source and target representations. This process associates facts and concepts between source and target representations, iteratively bounces between preliminary coordination and constructing the target, and transitions along the path of target skeleton, transitional target, and target representation.

*Determining Equivalence*

Determining representational equivalence necessitates considering source and target representations both independently and connectedly and evaluating whether necessary notions are simultaneously denoted in each representation such that the target representation commensurately denotes the relevant information from the source information. This necessitates recognizing: ideas shared between the two representations; ideas unique to each representation; ideas implicit within either representation and only discoverable through steps taken in the translation process; ideas that were previously ignored; and ideas from either the source or target that may lead to incorrect target representations. Lesh et al (1987) defines this process as restructuring of target representation and Sternberg (1984) employs the term selective comparison (rendering newly encoded or combined information meaningful by perceiving its relations to old information previously stored).

## Methodology

*Participants*

Participants in this study were two undergraduate preservice mathematics education teachers at a university in the southeastern United States. Participant involvement in the study was completely voluntary with no inducements provided to the students. Both of these students had previously completed a course in synthetic and metric Euclidean geometry. It was anticipated that these students possessed sufficient mathematical background to complete the research activities. Both students had extensive previous experience with Geometer's Sketchpad (GSP), a DME. GSP was selected as the DME for this study because students had extensive experience with GSP. In order to learn the basics of Boolean operations within the GSP environment, study participants were provided two days of researcher-directed, in-class training activities supplemented with online video tutorials, instruction, and activities that were provided for the duration before the performing of the research activities.

Combining participants' knowledge of geometry, GSP, and Boolean applications in GSP, it was anticipated that participants' problem-solving processes could be investigated without the distractors of limited knowledge of these domains. Although the research task was intended to be challenging, it was planned that none of these domains would be a significant hindrance to problem-solving strategies and processes employed by the participants.

### *Research Activity*

Because the training in both GSP and in Boolean applications in GSP was extensive, pre-assessment regarding participant knowledge of such was unnecessary. The research activity was defined as:

> *Circle and Triangle Task*: Using Geometer's Sketchpad, construct an independent, arbitrary, and movable circle and triangle. Use Boolean operations and create a method to dynamically determine and report which of the following cases manifested at any particular instance; the circle is inside the triangle, the triangle is inside the circle, the two shapes are overlapping, or the two shapes are disjoint.

The proposed results of the research activity are provided in Figure 1. Notably, among the four cases, far more difficult was the last, to determine when the two figures are disjoint.

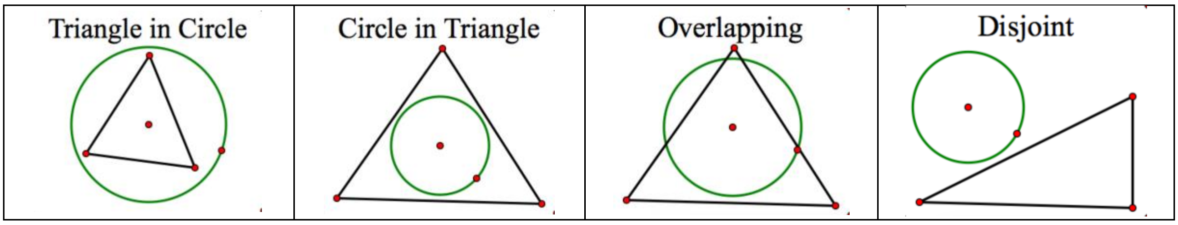


*Figure 1.* Proposed results of the research activity (Figure from Jobrack, Bossé, Chandler, and Adu-Gyamfi, 2018). Use the URL http://appstate.edu/~bossemj/DMEProcesses/DBM/ to preview the activity solution

Investigating different aspects of student work and understanding, versions of this task have been previously employed in research projects (Brown, Bossé, & Chandler, 2016; Jobrack, Bossé, Chandler, & Adu-Gyamfi, 2018), with different sets of participants, and foci on different perspectives and results. These have included significantly expanding the number of observable error types that can be made by students problem solving in a DME and the recognition of the interplay of students' strength and weaknesses in respect to the realms of mathematics and technology when problem solving in a DME.

This task was selected because it had the following characteristics: it considered typical sophomore high school level mathematics; was topically integrated (e.g., measurement, fractions and number theory, and geometric congruence and similarity); entailed rich mathematical problem solving (open-ended, with numerous entry points, and numerous potentially successful heuristics); addressed Krutetskii's (1976) three dimensions of mathematical abilities (i.e. reversibility, flexibility, and generalizability); and involved Boolean operations within Sketchpad constructions. Selecting a task that required Boolean applications in GSP was deemed valuable in that this would exemplify the uniqueness, flexibility, and problem solving power of DGEs. Needed for task completion were skills and understanding regarding employing Boolean operations to make comparisons (e.g., lengths and angle measures) and making elements (e.g., points, lines, text) exist or not exist. Although this task could be completed in innumerable ways, five or six conventional heuristics were anticipated as most possibly leading to task completion.

To complete the task, participants worked independently in a computer lab and were allotted three hours to complete the activity. The task required that participants construct and measure geometric figures and employ Boolean binary and unary operators (e.g., AND, OR, and NOT) and comparisons (e.g., >, <, =) regarding various conditions and measurements in the sketch. These Boolean tools were provided via a file of GSP custom tools.

## *Data and Analysis*

Using a task-based interview design, participants were asked to complete the research task (Goldin, 2000). Participants were given instructions to complete the task and were asked to work individually as researchers circulated among them, inquiring regarding participants' constructions and heuristics. When participants were stymied in the task for a protracted period, they were occasionally assisted with precise technological details (not necessarily leading to solutions). For instance, when one of the students knew that she wanted to use parameters equal to 0° and 180°, since the idea of using parameters in this way was correct, the researcher gently reminded the student how to generate parameters in GSP.

As participants worked on the task, Camtasia (www.techsmith.com) was used to record their activities in two ways. All participant task-related activities in GSP were recorded along with all communications from the participants or between the researcher and the participants. Additionally, using the computer's webcam, the participants' facial expressions were recorded. This assisted in explaining pauses in recorded participant work on the task. In addition to the Camtasia recordings, students worked out equations and ideas on paper. This written work was also collected for analysis.

Figure 2 provides a screenshot of student work. While the screenshot demonstrates multiple levels of superimposed work, it does not include erased or pre-corrected work, the iterative or repetitious nature of the employed heuristic, or the reasoning behind steps taken. Therefore, the complexity of the work and the problem solving process necessitated that participants' work and thought processes be analyzed through step-by-step investigations of the Camtasia recordings and the participants' work.

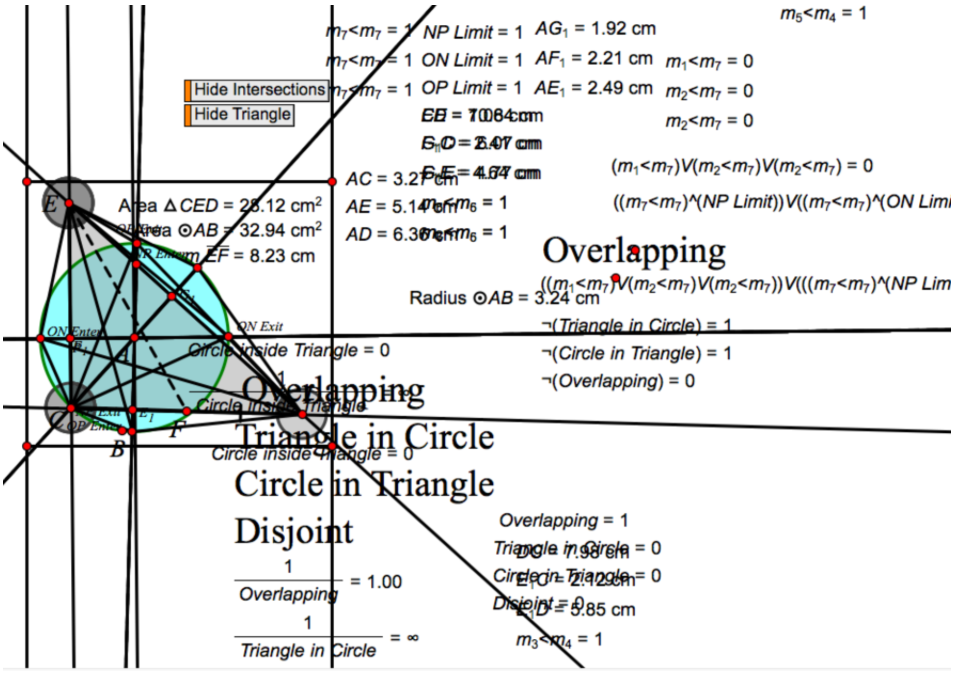


*Figure 2.* Example of student work before cleanup (from Brown, Bossé, & Chandler, 2016).

Two teams of researchers were involved in the analysis of the data (audio-video recordings). All the researchers worked at universities in the field of mathematics education and had experience with qualitative research techniques. Team A was familiar with an initial framework for this investigation. This framework included the stages of representational translations as defined by Bossé, Adu-Gyamfi, and Chandler (2014): unpacking the source, preliminary coordination, constructing the target, and determining equivalence. Team B was kept unaware of this framework. Together, the teams performed the following stages of analysis: First, the audio-video recordings were transcribed by two of the researchers. Second, all data (audio-video recordings, participant written work, and transcripts) were coordinated and reviewed to identify themes and categories (Bogden & Biklen, 2003; Creswell, 2003) and data was sorted into those categories. Team A employed the initial framework and Team B employed open coding (Flick, 2009) to accomplish this initial analysis. The initial coding structures were then compared and contrasted, leading to recognition of similar, different, and missing constructs.

This initial comparison of coding constructs led Team A to recognize that some of the participant actions could not be coded based singularly on the initial framework. Particularly missing was the preliminary construct *Problem Defined*. Consistent with Polya's (1945) first step in problem solving, *understanding the problem*, defining the problem entails: dissecting the problem statement; gaining understanding of the problem; and being able to globally understand what the nature of the solution may look like – albeit sans a heuristic to accomplish the task. Notably, at least the first iteration of this construct would occur before unpacking the source representation as defined in the initial framework.

This addition of one new construct to the initial framework, however, did not create a framework that fully coded for all the behaviors observed in the transcripts. The rationale that was proffered to explain these gaps is that the original framework was contextualized in the translation from one well defined mathematical representation to another well-defined representation; in open ended mathematical problem solving, both the problem and the solution may be more amorphous and processes to arrive at a solution are not as straight forward.

The two teams of researchers combined to reconsider the data and consider how developed constructs and codes could be syncretized. Researchers were able to employ the process of check-coding (Miles & Huberman, 1994) to clarify thinking, sharpen code definitions, and reach consensus (Strauss & Corbin, 1990). They arrived at the following constructs in respect to problem solving in a dynamic math environment:

**Problem Defined (PD):** (1) Interpreting the problem statement; (2) gaining and intuitive and non-formalized understanding of the problem (but not how to solve it); (3) being able to globally, but imprecisely, understand what the nature of the solution may look like.

**Problem Unpacked (PU):** (1) Recognizing larger mathematical and technological concepts within the problem and possible cases that may need to be considered (how to address these cases is not yet investigated); (2) recognizing possible intuitive, non-formalized global heuristics that may prove successful.

**Self-Assessment (SA):** (1) Assessing if one possesses sufficient mathematical knowledge of the problem and possible solution strategies (or the immediate ability to learn/construct the knowledge) necessary to produce a solution; (2) Assessing if one possesses sufficient knowledge of the technological environment regarding the problem and possible solution strategies (or the immediate ability to learn/construct the knowledge) necessary to produce a solution.

**Initial Process Development (IPD):** (1) Deeply investigating the problem posed and the proposed goal; (2) investigating necessary global mathematical ideas and more precise mathematical concerns; (3) determining which ideas are necessary and which are unnecessary to lead to a successful heuristic; (4) developing a process skeleton (cognitively constructed and/or diagrammatically represented); (5) envisioning the process from the initially posed problem to the solution.

**Technological Tool Investigation (TI):** (1) Investigating the availability and applicability of technology tools and techniques; (2) recognizing both the power and weakness of various technology tools, applications, and techniques; (3) investigating the precision involved with technology tools.

**Heuristic Development (HD):** (1) Synthesizing what is known mathematically and technologically into a detailed heuristic; (2) expanding the process skeleton to a complete heuristic.

**Heuristic Application (HA):** (1) Carrying out steps and procedures defined in the heuristic; (2) attending to details regarding mathematical and technological precision in the heuristic steps.

**Heuristic Revision (HR):** (1) Recognizing what mathematical ideas and technological tools and techniques do and do not work as previously desired; (2) revising the solution process based on better understanding of the problem and mathematical and technological information.

**Determining Equivalence (DE):** (1) Continually manipulating the problem result and ensuring that it produces correct results in all cases; (2) ensuring that the solution does not introduce additional dimensions to the problem that need to be addressed.

**Problem Completion (PC):** (1) Determining that the solution completely fulfills the problem; (2) Success is operationally defined.

With this framework, pairs of researchers, one from Team A and one from Team B, again analyzed the data using the codes PD*x*, PU*x*, SA*x*, IPD*x*, TI*x*, HD*x*, HA*x*, HR*x*, DE*x*, and PC*x*, representing the framework. This resulted in some of the following findings.

## Results

Initial results are provided via transcripts of student work and articulations. Within these transcripts, the research codes are inserted to provide the reader some insight as to how the codes were recognized, developed, refined, and associated to particular student behaviors. Additionally, some researcher notes are inserted into the transcripts in order to help the reader understand both the activity and the student work. These notes are interpretations of student behaviors, thinking, and heuristics. Researcher notes are denoted as *italicized* text within brackets. Additionally, since screen grabs of students, as in Figure 2, would be incomprehensible to the reader, simplified graphics depicting student work are provided.

Each transcript below covers approximately three hours of student work, after which students were asked to cease their activity even if incomplete. Neither student fully completed the research task in the time allotted.

**Student 1.**

> Student 1 reads the problem prompt, pauses, and then reads it again. [PD1] He looks into the air and draws some figures with his finger in the air. [PD2,3]. “This should be relatively easy. I think that all I really need to do is compare some lengths.” [SA1,2; PD2,3; PU1,2]. In GSP, he constructs a triangle and a circle as independent and distinct objects. He begins to drag the figures around the screen to observe the various cases of the triangle inside the circle, the circle inside the triangle, the figures overlapping on some

edges, and the figures being completely disjoint. [PD1,2,3; PU1]

He opens the file containing Boolean operations and comparisons and looks at some of his options. [TI1] [*Tools available in this file include (among others): a=b, a>b, a≤b, max(a,b) and min(a,b,c).*] "I think I know how to use all of these. I just need to use them in the right way." [SA1,2; TI1]

"I think that the primary issue is comparing the radius of the circle with measures from the center of the circle to parts of the triangle." [PU1,2; IPD1] He moves the triangle and circle on the screen. [PU1,2; IPD1,2,4] "Clearly, the triangle is inside the circle when the radius is greater than the distances from the center of the circle to the three vertices. [IPD1]. "But that would only take care of that case. [PU1,2; IPD1,2,4,5] I need to look at other cases." [PU1; IPD1]

He moves the circle to inside the triangle. "Checking to see if the circle is inside the triangle is more work. I think that I need to measure the distances from the center of the circle to the three sides and see if those are all three greater than the radius." [PU2; IPD1,3,4]

"I should get started with these. Then I'll look at the other cases." [IPD4,5] He manipulates the figure such that the triangle is completely inside the circle. [IPD2,3; HA1,2] He investigates his list of available Boolean operations and comparisons. [TI1] "I think I know how to use some of these." [TI3; HD1,2] He measures radius OD and distances OA, OB, and OC. (See Figure 3.) [HA1,2] "I think that I can compare the measurements using the $x<y$ tool." [SA2; TI3] He selects the ordered measurements OD and OA with the Boolean tool $x<y$. It produces a result of 0 (indicating a false condition). "Wait, that is wrong. It should be True. O, I have this backwards." [TI3] He pairs measurements to determine whether OA<OD, OB<OD, and OC<OD and then uses the AND Boolean operation to calculate the truth of

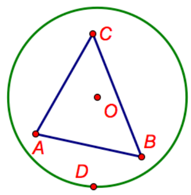

*Figure 3.*

$$(OA<OD)\wedge(OB<OD)\wedge(OC<OD).$$

It produces a value of 1 (indicating a True condition). [TI2; HA1,2] "That makes sense. Now we know when the triangle is inside the circle. I'll set that up with the words when I get all cases."

He moves the circle to inside the triangle. [HD2; HA1] "Now I need conditions for this. It is pretty clear that the distance from the center of the circle to the sides of the triangle must be greater than the radius of the circle. [PU2; IPD2,3,4,5; HD1,2] I think that I need to make perpendiculars from the center to each side of the triangle and measure those." [IPD1,2,3; HD1,2] He constructs and measures OE, OF, and OG. (See Figure 4.) [HA1,2] He pairs measurements to determine whether OD<OE, OD<OF, and OD<OG and then uses the AND Boolean operation to calculate the truth of

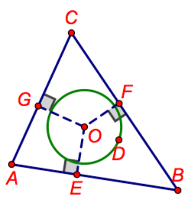

*Figure 4.*

$$(OD<OE)\wedge(OD<OF)\wedge(OD<OG).$$

It produces a value of 1. [TI2; HA1,2] "Now we have that case."

"I'll do the overlapping case. I should be able to use my measurements. I think that if OE, OF, and OG are less than OD, then the circle and triangle will be overlapping." He moves the triangle so that the circle overlaps the triangle on all three sides. (See Figure 5.) He pairs measurements to determine whether $OE<OD$, $OF<OD$, and $OG<OD$ and then uses the AND Boolean operation to calculate the truth of

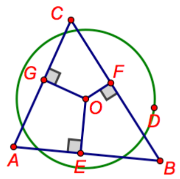


*Figure 5.*

$$(OD<OE)\wedge(OD<OF)\wedge(OD<OG).$$

It produces a value of 1. [TI2; HA1,2] "That was easy. Now to the last case."

He moves the triangle slightly to where the circle overlaps only one side of the triangle and notices that, while the shapes are still overlapping, the Boolean calculation produces a value of 0 (indicating a false condition). "Wait. That isn't supposed to happen. It should be True. [DE1,2] Ok, what's going on? [IPD1,2,3; TI2,3] Did I make a mistake?" [SA1,2] He looks at the Boolean calculation and wonders, "I think that I may need OR instead of AND. As long as one of these measures is less than the radius, then the shapes will be overlapping." [HR1,2; IPD3,4; HD1,2] He edits the Boolean calculation to

$$(OD<OE)\vee(OD<OF)\vee(OD<OG)$$

which produces a value of 1.

He moves the triangle and circle around the screen to test his cases. [DE1,2] In so doing, he notices that when the triangle is completely inside the circle, he gets an erroneous result of True for his last calculation. [DE1,2; TI3] "Now I'm getting 1 for both inside the circle and overlapping. That can't be. [DE1,2] So I need to adjust something." [HR1] He investigates his Boolean cases together. "I want it to be True when is it overlapping but False when the triangle is inside the circle. [PU2; IPD3,4,5; HD1,2] I already have both of those calculations. I wonder if I can combine them. [SA2; IPD1,2,4; TI1,2,3] I think that I can use an AND and a NOT. [HR1,2; TI1] But I need to be careful." [SA2; TI3] He sets up the calculation

$$((OD<OE)\vee(OD<OF)\vee(OD<OG))\wedge(\neg((OA<OD)\wedge(OB<OD)\wedge(OC<OD)))$$

which produces a value of 1. He moves the figures into and out of the conditions: triangle inside the circle; circle inside the triangle; and various forms of overlapping figures. [DE1,2] "Ok. Ok. That works. It looks great."

He finally, and possibly inadvertently, moves the shapes such that they are disjoint. (See Figure 6.) "Wait. [DE1,2] Something is wrong. My conditions are now saying that the triangle inside the circle is True. Did I do something wrong? [SA,1,2] Do I have to start all over? [HR1,2; PU1,2; IPD1,2,3,4,5]"

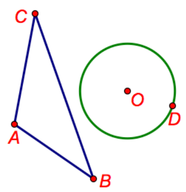


*Figure 6.*

After a protracted period investigating his previous cases [TI3], he decides to abandon his previous strategy and seek another heuristic for the disjoint case. [HR1; PU1,2; IPD1,2,3; TI1]. "I think that I can use ratios. [PU2; HR2] What if I extend the side of the triangle to a line? Then I can construct a segment from the furthest vertex to the circle center. If I find the intersections of this segment with the side line and the circle, I can set up a ratio that tells me if the circle edge is further away than the side of the triangle." [IPD2,4; TI1; HD1] He begins to develop his idea on one side of the triangle. He constructs intersection points H and I. [HA1] (See Figure 7.) "I need to be careful. [SA2] I think that I want the ratio to be OI:OH. Or is that OH:OI? [HA2; TI3] He sets up the ratio OI:OH and, by moving elements on the screen, the ratio is either 0<OI:OH<1 or 1<OI:OH. [HA1,2] "That's not what I want. I want positive and negative. [DE1] I think that HI:HO will work." [TI3, HD1,2] This produces HI:HO<0 or 0<HI:HO. [*He does this planning that 0<HI:HO indicates that the circle is beyond the side of the triangle and HI:HO<0 indicates that the circle crosses the side of the triangle.*]

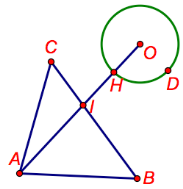

*Figure 7.*

[*The allotted time for the research task ends. However, unrecorded, Student 1 takes the task on as a challenge and continues to work, hoping to complete it. This work is not transcribed.*]

**Student 2.**

Student 2 reads the problem prompt [PD1] and immediately begins to investigate the list of Boolean tools at her disposal in GSP. [TI1] She constructs an independent circle and triangle and alternates her gaze from the Boolean tools to the figure. [PD2,3; TI1] She moves the two figures about the screen to observe/investigate various possible cases. [PD1,2,3; PU1,2] [*It does not appear that any specific heuristic is coming to her mind.*] "I'm not yet sure what to do." [SA1,2]

She clicks on GRAPH|SHOW GRID. [*Researcher asks why she did that.*] "I don't have an idea how to do this. [SA1,2] I was hoping to get an idea using the grid." [TI1] [*Unless she develops a truly unique heuristic unknown to the researcher, using a grid has no value.*]

She begins taking measurements between various elements: OA, OB, OC, and OD. (See Figure 3.) She makes a long list of Boolean comparisons: OA<OD, OA>OD, OB<OD, OB>OD, OC<OD, and OC>OD. She then moves the figures about the screen, transitioning through various cases, and inspecting changes in the Boolean conditions. [*Notably, she does not seem to have a plan as to what she will do with these measurements and conditions. She is taking these actions not as steps in a heuristic, but to better dissect and understand the problem.*] [TI1,3; PD1,2; PU1]

She measures the area of the circle and triangle and uses GSP to determine the coordinates of points A, B, C, O, and D. [TI1, PU1] [*The researcher, again, cannot imagine an effective heuristic based on the areas. She, again, seems to be taking these actions not as steps in a heuristic, but to better dissect and understand the problem.*] She moves the figures about the screen and observes the changing of all the measures. [PU1,2; TI1,2] [*Notably, all measurements and Boolean conditions thus far are left unused. Their primary role seems to be that of a running list of observations in search of applications.*]

"I think that I'm going to try to use the technique [HR1; HD1] we learned about a point being on one side of a line or the other." [*This technique employed angle measures to provide a Boolean value of 0 or 1, denoting False or True, respectively regarding, for instance, point B being on the A-*

*side of a line or not being on the non-A-side of the same line. This method, albeit cumbersome and persnickety in precision, has the possibility of success.*] "But how did that go?" [SA2]

"I need a point that stays on one side of the line and does not move from that side. The centroid of the triangle would always stay inside the triangle." [PU2; IPD2,3,4,5; HD1] Constructing medians of the triangle, she constructs the centroid of the triangle and labels it point E. [HA1,2] She constructs segment AO, marks angle BAO, reads the measure of angle BAO, and sets the preference to the angle mark as counter clockwise. She sets parameters of P1=0°, P2=180°, and P3=360°. She sets up the Boolean conditions 0°<m*BAO*<180° and 180°<m*BAO*<360°. [HA1,2] [*The heuristic she is attempting to use is: If* 0°<m*BAO*<180°, *O is not on the E-side of line AB and if* 180°<m*BAO*<360°, *O is on the E-side of line AB. To see the correct application of this completed heuristic, use this URL* *http://appstate.edu/~bossemj/DMEProcesses/ESide/* *to a dynamic applet.*] She moves the circle around the screen and the Boolean conditions report when point O is on the E-side of segment AB or not. [DE1; TI1]

She replicates this process for side BC of the triangle and sets up the Boolean conditions as 0°<m*CBO*<180° and 180°<m*CBO*<360°. She again replicates the process for side AC and sets up the Boolean conditions as 0°<m*CAO*<180° and 180°<m*CAO*<360°. [HA1,2] [*Notably, this last setoff Boolean conditions is backwards. The calculations should be* 0°<m*ACO*<180° *and* 180°<m*ACO*<360°.] She moves the circle around the screen to test her work. [DE1,2] The Boolean conditions rightly inform her when point O is on the E-side of segment AB and the E-side of segment BC. However, it incorrectly reports regarding side AC. I don't know what is wrong. Is it working wrong, or did I mess it up?" [SA1,2] She continues to move the circle around the screen. [DE1; TI1,2,3; HR1] "Wait. I have it." She corrects her last set of Boolean conditions [HR2, HA1,2] and, when she moves the circle around the screen the conditions correctly report what she wants. [DE1,2] She sets up the Boolean condition [HA1,2]

$$(0°<mBAO<180°)\wedge(0°<mCBO<180°)\wedge(0°<mACO<180°).$$

This always produces a value of 0 indicating that the conditions is false. "That isn't right. It should be true at times. [TI2,3; IPD2,3,4] I'll try…" [TI2,3; HD1,2; HA1,2]

$$(0°<mBAO<180°)\vee(0°<mCBO<180°)\vee(0°<mACO<180°).$$

She again moves the circle about the screen and notices that, although it now reports the correct value of 1=True when O is outside of the triangle, it only considers the center of the circle and not the circle itself. Overlapping conditions are not considered. She moves the figures about the screen and considers different cases. [PU1,2; IPD1,2,3; TI2,3]

I think that I need to consider the closest edge of the circle to the triangle in each case. [HR1; PU1,2; IPD1,2,3,4; HD1] But how do I find the closest point of the circle to a side? [SA2] If I can do that, then I can check if the center and the closest point are both on one side or another for each side. But, that closest point would always be different according to what side I am looking at. [IPD3,4,5; HD1] This will be tough."

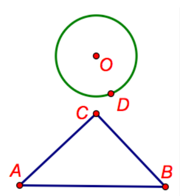

*Figure 8.*

[SA2] [*While this can lead to a workable heuristic, without further modification it will often fail when the circle is in the position shown in Figure 8.*]

[*The allotted time for the research task ends. Student 2 is glad the session is over and quickly departs, having no interest in going any further.*]

## Discussion

Figures 9 and 10 provide a graphical representation of the occurrences of each cognitive process coded in this study for Student 1 and Student 2, respectively. Importantly, the horizontal axis of the graph depicts the entire duration of the problem solving activity. For each of these two students, this was approximately three hours in duration. The separation of the horizontal axis is meant to demonstrate subsequent groupings of observed cognitive processes and not sets of cognitive processes equally spaced along the timeline. Table 1 represents numerical counts and percentages associated to different cognitive processes and groups of cognitive processes in respect to the work of each student. From this data alone, it can be readily seen that these two students differed in the frequency of processes they employed while problem. These figures and the table are analyzed in subsequent paragraphs.

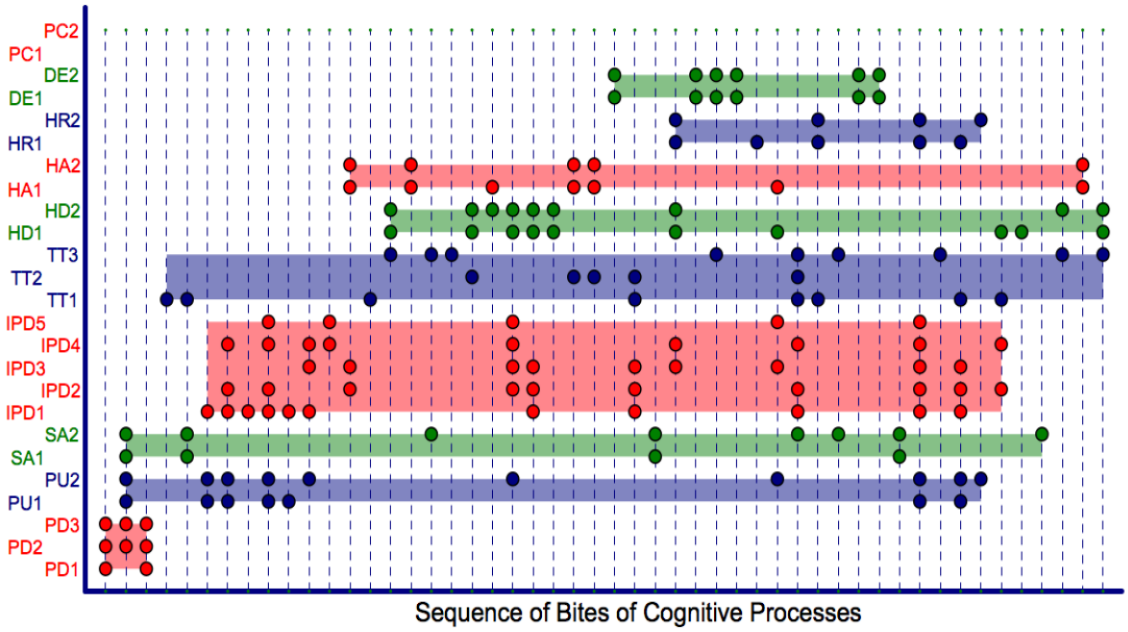


*Figure 9.* Occurrences of cognitive processes in the work of Student 1.

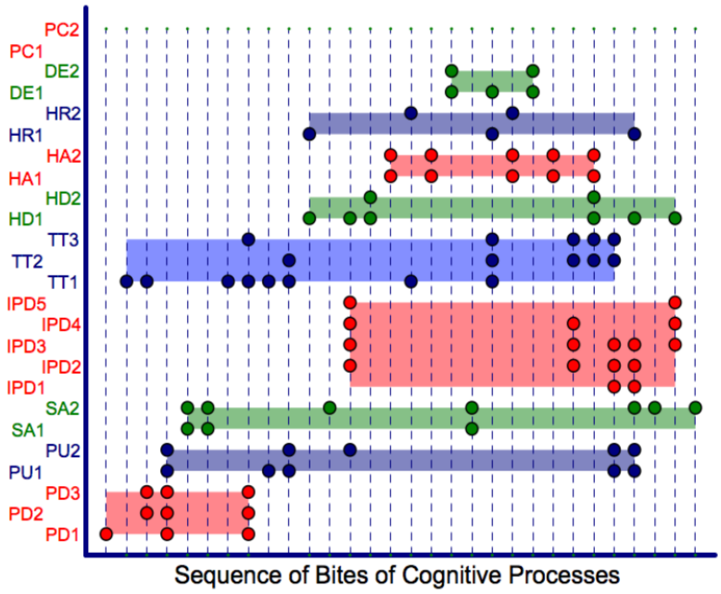


*Figure 10.* Occurrences of cognitive processes in the work of Student 2.

The discussions herein consider a number of aspects from the data. The first are born from considerations of the data. The following are more global in context.

## *Issues Regarding Counts in the Data*

Figures 9 and 10 are represented in Tables 1, 2, and 3. In each table the data is considered in different ways. These are addressed below.

*Counts of Cognitive Processes.* As shown in Table 1, in this dimension, all counts of each cognitive process is provided. This reveals that over the course of approximately three hours, Student 1 demonstrates 151 cognitive processes and Student 2 demonstrates 86 cognitive processes. This equates to Student 1 demonstrating approximately 76% more cognitive processes in the same duration of time. As quite apparent through Figures 9 and 10, Student 1 has a higher density of cognitive processes during the time span than did Student 2.

Therefore, it might be assumed that Student 1 may show 76% higher counts in all of the cognitive processes across the board. This does not occur. Quite a few cognitive processes

Table 1.
*Counts of Cognitive Processes*

| | PD | | | PU | | SA | | IPD | | | | | TI | | | HD | | HA | | HR | | DE | | PC | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 1 | 2 | 1 | 2 | 1 | 2 | 3 | 4 | 5 | 1 | 2 | 3 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 |
| Student 1 | 2 | 3 | 3 | 8 | 10 | 4 | 8 | 11 | 10 | 9 | 10 | 5 | 7 | 5 | 10 | 6 | 8 | 8 | 7 | 3 | 3 | 6 | 5 | 0 | 0 |
| Student 2 | 3 | 3 | 2 | 6 | 5 | 3 | 5 | 2 | 4 | 4 | 3 | 1 | 8 | 5 | 5 | 4 | 1 | 6 | 6 | 3 | 1 | 4 | 2 | 0 | 0 |
| | | | | | | | | | | | | | | | | | | | | | | | | | |
| | **Percent of Count of Cognitive Processes Group of All Processes** | | | | | | | | | | | | | | | | | | | | | | | | |
| Student 1 | ~5% | | | ~12% | | ~9% | | ~30% | | | | | ~15% | | | ~9% | | ~10% | | ~4% | | ~7% | | 0% | |
| Student 2 | ~9% | | | ~13% | | ~9% | | ~16% | | | | | ~21% | | | ~6% | | ~14% | | ~5% | | ~7% | | 0% | |

demonstrate cases where the percentage increase of Student 1 over Student 2 is less than 76% (e.g., PD1,2,3; PU1; SA1; TI1,2; HD1; HA1,2; and HR1) and others with percentages over 76% (e.g., PU2; SA2; IPD1,2,3,4,5; TI3; HD2; HR2; and DE1,2). These calculations alone demonstrate that the cognitive processes between Students 1 and 2 differ both qualitatively and quantitatively. We address these more in additional ways.

*Percent of Count of Cognitive Processes Group of All Processes.* Grouping each individual cognitive process according to its type (e.g., Student 1 P D= $(2+3+3)/151$ » 5% and Student 2 IPD= $(2+4+4+3+1)/86$ » 16%) comparing those groups to the entire number of recognized cognitive processes provides further insight into student differences (see Table 1). For instance, the results between Student 1 and Student 2 in respect to PU, SA, HD, HR, DE, and PC are nearly identical between Student 1 and Student 2. However, other factors, differ quite notably. For instance:

- The cognitive processes under PD is 80% higher for Student 2 than for Student 1.
- The cognitive processes under IPD is 87.5% higher for Student 1 than for Student 2.
- The cognitive processes under TI is 40% higher for Student 2 than for Student 1.
- The cognitive processes under HA is 40% higher for Student 2 than for Student 1.

While these may seem like random statistics, more careful analysis is revelatory. It is first important to note that Student 1 was more successful in completing the task than Student 2. This may be associated to Student 1 having a higher percentage of cognitive processes in the domain of IPD, where the student: deeply investigates the problem posed and the proposed goal, investigates necessary global mathematical ideas and more precise mathematical concerns, determines which ideas are necessary and which are unnecessary to lead to a successful heuristic, develops a process skeleton, and envisions the process from the initially posed problem to the solution. IPD is an action filled component of the problem solving process. It is plausible that a student demonstrating a higher percentage of IPD may be better at problem solving than a student demonstrating a lesser percentage of IPD.

The fact that Student 2 had higher percentages of PD, TI, and HA may again have explanatory power. PD includes: interpreting the problem statement, gaining informal understanding of the problem, and globally, but imprecisely, understanding the nature of the problem solution. TI includes: investigating the available technology tools and techniques; recognizing the power of various technology tools, applications, and techniques; and investigating technology tool precision. HA includes: carrying out steps and procedures defined in the heuristic; (2) attending to details regarding mathematical and technological precision in the heuristic steps. The higher percentage of PD indicates that the student who struggled more in the problem solving process (Student 2) more often needed to revert to dissecting the meaning of the problem. This makes sense, as does the fact that Student 2 had higher percentages of TI and HA processes. The greater percentages in TI and HA demonstrate that Student 2's struggles with technology led to spending more time

Table 2.
*Cognitive Processes Rectangle Width*

| | PD | | | PU | | SA | | IPD | | | | | TI | | | HD | | HA | | HR | | DE | | PC | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 1 | 2 | 1 | 2 | 1 | 2 | 3 | 4 | 5 | 1 | 2 | 3 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 |
| | **Percent of Width of Cognitive Processes Group Rectangle Width** | | | | | | | | | | | | | | | | | | | | | | | | |
| Student 1 | 6% | | | 86% | | 92% | | 80% | | | | | 94% | | | 72% | | 74% | | 32% | | 28% | | 0% | |
| Student 2 | ~27% | | | 80% | | ~87% | | ~57% | | | | | ~83% | | | ~63% | | ~37% | | ~57% | | ~17% | | 0% | |

investigating technology tools and attempting to apply developed heuristics – with varying levels of success – to the problem.

*Percent of Width of Cognitive Processes Group Rectangle Width.* As seen in Figures 9 and 10, the groupings of cognitive processes by domains produced rectangles. The widths of these rectangles were compared to the width of the entire problem solving experience (see Table 2). For instance, Student 1 HR width% =

$$\left(\text{HR rectangle width} = 16\right)\big/(\text{width of problem solving experience} = 50) = \tfrac{16}{50} = 32\%$$

and Student 2 PD width% =

$$\left(\text{PD rectangle width} = 8\right)\big/(\text{width of problem solving experience} = 30) = \tfrac{8}{30} \text{ » } 27\%.$$

When considering these percentages of rectangle widths, more insight is afforded to student understanding. The differences in these rectangles are most obvious in the domains of PD, IPD, HA, and HR. These differences again make sense. Student 1’s narrow PD rectangle demonstrates that this student returned to dissecting the stated problem only in the earliest stages of solving the problem. Thus, the problem was better understood by Student 1 early in the problem solving process – leading to more efficient problem solving. Student 1’s wider IPD rectangle demonstrates that he spent more time investigating the problem and planning its solution; the heuristic is well planned and not haphazard. Student 1’s wider HA rectangle demonstrates his more continual focus on applying the developed heuristic – a natural outcome of a better formed heuristic. Additionally, Student 2’s wider HR rectangle demonstrates that she needed to more often revise her heuristic. This, again, may be the result of both lesser understanding of the original problem and less than adequate heuristic planning.

Table 3.
*Percent of Cognitive Processes in a Domain Among All Possible in that Domain*

| | PD | | | PU | | SA | | IPD | | | | | TI | | | HD | | HA | | HR | | DE | | PC | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 1 | 2 | 1 | 2 | 1 | 2 | 3 | 4 | 5 | 1 | 2 | 3 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 |
| Student 1 | ~5% | | | 17% | | 12% | | ~18% | | | | | ~15% | | | 19% | | 12% | | 9% | | 12% | | 0% | |
| Student 2 | 5% | | | 17% | | ~17% | | 10% | | | | | 20% | | | ~13% | | ~17% | | ~8% | | ~8% | | 0% | |

*Percent of Number of Cognitive Processes in a Domain Among All Possible in that Domain.* We present one example to better represent this statistic. Since, for instance, there are three constructs in PD and Student 1 has 50 recognized bites of cognitive processes along the path of problem solving, there is a total of 150 possible occurrences of the PD constructs. Thus, Student 1 demonstrates eight PD constructs out of a possibility of 150 occurrences, or approximately 5% of the possible PD constructs.

Notably, Student 1’s lesser use of SA indicates greater confidence in being able to solve the problem and his greater IPD demonstrates greater focus on developing an informal heuristic, which becomes refined more efficiently. Student 1’s lesser focus on TI possibly means that he has a greater mastery of the technological tools available. Student 1’s greater focus on HD and lesser focus on HA, as compared to Student 2, indicates that the additional time he spent developing heuristics led to less time needed to apply the heuristics.

### *Additional Observations and Reconnecting to the Literature*

The data demonstrates some additional observations. First, the cognitive processes associated with problem solving are nonlinear. Problems are investigated, heuristics are developed, applied, and tested, difficulties are encountered, the problem is reinterpreted and heuristic revisions are made, and the process circles around on itself. Thus, in addition to being nonlinear, the process is iterative throughout all or parts of the process.

While the nonlinear and iterative nature of problem solving may complicate the assessment and observation of cognitive processes associated with student problem solving, the two student cases analyzed above are probably sufficient to demonstrate that the problem solving process is idiosyncratic in nature and may vary by individual. It must be wondered if any two students have identical paths of cognitive processes associated with problem solving.

Since problem solving is performed by individuals and problem solving is a sense-making tool (Schoenfeld, 1994), it is reasonable to believe that problem-solving processes would be different in a number of dimensions between different individuals. As particularly seen through the work of Student 1, the problem-solving activity and processes can lead the student to better understand and apply problem-solving techniques (Carpenter, 1989; Stacey & Groves, 1985; Thompson, 1985) and to greater conceptual understanding of the problem (Lester, et al., 1994). Although Student 2 clearly had more difficulty with the task and experienced more frustration, both students worked for three hours on the task. This demonstrates that, in a sufficiently stimulating problem-solving task, students can be motivated to learn (Stanic & Kilpatrick, 1989) can experience greater task engagement, time on task, and perseverance (Olkin & Schoenfeld, 1994). Although both participants in this study experienced the same course in Geometry and background with GSP and Boolean applications in GSP, Student 1 demonstrated superior understanding of mathematics, GSP operations, and Boolean structures; this led to stronger problem solving in the task. This parallels the argument by Pajares and Kranzler (1995) that greater mathematical knowledge leads to better mathematical problem solvers and that stronger problem- solving skills lead to more efficient mathematics learning.

As seen through the participants' work, the dynamic nature of the DME – the allowance of dragging elements about the screen and instantaneously and continuously observing relationships – was instrumental to both students (Arzarello, Micheletti, Olivero, Robutti, Paola, & Gallino, 1998; Goldenberg & Cuoco, 1998; Pea, 1985; Ruthven, Hennessy, & Deaney, 2008). Particularly for Student 1, this allowed him to concretize his ideas.

This study may imply that the use of technology in problem solving is far from a panacea. Jobrack, Bossé, Chandler, and Adu-Gyamfi (2018) found that students' use of technology may exacerbate problem solving difficulties by introducing additional factors into the problem-solving situation. This study may demonstrate that cognitive processes associated with problem solving in a DME – particularly in respect to the individualistic nature of these cognitive processes – may be too complex to make definitive statements regarding positive effects of DME use in problem solving. Too much remains unknown in this realm. Far more research is needed in respect to the effect of technology in problem solving in general and, more specifically, cognitive processes involved in problem solving in DMEs.

## Implications

The findings of this study have numerous implications, not all of which can be considered in detail here. Therefore, only a brief list will be considered. One fundamental and salient implication from this simple investigation is that research into the cognitive processes underlying student problem solving behavior is still a nascent area of research, much remains to be learned regarding cognitive processes involved in problem solving.

This investigation considered student problem solving in the context of a technologically rich dynamic math environment. While the DME introduced certain unique aspects to the problem solving experience, it may have also acted as an amplifier to make cognitive processes more readily observable. It is likely that the cognitive processes described and analyzed in this study, with the possible exception of Technological Tool Investigation (TI), are all applicable to other problem solving scenarios. This is an area in which further research is warranted. Likewise, the definition of these cognitive processes and their application in analyzing students' problem solving is here applied to mathematical problem solving. However, it is worth considering whether these same cognitive processes, perhaps with some modifications, could be applied to problem solving in other disciplines or learning contexts.

Over the decades, increasing curricular emphasis has been placed on student problem solving. While this study does not contradict the idea that this is a valuable focus, it does reveal that the process is far more complex than simply providing students with problem solving experiences. It is important to recognize that problem solving is nonlinear, iterative, and idiosyncratic, though this may make the posing of problems and the assessment of student problem solving skills more difficult and less straightforward. The idiosyncratic and individualistic nature of problem solving found in this study, particularly in respect to the cognitive processes involved, also increases the complexity of assessment of student problem solving. This study implies that assessing student problem solving skills by whether or not a student can solve a problem is woefully inadequate. Deeper assessment would need to examine which cognitive process the student is actually struggling with.

The preceding two points lead to the notion that resolving student difficulty regarding problem solving may require wildly new strategies. Student difficulties with particular cognitive processes may only be ameliorated through very precise and targeted activities. Notably, if the preliminary findings of this study are borne out, such activities would be highly beneficial to the learning processes but have yet to be thought out and invented.

Results in this study may share similar implications with Jobrack, Bossé, Chandler, and Adu-Gyamfi (2018). Both studies may imply that the use of technology in problem solving is far from a panacea. Indeed, technology use may exacerbate problem solving difficulties by introducing dimensions which interact with other content domains associated with the problem. Therefore, altogether, the use of technology may either help or hinder problem solving. Nevertheless, far more research is needed in respect to the effect of technology in problem solving in general and problem solving in DMEs.

Questions naturally arise regarding the validity of research that involves a limited number of subjects. However, this research is not intended to provide sweeping generalities. It was merely intended to observe and analyze the problem solving behaviors of two students. Nevertheless, with the limitation of considering only two students, significant findings were made. It is hoped that future research on larger groups both confirms these findings and extends upon them.

## Conclusions

Cognitive processes associated with student problem solving are complex but can be observed, described, and categorized. In this study, twenty-five distinct cognitive processes were discovered and grouped under ten categories. The process employed through a problem-solving task was found as nonlinear, iterative, and idiosyncratic. This study demonstrated that students vary greatly regarding how they solve problems in the context of a dynamic math environment.

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