

Developing Knowledge of Student Thinking: Understanding Big Ideas behind Students' Difficulties with Connecting Representations in Algebra

Kyunghee Moon
University of West Georgia
kmoon@westga.edu

This study concerns preservice teachers' knowledge of student thinking at the secondary level. In particular, it examines in what ways a three-week unit integrating content and student thinking was beneficial in developing preservice teachers' knowledge of student thinking associated with big ideas in algebra—such as variation, Cartesian Connection, and the graph as a locus of points. In the early stage of discussion of student work, preservice teachers tended to focus mostly on students' errors/mistakes, correct ways the problems should have been solved, or the types of problems that students had. However, with the progress made in discussion, they were able to attribute students' difficulties to a lack of understanding of qualities such as definitions, properties, and concepts, and further to a lack of the understanding of big ideas underneath the qualities, with the help from instructors. The majority were also able to design contextualized tasks that could stimulate students' interests; yet they struggled to explain how they would use the tasks to promote students' development of big ideas by connecting multiple representations.

Introduction

Knowledge of student mathematical thinking is a kind of knowledge that teachers need for effective teaching. Accordingly, many teacher preparation programs design their curriculum or units to incorporate such knowledge in their content or methods courses (Sowder, 2007). Using multiple sources of student work—such as video cases, written case studies, and students' written work—these interventions provide preservice teachers with opportunities to examine students' conceptions, misconceptions, and/or strategies. Studies report positive effects of such interventions on preservice teachers' content knowledge, knowledge of student thinking, and/or beliefs (Lampert & Ball, 1998; Phillip et al., 2007; Tirosh, 2000). However, most of such efforts have been made at the elementary level, and not much is known about the types of interventions attempted or how and in what way they are beneficial at the secondary level (Doerr, 2006).

This study addresses the under-researched area of knowledge of student thinking at the secondary level with the research question: In what ways preservice teachers developed knowledge of student thinking on big ideas associated with connecting representations in algebra? Connecting representations is extremely important in mathematical teaching and learning, as identified in prior research (Brenner et al., 1997; Fonger, 2019; Hiebert & Carpenter, 1992; Goldin, 2020; Moschkovich et al., 1993) and as described in the Principles and Standards for School Mathematics (NCTM, 2000) and the Common Core State Standards. However, many individuals, including teachers, have a compartmentalized understanding of mathematical representations (Moon et al., 2013; Moon, 2020; Presmeg & Nenduradu, 2005), and furthermore, many teachers do not understand students' difficulties with connecting representations (Postelnicu (2011; You, 2006).

Theoretical Underpinnings

Knowledge of Student Thinking

Teachers need to be equipped with content knowledge: knowing something is so and why it is so on what warrants and under what conditions (subject matter knowledge);

knowing how to represent and formulate the subject to make it comprehensible to students (pedagogical content knowledge); and knowing the full range of curricula that they teach as well as other alternatives available for instruction (curricular knowledge) (Ball & Bass, 2000; Hill et al., 2007; Shulman, 1986). In addition, they need knowledge of student thinking—understanding students’ strategies, ideas, and errors—and the ability to plan instruction based on student thinking (Carpenter et al., 1988; Fennema et al., 1996; Fosnot & Dolk, 2001; Jacobs et al., 2007). Studies have shown that Professional Developments (PDs) that integrate content and student thinking are effective (Bell, et al., 2010; Carpenter et al., 1988; Cohen, 2004; Fennema et al., 1996; Jacobs et al., 2007). These interventions improve teachers’ knowledge of content (Bell et al., 2010; Jacobs et al., 2007) and student thinking (Bell et al., 2010; Jacobs et al., 2007). Additionally, they enhance students’ performance (Carpenter et al., 1988; Jacobs et al., 2007). According to Carpenter et al. (1988), however, teachers understand and predict students’ strategies and performance better when dealing with their students’ work, as opposed to dealing with the work of general students in video cases.

Meanwhile, studies have shown that preservice teachers tend to evaluate students’ mathematical work based on their own mathematical knowledge (Crespo, 2000; Tirosh, 2000)—knowledge that is rule-based, and for which they do not understand the underlying reasons those rules work (Ball, 1990; Even & Tirosh, 1995; Kinach, 2002; Nathan & Petrosino, 2003; Tirosh, 2000; Van Dorreen et al., 2002). To intervene in the issues, researchers have incorporated knowledge of student thinking into their methods/content courses using various methods—using students’ written work/ video cases (Crespo, 2000; Wilson et al., 2011), common errors from research (Tirosh, 2000), or lesson studies (Jansen & Spitzer, 2009)—and documented the effects of the interventions. For example, Crespo (2000) and Tirosh (2000) demonstrated that communication with students through letters or a course focusing on content and student thinking helped preservice elementary teachers focus on the meanings of students’ work and speculate about students’ reasoning behind their errors. Additionally, Jansen and Spitzer (2009) showed that lesson designing and implementing helped preservice teachers understand students’ thinking with the specificity on the topic of fractions.

Big Ideas in Mathematics

The notion of big ideas varies depending on disciplines or contexts; however, their importance is emphasized by many researchers and national standards such as NCTM (2000). In the discipline of mathematics, big ideas are generally understood as mathematical ideas overarching numerous mathematical understandings into coherent wholes (Baroody, 2007; Charles, 2005; NCTM, 2000; Shifter & Fosnot, 1993). They are a critical aspect of understanding the nature, structure, and connections within and among domains (Charles, 2005), and they provide a basis for structuring and restructuring knowledge (Baroody, 2007). For example, equivalence—a mathematical object can be represented in different ways without changing the value or solution—is a big idea in mathematics. It connects various mathematical topics—such as numbers, expressions, and equations—and processes—such as applying properties of equality and solving equations—into a coherent whole, and it is integral to achieving a deep understanding of both concepts and procedures (Charles, 2005; Fosnot & Jacob, 2010).

Researchers and practitioners have claimed that big ideas can serve as the framework for mathematical teaching/learning, research, and curriculum design (Charles, 2005; Baroody, 2007; Fosnot & Dolk, 2003; Fosnot & Jacob, 2010; Shifter & Fosnot, 1993). For example, Charles (2005) presented a set of 21 big ideas at the K-8 level, derived from the content analysis of a curriculum, and claimed that big ideas should serve as the anchors for

everything. Additionally, Fosnot and colleagues developed a K-8 curriculum, Math in the City (Fosnot & Dolk, 2003; Fosnot & Jacob, 2010), and a PD to help teachers implement the curriculum. Influenced by the social constructivism (Cobb & Yackel, 1996) and realistic mathematics education (Freudenthal, 1991), Fosnot and colleagues believed that mathematics is a human activity of mathematizing—modeling and structuring the world mathematically. As such, learners construct big ideas by progressively developing strategies and representations while working on tasks that serve as the starting point for mathematical investigation.

Big Ideas in Algebra

The big ideas this study particularly focuses on are variation, the Cartesian Connection, and the graph as a locus of points. Variation is the idea that variables represent varying quantities, not just unknown quantities—a critical idea for transitioning from arithmetic to algebra (Herscovics & Linchevski, 1994) and for understanding the function concept (Carlson et al., 2002; Sfard, 1992). The Cartesian Connection is the idea that a point is on the graph of a mathematical relation if and only if its coordinates satisfy the relation. The notion was originally coined by Moschkovich et al. (1993) for the linear function and extended later by Moon and colleagues for other mathematical relations, including inequalities (Moon et al., 2013; Moon, 2020). The graph as a locus of points is the idea that a graph of a mathematical relation is a locus of points whose coordinates satisfy the relation. Originating from the definition of graph (Borowski & Borwein, 2002), it is shown to be a big idea that integrates variation and Cartesian Connection (Moon et al., 2013; Moon, 2020).

Studies have shown that students and teachers have difficulty applying big ideas in problem solving. For example, 90% of students used an inefficient, algebraic approach rather than an efficient geometric approach using the Cartesian Connection—simply reading the x and y coordinates off from the given graph—when asked to find a solution of a linear equation (Knuth, 2000). Additionally, only 27 of 104 elementary and middle school teachers provided a correct answer for the question in Knuth’s study (2000), with 7 teachers offering a solution by applying the Cartesian Connection (You, 2006).

Students also struggle with the big ideas of variation and the graph as a locus of points. They may plot and trace some points to visualize functions but do not understand the continuity of the graphs (Kerslake, 1981; Leinhardt, Zaslavsky, & Stein, 1990). They explain graphs at certain points but not over intervals (Carlson, 1998). They incorrectly represent the table of a non-linear function as a line graph, even when the points show a non-linear pattern (Presmeg & Nenduradu, 2005). Moreover, they have difficulty explaining the $y = x + z + 3$ graph as a set of points satisfying the equation (Trigueros & Martinez-Planell, 2010).

Many teachers also do not understand students’ struggles with the big ideas. In a study by Asquith et al. (2007), only 3 of 20 middle school teachers understood that students’ difficulty with the question—what is larger between $3n$ or $n+6$?—was related to the big idea of variation. Most teachers attributed students’ difficulties to other issues, such as not testing more than one value for n , being less skilled in substitution, and being unaware that n stood for numerical values. Postelnicu (2011) also found that teachers could not associate students’ difficulties with a lack of understanding of big ideas. For the problem asking if the point (2, -8) is on the graph of the line, $y = 3x - 4$, only one of twenty teachers attributed students’ difficulty to a lack of understanding of the Cartesian Connection. For another question asking to create a table of values for the equation, $y = -2x + 3$, with no x or y value provided, only two teachers attributed students’ struggles to a lack of understanding of the big idea of variation. Many teachers instead attributed students’ difficulties to other factors, such as the misuse of formulas, the absence of x -values, or the novelty of the problem format.

To address aforementioned struggles of students and teachers, a team of three

mathematicians and teacher educators designed and implemented the Algebra unit for preservice secondary mathematics teachers (PSMTs). The team has already published their findings on PSMTs' content knowledge on big ideas (Moon et al., 2013). In this paper, the author, who is one of the team members, presents the findings on PSMTs' knowledge of student thinking.

Methodology

Setting

The research was conducted at a large public university in the western U.S. A sequence of two mathematics content courses, Math A and B, were designed for PSMTs incorporating the views of *realistic mathematics education* (RME, Gravemeijer, 2008) and the *social constructivist theory* (Cobb & Yackel, 1996). Each course consisted of three units, with each unit focusing on various topics. Each unit was composed of solving collegiate level problems and investigating student work at the grades 5-12 level, so that PSMTs would connect big ideas at different levels. The courses were taught by three instructors (Dr. B, Dr. C, and Dr. K), with Dr. B as the main instructor. The role of the instructors was more of a facilitator of the designed activities than a transmitter of knowledge. The instructors visited tables to listen to PSMTs' ideas, posed questions that could redirect PSMTs' mathematical thinking, and offered suggestions and shared information when it was essential to move PSMTs forward (Hibert et al., 1997).

Participants

The participants of this study were twenty PSMTs enrolled in Math B. All participants were in their junior or senior year: Seventeen of them were mathematics majors with emphasis in secondary education and three science or engineering majors with secondary teaching as a minor. The twenty PSMTs were assigned to sit in groups of four for collaborative learning. Factors, such as PSMTs' mathematical abilities and their tendencies to work with other PSMTs, were considered for the seating arrangement.

The Algebra Unit

The Algebra unit was the second unit of Math B and comprised two in-class tasks. Task 1 involved solving an advanced level problem that would bring up big ideas in algebra, while Task 2 involved the analysis of student work at the secondary level. A writing assignment followed the in-class tasks. For this study, the data from Task 2 and the writing assignment were utilized, as they were relevant to knowledge of student thinking. The description of Task 1, Task 2, and the writing assignment is provided as follows.

The CONTEXT SCALE

Description: The **Use of Context Scale** depicts teacher's development from a mechanical use of context merely as a locus for applying taught procedures, towards the use of (realistic: i.e. realizable, imaginable) contexts and 'truly problematic' situations both as starting points for mathematical constructions and as a didactic to facilitate mathematical development.

1. Lack of context, or mechanical use of context: either the mathematical work is done entirely within the domain of bare numbers (no context at all), or contexts—mostly limited to stereotypical word problems—are used for the application of previously learned concepts and procedures.

Operational behaviors to look for when coding: Teacher explains to children that they have been working on a topic, i.e. addition, and now they will do some problems where it is used. No context is used at all, or when used, problems are trivialized "school type" word problems to see if children can apply the operations and procedures they have already been taught.

2. Word problem types of contexts are used as a *starting point* for construction, in contrast to application of previously learned knowledge as depicted in level one. But this serves merely the purpose of motivation or to elicit children's thinking; no attention is paid to the process whereby mathematical ideas and/ or strategies may emerge or originate from suggestions or constraints in rich contexts.

Operational behaviors to look for when coding: Teacher embeds "school math" into a word problem, e.g., how many bags can I make of yams, ten to a bag, if I have 32 yams?

Teacher expects the answer of 3, remainder 2. Children's names may be used in problems to motivate, spark interest, but context is trivial and not likely to generate new strategies or bring big ideas up for discussion or exploration.

3. Use of realistic contexts and truly problematic situations as a didactic.

Operational behaviors to look for when coding: Contexts are purposely designed so as to bring to develop mathematical big ideas, models, and strategies. Contexts have implicit potentially realized suggestions of ideas or strategies built-in, or context has potential constraints to learners' strategies built-in, e.g. beach umbrella obscures some tiles on a patio and children who are using counting strategies are asked to figure out how many tiles. Teacher adapts and modifies context as she works with different children, in relation to learner's reasoning. For example, teacher may decide to not allow cubes, or may place numerals, rather than dots, on a die, to encourage children to "count on." Teacher uses models like the open number line, or the open array, as a didactic to bridge from the informal to the formal.

Figure 1. The context scale of AFM

Task 1: The Omar Khayyam task. Omar Khayyam was the Persian poet and mathematician who found the solution of the equation, $x^3 + p^2x = p^2q$, with p and q positive integers, using a geometric approach. Khayyam reduced the cubic equation to equations of parabola and circle, $py = x^2$ and $x^2 + y^2 = qx$, and found the real solution of the cubic by intersecting the parabola and circle graphs. During the Khayyam task, PSMTs were exposed to the big ideas of variation, the Cartesian Connection, and the graph as a locus of points. For instance, they had opportunities to discuss the Cartesian Connection: If $x = y = 0$ was a solution of the equation, $x^2 + y^2 = qx$, $(0, 0)$ had to be on the graph of the equation. They also discussed variation in geometric representation—points (x, y) vary in graph as x and y vary in the equation $x^2 + y^2 = qx$ —and the graph of a parabola as the locus of points equidistant from a point (focus) and a line (directrix) (see Moon et al., 2013 for further details).

Task 2: Analysis of algebra students' work. The student work samples were created by Dr. K (the author of this study) based on her former students' responses to test and quiz problems in a college intermediate algebra course. The work samples included students' incorrect responses as well as correct responses with inefficient strategies. PSMTs discussed student work on six algebra problems. Among them, three were related to the big ideas of

variation, the Cartesian Connection, and the graph as a locus of points, and three related to other big ideas such as equivalence and expressions as objects: expressions are mathematical objects by themselves (Fosnot & Jacob, 2010).

Writing. The writing assignment had two prompts: W(a) Provide three examples that illustrate students' cognitive difficulties and big ideas in understanding mathematical expressions in algebra; and W(b) Design a task that allows learners to develop big ideas and discuss pedagogical moves with the task as a teacher. W(a) was a form of reflective writing assignment, which was incorporated into the curriculum throughout Math A and B. However, W(b) was a task design assignment assigned only once in Math B, but not in Math A.

Data Collection and Methods of Analysis

All sessions during the Algebra unit were recorded using two video cameras. The first camera captured a group of PSMTs—PSMT A, B, C, and D—entirely; and the second camera mostly captured the whole class. The group with PSMT A, B, C, and D was selected due to their proximity to a camera, and it represented a typical group of PSMTs. For the video data from Task 2, a selective coding strategy (Strauss, 1987) was used to examine whether the PSMTs related students' difficulties to big ideas or they focused on other issues, such as the correctness of student work and problem types, as shown in other studies (Crespo, 2000; Jansen & Spitzer, 2009; Nathan & Petrocino, 2003; Postelnicu, 2011; Tirosh, 2000).

For the writing assignment data, in W(a), the number of examples showing PSMTs' successful connections of students' difficulties to big ideas was counted. In W(b), the context scale of the Assessment of Facilitation of Mathematizing (AFM; Fosnot, Dolk, Zolkower, & Seignoret, 2006) was applied (see Figure 1). The context scale of AFM is a tool measuring teachers' abilities to design and use tasks in instruction. It measures whether tasks are realistic—truly problematic in the sense that they can serve as the starting points for mathematization—and how tasks are implemented by bringing strategies, representations, and big ideas together. The PSMTs in this study designed tasks and explained their hypothetical, pedagogical moves with the tasks, without implementing them. However, the scale was an appropriate instrument for evaluating their work. For the reliability, the author and a Ph.D. student in mathematics education, who had previously taken Math A and B, independently rated written work of five PSMTs for both W(a) and W(b). They then negotiated discrepancies in their ratings and reached an agreement. The author then rated the work of twelve remaining PSMTs based on the consensus made by the author and the Ph.D. student.

Results

The results are presented in three parts. The first two deal with PSMTs' knowledge of student thinking—the first analyzed from the video data of the video-recorded group with PSMTs A, B, C, and D, and the second from the writing assignment data of the entire class. The third deals with PSMTs' ability to design and explain tasks analyzed from the writing assignment data of the entire class.

PSMTs' Knowledge of Student Thinking in Class Discussion

When analyzing student work, the group normally started their discussion by describing whether the student work was correct or not, how the problems should have been solved, or how the types of problems confused students—similar to the traits of the teachers in other studies (Crespo, 2000; Postelnicu, 2011). They then evolved their discussion into questioning why students made such mistakes or approaches—the cause of students'

difficulties. However, unless intervened by the instructors, they tended to make hasty conclusions that students did not understand the qualities—such as definitions, concepts, and properties—without further investigating the big ideas behind them. Even in cases where they were intervened by instructors, they often could not articulate what big ideas were involved with students’ struggles or how they were related.

On the three problems related to the big ideas of variation, the Cartesian Connection, and the graph as a locus of points, the group associated big ideas with students’ difficulties quite successfully in two problems, and less successfully in one problem. Provided below are excerpts showing their progress in knowledge of student thinking in two problems. Excerpts 1 and 2 are for Problem 1 (Figure 2), one of the two problems showing successful connections to big ideas. Excerpts 3 and 4 are for Problem 2 (Figure 3), the one showing a less successful connection.

Problem 1: Sketch the graphs of two lines, $x = 2$ and $y = 3$, on the x, y coordinate plane.



Figure 2. Student work

Problem 1. The group was given two samples of student work responding to the question, “Sketch the graphs of the two lines $x = 2$ and $y = 3$ on the x, y coordinated plane.” One sample showed a drawing of two points $(2,0)$ and $(0,3)$, and the other a single point $(2, 3)$ in the plane (see Figure 2).

Initial response

(Excerpt 1)

PSMT A: They don’t understand that these are even lines. They read the line part, but they don’t grasp ... (the rest is inaudible).

PSMT B: They both got the idea that for the x -axis they have something to do with 2 and for the y -axis they have something to do with 3. But, basically this is... I don’t know. They both got the idea, but they didn’t have the line thing down.

PSMT C: This (the second drawing in Figure 2) is thought as a coordinate set like $(2,3)$. And the other one is like, x is this and y is this.

PSMT A: I don’t think they look at them as two lines. Whenever I look at a math problem, I focus on numbers rather than contexts other than numbers.

PSMT D: I also think it is weird because it is $x = 2$, but not like $y = mx + b$.

PSMT C: I think the problem is you are used to seeing $y = mx + b$ form, right? But this doesn’t look like that. Neither of them looks like that. So how could it be a line if it is not in that form?

PSMT A: So it came to like, oh, where do I draw the lines?

PSMT C: So if they see a number (a form) like this ($x = 2$), they think immediately it is a point. But if they see a $y = mx + b$ form, they see that it is a line. It is kind of like the way we did with the circle thing (in the Omar Khayyam task). It ($y = \sqrt{qx - x^2}$, which they converted from $x^2 + y^2 = qx$) wasn’t in the form (of a circle, $x^2 + y^2 = r^2$). So we thought that it couldn’t be a circle.

PSMT A: And also when you look at $x = 2$, it is not saying y equals anything. That is why you don’t think it is a line.

As shown in Excerpt 1, the group initially focused on what the students did wrong and how the problem could have been done correctly. But with PSMT C reminding the group of their own struggle with a circle graph in the Omar Khayyam activity, the group began to evolve their discussion to find a cause for students' mistakes. PSMT A pointed out that the students might not have understood that $x = 2$ means y equals anything—a relevant idea to variation. The group, however, did not make any effort to articulate what big idea was involved in “ y equals anything” until Dr. K joined their discussion.

Transitioning to big ideas

(Excerpt 2)

Dr. K: So, what are the big ideas they are missing?

PSMT A: I don't think they understand that when they say $x = 2$, y could be equal to anything else. So then $x = 2$ was just like $x = 2$.

Dr. K: How do you know when $x = 2$, y could be anything else?

PSMT A: So, if $x = 2$, and if you put it in the form $y = mx + b$, then it could be, uh ... (She trailed off.)

Dr. K: So, how do you interpret this algebraic equation, $x = 2$, graphically (asking the group)?

PSMT C: It is all the points that are $x = 2$ in the plane.

Dr. K: All the points?

PSMT C: All the points (x, y) with $x = 2$. All the points, $(2, y)$.

Dr. K: So, what kind of difficulties did the students have in sketching those graphs?

PSMT C: They didn't understand that y was a variable that didn't show. So, it could take any value.

As shown in Excerpt 2, PSMT A knew that the equation $x = 2$ in the two-dimensional context meant y could be anything. But when asked why the equation $x = 2$ meant y could be anything, she was confounded by her algebraic image of the line, $y = mx + b$, and could not articulate the role of y in the equation $x = 2$. It was possible that her knowing of “ y could be equal to anything” was from her memorization rather than from her understanding of the role of variable in the equation. Seeing her struggle, Dr. K invited other group members to join the discussion. PSMT C then clarified the idea, claiming that the geometric meaning of $x = 2$ was “all the points (x, y) with $x = 2$ ”—associating it implicitly to the big idea of the graph as a locus of points. He also pointed out that students did not understand the role of variable in the equation, $x = 2$ —connecting students' errors to the big idea of variation.

Problem 2. The group discussed student work (Figure 3) on the problem: “Find the x -intercepts of the graph of the quadratic function, $f(x) = (x+1)(x+5)$.” The sample of students' work showed two responses to the problem—the first showing an incorrect answer of 1 and 5, and the second showing an almost correct answer of -1 and -5 (the correct answer should have been $(-1, 0)$ and $(-5, 0)$) with an inefficient approach.

Initial response

(Excerpt 3)

PSMT B: Do you even know why they didn't put this $[(x+1)(x+5)]$ equal to 0? Well, I think when they saw the word quadratic, they were probably thinking that they had to try the formula.

Others: Yeah!

PSMT B: When they saw the word quadratic, they thought that they needed to use the quadratic formula. So, they expanded it and used it.

Others: Yeah!

PSMT D: This student (the first student) knew the procedure. But he forgot the negatives.

PSMT A: This person (the second student) didn't even put this $(x+1)(x+5)$ equal to 0. If he knew that ... (PSMT A is then interrupted by PSMT B.)

PSMT B: So, he memorized the procedure, but did slightly wrong.

PSMT A: Yeah.

PSMT C: I think for both, what is actually lost is, what the x -intercept is.

Others: Yeah!

PSMT C: Because if he knew that x -intercept means y -coordinate is 0, then he could figure out this easily.

Others: Yeah!

Problem 2: Find the x-intercepts of the graph of a quadratic function
 $f(x) = (x+1)(x+5)$.

Student 1

1, 5

Student 2

$$(x+1)(x+5) = x^2 + x + 5x + 5 \\ = x^2 + 6x + 5$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} = \frac{-6 \pm \sqrt{16}}{2}$$

$$= \frac{-6 \pm 4}{2}$$

$$\frac{-6+4}{2} = \frac{-2}{2} = -1$$

$$\frac{-6-4}{2} = \frac{-10}{2} = -5$$

Figure 3. Student work

The group started with a question why the students did not even include the equation $(x+1)(x+5) = 0$ in their work. But rather than pursuing cognitive issues related to student work, the group initially followed the same traits as they had in Problem 1. The group attributed students' difficulties to an incorrect application of a procedure and students' confusion over the word "quadratic." The group then changed the direction of the discussion to find the cause of students' difficulties and agreed that students' lack of understanding of the meaning of x -intercept was the cause for students' difficulty. However, being (seemingly) content with their conclusion, the group ended their discussion without bringing the big idea of the Cartesian Connection. Dr. B then visited the group and intervened their discussion as shown in the following excerpt.

Transition to big ideas

(Excerpt 4)

Dr. B: There is a big idea embedded in here. Don't you think? I mean, whether you want to write this as -1 and -5 or $(-1, 0)$ and $(-5, 0)$, to be able to look at that and to be able to write it down, what idea is involved?

PSMT C: Understanding what x -intercept means?

Dr. B: That is certainly a part of it. But let's say they knew the x -intercept is where the graph hits the x -axis. So, what big idea seems to be missing here?

PSMT C: The definition of x -intercept is when y equals to 0. And so the equation what they are basically looking for is, with $f(x)$ equals to 0, $(x+1)(x+5)$ equals to 0. And I don't think they got that concept.

Dr. B: So, what concept is it?

PSMT A: Like the Zero Product Property.

Dr. B: The Zero Product Property?

PSMT A: Yeah, the Zero Product Property. Because when $x+1$ equals 0 or $x+5$ equals 0, 0 times another term, 0 times anything is 0.

Dr. B: So, the big idea of the Zero Product Property seems to be missing here. At least it is not used. Is there anything else?

(There is a brief silence.)

PSMT B: I think they are used to $f(x)$ equation looking like $(x-1)(x-5)$. So, they would just write, x -intercepts are 1 and 5. And in this problem, it changes to addition, so they were just thinking, oh it is the number that is in each one of those terms, so it is 1 and 5.

Dr. B: Yeah, that could be part of it. But let's think about why you should make it equal to 0.
 PSMT C: Because you are looking for where the graph passes the x -axis.
 Dr. B: Right. So, to be able to write the equation $(x+1)(x+5) = 0$ from the information, what idea is needed?
 PSMT C: They need either one of them equals to 0.
 Dr. B: That is the Zero Property again. But there is one other thing they need to understand.

...
 Dr. B: They need to understand this idea of the Cartesian Connection here. When the graph hits the x -axis, the x -intercepts should have the coordinates of $(x,0)$ as they are on the x -axis. So by the Cartesian Connection, the coordinates of $(x,0)$ must satisfy the equation $y = (x+1)(x+5)$, yielding $(x+1)(x+5)=0$.

It was noticeable that Dr. B and PSMT C described the x -intercept differently. While Dr. B described it as “where the graph hits the x -axis” (a geometric description, which is the formal definition of x -intercept), PSMT C described it as “when y equals to 0” (an algebraic description). Since PSMT C claimed that students were unable to set up the equation, $0 = (x+1)(x+5)$, because they did not know y had to be 0, which was seen as more of a procedure than an understanding, Dr. B asked the group what big idea students were missing there. He hoped that the group could identify the Cartesian Connection as the big idea connecting the geometric and algebraic interpretations of the x -intercept. Instead of the Cartesian Connection, however, PSMT A brought up the Zero Product Property—if $xy = 0$, then either $x = 0$ or $y = 0$ —as the cause for students' difficulty.

The Zero Product Property was indeed an important idea that should have been used to solve the problem. However, it was not the main source of the difficulty for the students who could not set up the equation $0 = (x+1)(x+5)$. So, Dr. B asked the group again if there were any other ideas involved in the students' difficulty. Yet the group reverted to the initial stage, as PSMT B attributed students' difficulty to the type of problem students had— $f(x) = (x+1)(x+5)$ —which was not in a typical form of $f(x) = (x-a)(x-b)$. The group spent a few more minutes to discuss it, but their ideas circled around the definition of x -intercept and the Zero Product Property without making progress. In the end, Dr. B explained the group that the Cartesian Connection was the idea that students had not understood.

PSMTs' Knowledge of Student Thinking in Writing

The analysis of PSMTs' writings on W(a) reveals that sixteen of the seventeen PSMTs who submitted the writing assignment were able to connect students' difficulties to big ideas in at least one of the three examples they provided—with five PSMTs providing 1, six providing 2, and five providing 3 successful examples. The analysis also finds that PSMTs had a higher success rate in describing big ideas when their examples were similar to those discussed in the course, compared to when they were not similar. The success rate for the similar examples was 72.2% (26 of 36), while for the not similar examples it was 40% (6 of 15).

Of the 51 examples provided by PSMTs, 28 examples were about the big ideas of variation, the Cartesian Connection, and/or the graph as a locus of points. Out of those 28 examples, 22 examples (78.5%) successfully described the big ideas embedded in the examples including 6 of 7 examples about Problem 1 (graphs of $x = 2$ and $y = 3$), 10 of 15 examples about Problem 2 (the y -intercept of the graph of $y = (x+1)(x+5)$), and all 6 examples about the Omar Khayyam task.

Provided below are quotations of three, successful examples. The first is for Problem 1, the second for Problem 2, and the third for the Omar Khayyam task.

PSMT E: Her students did not seem to grasp the concept that if $x =$ a number, such as 2, then y could be equal to an infinite number of numbers, on the same line, therefore making their graphs as shown in Figure 3. ... The concept of function, in which a function with one variable determines the value of the other variable within that same function, should be considered... If students had understood this concept of variation, they may have been able to reason that if a function, for instance $x = 2$ represents

the point (2,0) on the x -axis, then the value of y along that same line, although not shown in the function itself, is infinite and therefore appears as so in Figure 3.

PSMT F: The student is missing the big idea that an x -intercept is a point, or points, on the graph where there is some value for the x -coordinate and a value of zero for the y -coordinate. They do not see that to find a point where a graph intercepts the x -axis one must understand that this value is a point on the graph with an x and a y coordinate. They are not corresponding the equation in front of them to a graph they are probably familiar with. They are not interpreting the graph as a collection of points that all satisfy the same equation. This is the big idea they are missing, that a graph is a locus of points all satisfying the same condition.

PSMT G: In our own class, we demonstrated a lack of understanding that a circle is the set of all points a fixed distance (in radius) from a given point (the center). Due to the compartmentalization of the geometric and algebraic concepts of a circle, we could not sketch the graph of the equation $x^2 + y^2 = qx$. Looking at the equation, we thought that the radius was simply $\sqrt{qx}$ and the center was at (0,0). We did not realize that the radius of a circle must be fixed and cannot depend on x or y . The big idea we struggled with was that if (0,0) satisfies the equation, it must be on the circle, because a circle is a set of points (geometric) that satisfy the equation.

PSMTs' Ability to Design and Explain Tasks

Twelve of the seventeen PSMTs designed a task for students who did not understand the big idea of variation, the Cartesian connection, and the graph as a locus of points; of those four rated at Level 1, five at Level 2, and three at Level 3 (see the levels in Figure 1). Provided below are explanations for three tasks that PSMTs designed and commented, one for each level.

Level 1. PSMT H provided the following task:

There were two cars traveling on two different roads. The red Porsche's path can be drawn by graphing the line $x = 2$, and the black Ferrari's path can be traced by the line $y = 3$. These two lines create perpendicular paths through the axis that they intersect. Determine whether or not these two cars will collide while on their separate paths.

This task already assumed that students knew how to represent the equations, $x = 2$ and $y = 3$, as lines. Moreover, the role of variables x and y was totally missing in this task (although PSMT H mentioned variable). Even though the task was contextualized, it was not an investigational task that could serve as the starting point of mathematization. It had very little potential to help students develop the big idea of the graph as a locus of points, which she aimed for.

Level 2. PSMT I wrote that Dr. K's students displayed "a clear lack of understanding of what a line is." He said, "a line is an infinite collection of points that satisfy a relationship between x and y . This is an idea that hinges on the concept of variation. As the variable x changes, the variable y will change in an amount proportional to x ." To address this issue, he created a task about "the distance a car has traveled over time." He would first give students two points and then ask students "to graphically represent another point." He would also ask them "to determine the slope of the line in order to find a useful equation for the line." He believed that "the construction of a line by mapping out many points would start to help them make the realization that a line is a collection of points."

The task by PSMT I, which used a distance and time context, had some good components. It connected algebraic and geometric representations and had potential to serve as a starting point of an investigation of the big ideas. However, his pedagogical moves with the task seemed to lack details about how students develop the big ideas. "Mapping out many points" may help students realize that there are many points involved in the graphing activity. However, as the activity still involves finitely many points, it would not be sufficient enough to develop student understanding of variation or the graph as a locus of points. He also wrote

that he would have students find the slope of the line. However, he did not explain what slope meant and how it could be done in the given context. Although his task was somewhat beyond an application of known facts, his instructional plan lacked elaboration and should have been supplemented by subtasks to achieve his goal.

Level 3. PSMT J wrote that his task was to address the big idea of the graph as a locus of points, with which PSMTs had struggled in the Omar Khayyam task. He said that the relationship between the Pythagorean theorem and the distance formula, as well as the big idea of variation, was the critical idea for understanding circles. For a leading activity, he would first ask students to draw triangular gardens at the four corners of a rectangular backyard and then to find the perimeters of gardens, in order to help students understand how the Pythagorean theorem is “related to lengths and distances.” He would then ask students to connect those gardens by line paths and to find the distances so students could use the Pythagorean theorem for various distances. Afterward, he would ask students to locate possible spots, 6 meters away from the center of the backyard, where a fountain could be created. He would then ask students to find an equation describing the distance of 6 for the various locations of the fountain, expecting students to come up with a circular shape.

He anticipated that students would come up with something similar to a “standard equation of a circle,” as the distance derived from the Pythagorean theorem would provide such a form. He also expected that students would “think there are many different equations,” as their equations would look slightly different depending on the location of the fountain. If so, he would ask students to compare those equations so that they could see that “all of them are equivalent.” He would also help students to “see that the x -distance and the y -distance from the center create two sides of a right triangle where the hypotenuse is constant, which by definition of a circle (set of all points equidistant to a center) is a circle.” His work was rated at Level 3 due to the sophistication of his context and his pedagogical moves connecting representations and big ideas. His work could serve as the starting point of meaning-making; included the critical idea of the distance formula as an extended idea of the Pythagorean theorem; could help student understand the big idea of variation by having them locate all “possible spots” for fountains; and connected verbal, graphical, and algebraic representations of circle.

Discussion and Conclusions

The analysis suggests that the intervention was beneficial in developing preservice teachers’ knowledge of student thinking associated with big ideas in algebra. At the beginning of their discussion of student work, preservice teachers tended to focus on students’ errors/mistakes, correct ways the problems should have been solved, or the types of problems that students had, similar to the teachers in other studies (Jansen & Spitzer, 2009; Nathan & Petrocino, 2003; Postelnicu, 2011; Tirosh, 2000; You, 2006). However, as their discussion progressed, they often attributed students’ difficulties to a lack of understanding of qualities such as definitions, properties, and concepts. In some cases, with the help from instructors, they were able to associate students’ difficulties with the big ideas underlying multiple representations of the qualities.

The preservice teachers also designed tasks that incorporated the views of RME (Freudenthal, 1991) to some extent. Sixteen of the seventeen preservice teachers designed a contextualized problem and twelve designed a task beyond a simple application of learned facts using multiple representations. This can be interpreted as a positive effect of the intervention, considering that typical middle grade or secondary teachers tend to be symbolic-oriented in their own problem solving and have the preference of algebraic

representations in student work (Moyer, 2001; Nathan & Petrosino, 2003; Stylianou, 2010; Van Dooren et al., 2002).

Another positive effect of the intervention, and of the curriculum, was that many preservice teachers exhibited views aligned with those expected in today's mathematics education. Ten (out of 17 teachers) preservice teachers wrote about the importance of students' collaborations or communications in developing mathematical ideas, and ten teachers the importance of teachers' questioning. It seemed that their learning environment, which was based on an inquiry-based instruction with educational and philosophical foundations in social constructivism (Cobb et al., 1992) and RME (Freudenthal, 1991), coupled with their analysis of student work, influenced their views on mathematical teaching and learning.

The findings additionally suggest that there are issues that need to be addressed for the interventions focusing on big ideas. First, teacher educators should find ways to assist preservice teachers in conceptualizing the notion of big ideas. The main reason for preservice teachers' difficulties in identifying big ideas in this study was their lack of understanding of the notion of big ideas. As observed in discussions and writings, and also in the interview conducted after the course, many preservice teachers were unsure how big ideas were different from qualities such as definitions, properties, theorems, and concepts. Hence, once they made a conclusion that students' difficulties were originated from insufficient understanding of the qualities, they ended the discussion without further investigating the big ideas behind students' difficulties with the qualities. Moreover, many preservice teachers perceived mathematical qualities as mere memorization of facts rather than comprehending them based on big ideas underlying them, impeding their ability to identify big ideas behind student work.

Second, teacher educators should assist preservice teachers in becoming action researchers (Miller & Pine, 1990) on big ideas. Studies have shown that teachers tend to analyze student thinking beyond errors and mistakes and delve into the cognitive issues behind students' difficulties when dealing with difficulties of their own students or of students they interact with (Carpenter et al., 1988; Crespo, 2000; Fennema et al., 1996). Studies have also shown that identifying big ideas and gaining insight into the role of big ideas in student thinking can come from listening to students (Schifter et al., 1999). Therefore, for preservice teachers to become action researchers on big ideas, they must be provided with opportunities to analyze their own students' thinking, identify big ideas underlying students' struggles, and plan instruction accordingly.

Third, task design activities should be integrated more extensively and carefully into interventions. In this study, most of the preservice teachers (12 of 17 teachers) could not design tasks that could serve as the starting point of the construction of big ideas or effectively plan their pedagogical moves with the task—possibly due to the limited opportunity for task design activities. The previous studies suggest that extensive and comprehensive task design practice could alleviate this situation to some extent (see for example, Barlow & Cates, 2006). However, teachers, particularly novice teachers, often cannot design tasks to the level that they believe the tasks should be (Lee et al., 2018). Therefore, further research is needed to determine how task design activities can be effectively integrated into interventions to help preservice teachers understand the role of tasks as the starting point of the construction of big ideas and the importance of pedagogical moves in task implementation.

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