

Investigating Prospective Mathematics Teachers' Strategies to Generate Definitions of Special Quadrilaterals

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This study examined how prospective mathematics teachers (PSTs) enrolled in different years of a teacher education program at a public university in Turkey generated definitions of special quadrilaterals. By examining the strategies PSTs used to generate definitions, this study aimed to investigate the extent to which participating teachers could distinguish the necessary and sufficient conditions for each special quadrilateral. Analyzing the details of the characteristics of definitions generated and the strategies employed by the PSTs who had different years of experience in a teacher education program was essential to further delve into the complex defining processes and to better conceptualize the types of struggles that the PSTs experienced at the beginning and near the end of the program. A questionnaire was administered to 52 senior and 52 freshman prospective teachers, and six prospective teachers from each grade level further participated in the individual interviews. The results indicated that both senior and freshman PSTs struggled to generate more than one definition for each special quadrilateral, especially for a kite and isosceles trapezoid. Furthermore, the strategies participants used differed across years of the program and illustrated their different ways of reasoning.

Keywords: definitions, geometry, mathematical reasoning, prospective teachers, special quadrilaterals

Introduction

The essential roles of definitions in teaching and learning geometry have been long acknowledged (Haj-Yahya, 2019; Fujita & Jones, 2007; Miller, 2018; Usiskin & Griffin, 2008; Zazkis & Leikin, 2008). Researchers have highlighted these essential roles of definitions in mathematics classrooms as exemplifying, classifying, generalizing, and/or justifying (Leikin & Winicki-Landman, 2001; Selden & Selden, 2008; Stylianides, 2007). Harel and Tall (1991) echoed a similar theme, stating that “the definition is more than a naming. It is the selection of generative properties suitable for deductive construction of the abstract concept” (p. 39). Definitions foreground the logical relationships between mathematical statements and play essential roles in deductive reasoning, since all parts of a definition should be considered when constructing an argument (Haj-Yahya, Hershkowitz & Dreyfus, 2023; Selden & Selden, 2008; Stylianides, 2007). Haj-Yahya, Hershkowitz, and Dreyfus (2023) have shown that students' difficulties in understanding the roles and logical structure of different definitions when constructing and/or evaluating proofs affect their proving processes. Thus, the definitions employed and generated in a classroom environment indeed shape the structure of learners' thinking and reasoning processes. Given the centrality of definitions in teaching and learning geometry, how definitions should be introduced and presented to students in mathematics classrooms has sparked interest among many researchers.

Zazkis and Leikin (2008) argue that learner-generated definitions reflect learners' understanding of underlying mathematical concepts and can serve as a research tool. Because learner-generated definitions can serve as a lens into learners' understanding of concepts, they may also serve as a teaching tool in mathematics classrooms. Leikin and

Winicki-Landman (2000) have advocated using mathematical definitions to foster learners' mathematical thinking, stating: "One of the elements for active learning is encouraging students to realize that alternative definitions for the same concept are possible" (p.24). Similarly, many researchers and current educational reform movements have advocated learner-generated definitions to teach geometrical concepts (Avcu, 2023; CCSSM, 2010; NCTM, 2000). However, implementing CCSSM's recommendation to "trying to communicate precisely and to use clear definitions" (p. 7) in mathematics classrooms at all levels is challenging for teachers, especially prospective teachers.

To generate mathematical definitions succinctly and use correct terminology precisely, as envisioned by researchers and educational reform movements, prospective teachers must first understand and be sensitive to definitions' characteristics and roles. Yet research in mathematics education has repeatedly shown that students at all levels, even prospective teachers, have difficulty interpreting new definitions and generating definitions properly (e.g., Avcu, 2023; Zaslavsky & Shir, 2005). Moreover, much remains unknown about prospective teachers' act of defining some quadrilaterals (i.e., kite or trapezoid), as many researchers have pointed out (Avcu, 2023; Miller, 2018; Törnükü, Gündoğdu Alaylı & Akkaş, 2013). Thus, this study investigates prospective mathematics teachers' (PSTs) strategies for generating definitions of special quadrilaterals, including rectangles, rhombuses, squares, parallelograms, trapezoids, kites, and isosceles trapezoids. To investigate prospective mathematics teachers' strategies for generating definitions of special quadrilaterals, this study first analyzes whether PSTs can distinguish between necessary and sufficient conditions for special quadrilaterals. The following research questions guided this study:

1. To what extent do prospective mathematics teachers generate appropriate definitions of special quadrilaterals?
2. What strategies do prospective mathematics teachers employ while generating definitions?

Theoretical Framework

Studies that have explored learners' defining acts in depth have pointed out various factors that might influence learners' ability to generate valid definitions (e.g., Avcu, 2023; Fujita, 2012; Haj-Yahya, Mahameed & Haj-Yahya, 2024). Although many factors could be cited to explain why learners struggle to generate definitions, two of these factors stand out: (1) language and (2) concept image, which will be addressed in this section.

The Role of Language in Defining and Understanding Geometric Concepts

Language presents several significant challenges to students when learning and doing mathematics (Schleppegrell, 2007). When students encounter technical words in mathematics classrooms with meanings that relate to, or differ from, their colloquial use, it can be a major challenge (Thompson & Rubenstein, 2000; Seah, 2018). For instance, Avcu (2023) has argued that the Turkish name "Yamuk (Trapezoid)" conveys the meaning of "crooked objects" in everyday register, while it conveys a totally different meaning (a quadrilateral with at least one pair of opposite parallel sides) in the mathematics register, which might contribute to low performance in defining trapezoids properly. Similarly, many researchers have investigated several terms in mathematics such as "trapezoid, tangent, similar, right, rectangle, or regular" and have documented that these terms could convey different meanings to students in mathematics classrooms and in their real lives (e.g., Haj-

Yahya & Haj-Yahya, 2024; Seah, 2018; Türnüklü, Gündoğdu Alaylı & Akkaş, 2013). Beyond colloquial use in mathematics, different meanings associated with a term within the mathematics register may also lead students to interpret concepts differently and incorrectly (e.g., Riccomini, Smith, Hughes & Fries, 2015). For instance, fundamental geometric concepts such as base and height are mathematical terms used with subtly different meanings at different times (Otten & Herbel-Eisenmann, 2009). This phenomenon is described as lexical ambiguity or polysemy; both terms refer to several meanings associated with a single word or phrase. Lexical ambiguity or polysemy can present unique challenges for both students and teachers (Barwell, 2005). Lexical ambiguity might then act as a barrier to generating appropriate definitions.

Although the challenges language presents in learning and teaching mathematics have been well documented, some researchers suggest it should also be conceptualized as a resource and a contributing factor in learning mathematics. This growing body of research indicates that language presents both challenges and opportunities for mathematical learning and teaching (Planas, 2014; Planas & Setati-Phakeng, 2014). For instance, Martin and Bassok (2005) coined the term “semantic cue effect” to describe how semantic relationships between objects influence word-problem solving. Similarly, Bieswanger and Becker (2017) have mentioned endocentric compounds (i.e., the meaning of a compound can be guessed by combining the meanings of its components) and exocentric compounds (i.e., the meanings of a compound cannot be guessed by combining the meanings of its components) to highlight the roles of language in learning (p. 92-93). Furthermore, some researchers have highlighted the effects of syntactic, semantic, and lexical features of the given names of quadrilaterals when generating definitions for these quadrilaterals (Avcu, 2023; Haj-Yahya & Haj-Yahya, 2024; Usiskin & Griffin, 2008). Avcu (2023) documented that the Turkish names for parallelogram (paralelkenar) and rhombus (eşkenar dörtgen) are endocentric compound words that may contribute to definition generation. Haj-Yahya (2021) also documents that students preferred formal definitions that captured the core meaning of a concept name. Thus, statements as “We call it parallelogram, parallel, the word parallel should be included” (p. 624), while defining a parallelogram as mentioned by Haj-Yahya, Hershkowitz and Dreyfus (2023), demonstrate the importance of the semantic sense of concept names and the student’s preferences for properties used in defining the concept. From this perspective of seeing language as a resource, language does not consist of mere grammar and vocabulary but is also seen as a semantic system or a system of meaning that allows learners to create webs of relationships among and between ideas. Thus, the semantic relationships between a concept's name and definition might influence the ability to define it. Another factor that might influence the ability to define is the concept images of the quadrilaterals being defined, which will be addressed next.

The Role of Concept Image on Defining and Understanding Geometric Concepts

Tall and Vinner (1981) proposed a theory of concept image as “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (p.152). They argue that concept images of learners are personal and built up over the years through experiences of all kinds. Mental representations of geometric concepts play a crucial role in shaping how learners conceptualize them. Researchers demonstrate that difficulties in understanding concept definitions can arise when the concept image is incomplete or inaccurate (e.g., Pickreign, 2007). Thus, the concept image is essential to generating mathematical concept definitions.

Hershkowitz (1990) notes that prototypical examples of mathematical concepts influence how learners form concept images. Because learners acquire prototypical examples first and therefore form the concept image most learners hold, their specific attributes may serve as the concept definition and the criterion for classifying examples (Hershkowitz, 1990). The specific attributes of the prototypical example are usually highly visual, which may lead learners to judge quadrilaterals solely based on these prototypical examples and to understand inclusion relationships improperly (Fujita, 2012; Fujita & Jones, 2007). Hershkowitz (1990) has identified several approaches in which learners use prototypical examples to judge examples and non-examples of a geometric concept, such as: Visual Judgment, Prototype Attribute Judgment, and Analytical Judgment. In Visual Judgment and Prototype Attribute Judgment, the prototypical examples' visual appearances and specific attributes may play a critical role, while in Analytical Judgment, the concept's mathematically defined properties fulfill this role.

Similar to the approaches Hershkowitz (1990) identified, van Hiele's theory suggests that learners first acquire the visual appearances of geometric shapes at the Visualization Level and then move to comprehend their properties at the Analytical Level (van Hiele, 1986). However, judging relationships between different geometric shapes and recognizing their critical features occur later in their learning journey (aka Informal Deduction Level). Given that difficulties in understanding concept definitions might be influenced by incomplete or inaccurate concept images of the learners (e.g., Pickreign, 2007; Tall & Vinner, 1981), the learning path of learners' geometric thinking with different but hierarchical levels, which occurs from conceptualizing geometric shapes as whole to focusing on properties of geometric shapes and then recognizing interrelationships among geometric shapes, might then characterize the learners' ways of generating definitions. Thus, this study hypothesizes that participants' defining acts could be categorized based on whether learners' defining strategies are affected by prototypical examples of the concepts being defined. For instance, as in the example strategy, participants might define quadrilaterals based on a prototype's specific visual appearance, or, as in the property strategies, they might define quadrilaterals based on the specific features of these prototypes. The following section addresses the different strategies participants could employ.

Method

This study was planned as a collective case study (Stake, 1995) to understand how prospective teachers distinguish necessary and sufficient conditions while generating definitions and the strategies they employ during this process. Stake (1994) defined collective case studies as the "study [of] a number of [individual] cases jointly in order to inquire into [a] phenomenon, population, or general condition" (p. 237). Thus, this study examined individual prospective teachers' strategies as they generated various definitions of special quadrilaterals by addressing interrelated skills, including determining the necessary and sufficient conditions for each special quadrilateral.

Participants

A total of 104 prospective mathematics teachers (PSTs), 52 freshmen and 52 seniors, who enrolled in a teacher education program at a public university in Turkey, participated in the study. Thus, the study included all prospective teachers in the first and last year of the program. The PSTs were enrolled in a teacher education program to be certified to teach

mathematics from grades 5-8. The study conducted individual interviews with 12 prospective teachers (6 freshman and 6 senior PSTs). The study used purposive sampling to select participants. According to Patton (2002), “the logic and power of purposeful sampling lie in selecting information-rich cases” (p. 230).

The purpose of conducting the study with senior and freshman PSTs was not to compare their ability to define special quadrilaterals. The study hypothesized that senior prospective teachers would perform better in defining special quadrilaterals than freshman prospective teachers, given that senior prospective teachers had completed more education and mathematics courses in the program. Instead, the study analyzed the characteristics of the definitions participants constructed and the strategies they used to generate them, to better understand the complex definition process and to better conceptualize the types of struggles PSTs experienced at the beginning and near the end of the program.

We administered a questionnaire to all prospective mathematics teachers in their first and final year of the teacher education program. The PSTs were asked to construct two different definitions for each special quadrilateral. We analyzed the responses of prospective teachers who volunteered and agreed to participate in individual interviews, and we identified 6 PSTs from each grade level who were predicted to use different strategies in the definition-construction process.

Data Sources

We administered a two-part questionnaire to all prospective teachers in freshman and senior year at a public university in Turkey (104 PSTs) in the spring semester of the 2023-2024 academic year. Later, I conducted individual interviews with six PSTs at each grade level to further explore their responses to the questionnaire questions. Thus, the data sources were the PSTs' questionnaire responses and individual interviews. In the first part of the questionnaire, PSTs were asked to draw two examples of each of the following special quadrilaterals: rectangle, rhombus, parallelogram, square, trapezoid, isosceles trapezoid, and kite. Although the PSTs' quadrilateral examples were not included in the data sources, the researcher collected them to better understand participants' conceptual images of quadrilaterals. In the second part of the questionnaire, PSTs were asked to generate two different definitions of the given quadrilaterals and state how the definitions differed. The researcher administered the questionnaire to the PSTs in a classroom setting, and it took approximately 75 minutes.

The researcher examined the definitions of special quadrilaterals proposed by prospective teachers who volunteered and were willing to participate in individual interviews and conducted semi-structured individual interviews with 6 prospective teachers from each grade level who were thought to have different thinking structures and definition-forming strategies. During the individual interviews, the PSTs were asked to generate additional definitions beyond those they provided in the questionnaire for each special quadrilateral and to explain their definition-generation strategies. “Can you generate another definition?” “Can you explain how you decided on this definition?” and “What is your strategy while generating a definition?” were among the questions used during the individual interviews. The researcher conducted the individual interviews in an empty classroom, and all were video-recorded. The individual interviews took approximately 75 minutes since constructing various definitions was a highly demanding task for the participants. The researcher recorded all definitions participants generated during the individual interviews and on the questionnaire in Turkish and translated them into English for this manuscript.

Analyzing PSTs' Responses to the Questionnaire Questions

Zazkis and Leikin (2008) classified learner-generated definitions into categories such as accessibility and correctness. In the accessibility category, they considered whether the definitions were generated with ease or with struggle. As they cautioned researchers who might adopt this category, stating: “The method of data collection in this study (limited time for written response) did not allow for the analysis of the accessibility category” (p.137), determining whether learners generated definitions with ease or struggle would be difficult. Thus, this study classified learner-generated definitions based solely on correctness. Zazkis and Leikin (2008) classified the correctness of a generated definition into four subcategories: Necessary and Sufficient, Necessary but not Sufficient, Sufficient but not Necessary, and neither Necessary nor Sufficient. We followed a similar approach, and Table 1 displays sample responses for each category.

Table 1
Sample Responses for Coding Categories in the Case of a Rectangle

Categories	Subcategories	Sample of Responses
Correctness	Necessary and Sufficient	A rectangle is a parallelogram with diagonals that are equal in length.
	Necessary but not Sufficient	A rectangle is a quadrilateral with opposite sides parallel and equal.
	Sufficient but not Necessary	A rectangle is a quadrilateral with all sides equal and all angles 90 degrees.
	Neither Necessary nor Sufficient	A rectangle is a quadrilateral with perpendicular diagonals.

In the first step of analyzing the correctness of the PSTs' generated definitions, we classified the definitions proposed by the PSTs as a finished product in the questionnaire as appropriate if they included necessary and sufficient conditions, and inappropriate if they did not include necessary and/or sufficient conditions. Later, they classified inappropriate definitions as including necessary but not sufficient, sufficient but not necessary, or neither necessary nor sufficient conditions.

Analyzing PSTs' Responses to the Individual Interview Questions

The data analysis of the PSTs' responses to the interview questions began with transcribing the individual interviews and reviewing participants' responses. Next, the study reviewed current literature on generating and/or evaluating mathematical definitions to identify the strategies PSTs could use when generating definitions during individual interviews. For instance, Leikin and Winicki-Landman (2000) documented two types of strategies participants used when dealing with mathematical definitions: set strategies and properties strategies. However, they have cautioned that “The properties strategy was rarely used when the teachers dealt with the definitions of quadrilaterals” (Leikin & Winicki-Landman, 2000; p.25). Researchers have also documented the influence of concept images, especially prototypical examples, as reference sets during definition generation (Fujita, 2012; Hershkowitz, 1990; Türnüklü, Gündoğdu Alaylı & Akkaş, 2013; Zazkis & Leikin, 2008). Furthermore, Haj-Yahya, Mahameed, and Haj-Yahya (2024) recommend using concept maps as an instructional strategy to foster learners' comprehension of inclusion

relationships among quadrilaterals. They found that concept maps helped learners construct appropriate, constructive definitions. Table 2 presents all strategies the participating prospective teachers might use to generate definitions.

Table 2

Strategies to Generate Definition of Special Quadrilaterals

Name of the Strategy	Characteristics of the Strategy
Example Strategy	Focusing on the sets of objects that exemplify each quadrilateral and employing the visual aspects of these examples during definition generation process (Hershkowitz, 1990; Leikin & Winicki-Landman, 2000).
Property Strategy	Focusing on the properties of each concept and relying on these properties without questioning whether these properties are necessary and/or sufficient during definition generation process (Hershkowitz, 1990; Leikin & Winicki-Landman, 2000).
Hierarchical Relationship Strategy	Focusing on the interrelationships between quadrilaterals and employing these relationships while generating definitions (Haj-Yahya, Mahameed & Haj-Yahya, 2024).
Logical Deduction Strategy	Focusing on the critical properties of each concept and knowing these properties could be enough to derive all other properties of the quadrilateral by employing logical deduction during definition generation process (Hershkowitz, 1990).

As shown in Table 2, the strategies participants could employ to generate definitions align well with previous studies investigating learners' comprehension of geometry (i.e., van Hiele Theory or Prototype Phenomenon). For instance, while the PSTs might focus on the sets of objects that exemplify special quadrilaterals and employ the visual aspects of these examples during the definition generation process in Example Strategy (corresponds well with the Visualization Level of van Hiele Theory or Visual Judgment), the PSTs might employ the properties of each quadrilateral in Property Strategy (corresponds to the Analysis Level in van Hiele Theory or Prototype Attribute Judgment). Similarly, the participants might choose to rely on the interrelationships among the quadrilaterals in Hierarchical Relationship Strategy, which is similar to the Informal Deduction Level in van Hiele Theory, or they might focus on deducing the critical properties of the quadrilaterals deductively (corresponds to the Deduction Level in van Hiele Theory or Analytical Judgment).

The data analysis process mainly occurred in several steps in this study. These steps involved concurrent processes, with analysis moving back and forth throughout (Miles, Huberman, & Saldana, 2014). The author served as the main coder in each step, and two graduate students served as second coders. The author independently reviewed all the responses and identified the definitions, characteristics of definitions, and common strategies employed while constructing definitions during the interviews; (2) graduate students coded a sample of the responses by using coding schemes constructed by reviewing the existing literature (see Table 1 and Table 2); and (3) the coders compared their coding and indicated any mismatches with the codes that could lead to the generation of new codes or adjustment of existing codes.

The researcher and the two graduate students conducted the analyses simultaneously, and the researcher calculated coding reliability to demonstrate coding similarities and differences between the coders. To obtain inter-coder reliability, the reliability formula specified by Miles and Huberman (1994) was employed ($Reliability = \frac{Consensus}{(Consensus + Disagreement)} \times 100$). Obtaining a reliability percentage of at least 70%

between two coders was considered necessary for the reliability of the data analysis process (Yıldırım & Şimşek, 2006). In this study, the coding reliability percentage was 0.92.

Results

Characteristics of the Definitions Generated by Prospective Teachers

Table 3 shows the cumulative characteristics of the definitions generated by senior prospective teachers. As shown in the table, prospective teachers provided appropriate definitions of a parallelogram and a rectangle, but struggled to define an isosceles trapezoid appropriately. “A trapezoid with at least one pair of opposite sides with equal length” was a common definition of an isosceles trapezoid generated by senior prospective teachers. Although this definition includes the necessary properties of an isosceles trapezoid (i.e., at least one pair of equal opposite sides), it is not sufficient because it would also include parallelograms and rhombi. Thus, it was coded as an inappropriate definition of an isosceles trapezoid.

Table 3

Characteristics of the Definitions Generated by Senior Prospective Teachers

Quadrilaterals	Appropriate	Inappropriate		
	Necessary and Sufficient	Necessary but not Sufficient	Sufficient but not Necessary	Neither Necessary nor Sufficient
Square	75	22	-	4
Rectangle	79	15	3	5
Parallelogram	82	4	1	11
Rhombus	68	13	7	5
Trapezoid	63	-	12	15
Kite	52	14	17	14
Isosceles Trapezoid	14	67	1	6

Table 3 also demonstrates that the PSTs struggled to generate alternative definitions of quadrilaterals. All PSTs provided at least one definition of special quadrilaterals (except Isosceles Trapezoid) that was coded as appropriate. Thus, if asked to generate one definition for each special quadrilateral, the senior prospective teachers could have performed well on such a task. However, when asked to generate two different definitions, they struggled and produced definitions coded as Necessary but not Sufficient. For instance, 14 of 22 definitions generated for a square, coded as Necessary but not Sufficient, included only side or angle properties of a square (i.e., "Square is a quadrilateral with all sides equal in length" or "Square is a quadrilateral with all right angles"). Similar tendencies have been detected for defining a rectangle, rhombus, and kite. “A rectangle is a quadrilateral with opposite sides equal,” which was classified as Necessary but not Sufficient, was commonly generated by the senior PSTs as a second definition. Because one of the definitions generated by the PSTs was classified as appropriate in these cases, it may demonstrate the complexity of understanding the critical properties of quadrilaterals.

“Kite is a quadrilateral that is composed of two isosceles triangles which have a common base” is a common definition proposed by the senior prospective teachers. However, researchers disagree on whether this definition is appropriate (e.g., Avcu, 2023). For instance, Avcu (2023) argues that “diagonals are not integral parts of kites (in general of

quadrilaterals). Thus, “two isosceles triangles” is not a relevant reference set for a kite (p.773-774)”. However, this definition is mostly proposed by high school textbooks in Turkey (Avcu, 2019; MoNE, 2018). Thus, we accepted it as appropriate for this study. Even though two isosceles triangles are accepted to be a relevant reference set for a kite definition, it was seen in the results that the PSTs were attempting to generate another definition for a kite by using isosceles triangles as a reference set, which resulted in proposing inappropriate definitions (i.e., Sufficient but not Necessary category). For instance, “Kite is a quadrilateral that is composed of two equilateral triangles which have a common base” formed an example of such a definition.

Table 4 shows the cumulative characteristics of the definitions generated by freshman prospective teachers. As shown in the table, prospective teachers struggled to define special quadrilaterals. Specifically, Isosceles Trapezoid and Kite were the special quadrilaterals PSTs struggled with most.

Table 4

Characteristics of the Definitions Generated by Freshman Prospective Teachers

Quadrilaterals	Appropriate		Inappropriate	
	Necessary and Sufficient	Necessary but not Sufficient	Sufficient but not Necessary	Neither Necessary nor Sufficient
Square	32	37	-	23
Rectangle	39	17	2	35
Parallelogram	43	4	3	36
Rhombus	49	7	10	18
Trapezoid	23	7	7	41
Kite	7	12	3	34
Isosceles Trapezoid	7	40	1	33

PSTs in the first year of the teacher education program most often generated definitions coded as Neither Necessary nor Sufficient. That is, they either included properties that were neither necessary nor sufficient for the defined quadrilateral (i.e., a rectangle is a quadrilateral with perpendicular diagonals) or used terms inappropriate for defining the quadrilateral (i.e., geometric structure, geometric solid, or polygon). In addition to being precise and using correct terminology, generating more than one definition for each quadrilateral was another struggle for freshman prospective teachers in this study.

The PSTs were asked to generate two different definitions of each quadrilateral, and Table 4 demonstrates that they struggled to propose two different definitions for each quadrilateral. For instance, the PSTs generated only 56 of 104 definitions for a kite, and only 7 were classified as appropriate. As shown in Table 4, freshman prospective teachers most often proposed definitions coded as Necessary but not Sufficient, followed by Neither Necessary nor Sufficient. For instance, for a square, “A quadrilateral with equal sides,” “A quadrilateral with opposite sides equal and parallel,” and “A quadrilateral with right angles” were among the definitions classified as Necessary but not Sufficient. These definitions classified as Necessary but not sufficient may reflect the struggles PSTs faced in identifying the critical attributes of quadrilaterals. These quantitative patterns in Tables 3 and 4 raise questions about the thinking processes underlying prospective teachers' definition generation. Interview data revealed four distinct strategies that help explain these patterns.

Defining Strategies

Six freshman and six senior prospective teachers participated in the individual interviews. During the individual interviews, participants generated various definitions of special quadrilaterals to understand the strategies they used. Table 5 below shows the strategies and the participants who used each strategy cumulatively. As shown in the table, prospective teachers in their first year of the program mostly used the example strategy when defining.

Table 5

Defining Strategies Employed by the Prospective Teachers

Year in the Teacher Education Program	Name of the Strategies Employed		
	Logical Deduction Strategy	Hierarchical Relationship Strategy	Example Strategy
Freshman	Ahmet ¹ Zeynep	Mira (Incorrect)	Ilknur Elvan Erdi
Senior	Feriha Rengin	Nuri Mahmut Meryem Ildem	

¹Note: All names used in the study are pseudonyms.

Participants who employed the example strategy attempted to generate definitions based on the examples they drew and thought of, as well as their visual appearances. For instance, Ilknur drew several examples of rectangles and used the properties they have in common to generate definitions.

Researcher: Okay, what was going on in your mind while you were generating these definitions? Can you tell me more about your strategy while defining?

Ilknur: I thought about what could fit in these examples [Pointing to rectangles that she drew] but not a square; I wanted to exclude square from this definition, because I needed to write a definition for a rectangle. I used the properties that these examples all have.

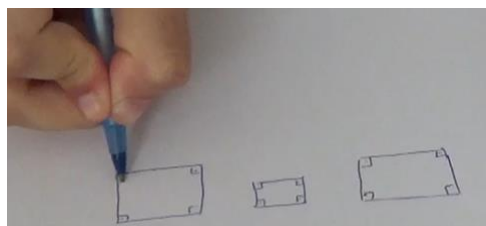


Figure 1. The examples drawn by Ilknur as cases of Rectangles

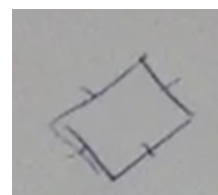
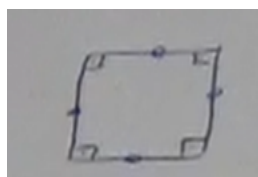
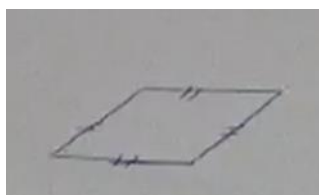
The prospective teachers who used the example strategy to generate definitions struggled most to define each concept. Furthermore, these PSTs mostly generated definitions that included additional superfluous properties to highlight the differences between the examples and non-examples of the concepts in their minds (aka non-economical definitions). For instance, Ilknur generated a definition of a rectangle as follows: “A quadrilateral that has consecutive sides different in length, opposite sides parallel, and all angles are right angles.”

Other PSTs who employed the example strategy also demonstrated similar tendencies while generating definitions.

Researcher: What is rhombus? How would you define it?

Elvan: I would say that it is a quadrilateral with opposite sides parallel and all sides equal in length.

This definition is compatible with these examples I drew [pointing to the examples in Fig. 2]. The opposite sides are parallel, and all sides are equal. It provides the first example. It is the same in the second example, so it provides the second example. I can't think of a counter example right now. I think it is the same in all rhombi I can draw. By changing the angles in this [the first example], for example, I can draw a shape like this [the second example], it [opposite sides] is still parallel, and its four sides are still equal. It provides the definition again. I would say that it is a quadrilateral whose opposite sides are parallel, and four sides are equal.



a. First example

b. Second example

c. Third example

Figure 2. The examples drawn by Elvan as cases of Rhombi

Although Elvan extended her concept image of rhombi by including a square as an example, she still attempted to define a rhombus by focusing on common attributes of the examples she considered cases of the defined quadrilateral (see Figure 2). Unlike the PSTs who used the example strategy, those who used the hierarchical relationship strategy tried to define quadrilaterals using inclusion relationships. Senior prospective teachers, except Mira, commonly used this strategy during individual interviews. Mira was the only freshman participant who used the hierarchical relationship strategy. However, she employed the strategy incorrectly, as evident in the excerpt below.

Mira: What I'm saying is that all properties of a square are also properties of a rectangle in this drawing [Pointing to the diagram with parallelogram (P), rectangle(D) and square (K) from outmost to innermost in Fig. 3]. With this drawing, I'm saying that all properties of a rectangle and a square are also properties of a parallelogram.



Figure 3. The diagram drawn by Mira to show the inclusion relationships between quadrilaterals

Researcher: So, can you explain your diagram further?

Mira: Yes. Every square is a rectangle, and every rectangle is a parallelogram. Yes, every rectangle is a parallelogram here. This is it!

Researcher: If I asked you to give me an example of a rectangle then by using this diagram, what would you draw as an example?

Mira: What did I say here? I said every square is actually a rectangle, every rectangle is actually a parallelogram. I would draw a parallelogram. Am I wrong? [Drawing a parallelogram]. I would draw a square, umm, Am I wrong? I am confused!

Researcher: What are you questioning right now? What bothered you in the parallelogram example?

Mira: I think I can't write both [Referring to square and parallelogram examples]. One of the two. I say every square is a rectangle, every rectangle is a parallelogram. I didn't draw the rectangle outside of the parallelogram [Pointing to the set of rectangles and squares in the diagram]. The rectangle is more specific. So, I'll go either from general to more specific or vice versa. I'll draw a parallelogram as an example. Every rectangle is a parallelogram. I will go with more general!

Researcher: Okay, Let's talk about definitions, how could you define a rectangle?

Mira: Well, I would say, it is a parallelogram whose sides intersect at right angles. But it also happens like this, it is a parallelogram whose consecutive sides intersect at right angles but different in length. What I mean by parallelogram is that the opposite sides are parallel and equal.

Researcher: If I asked you to propose another definition, what would it be?

Mira: I would say it is a quadrilateral with opposite sides parallel.

Researcher: So, what examples can we draw for this definition?

Mira: I can draw parallelograms. I can draw squares. We can also draw rectangles.

As shown in the excerpt above, Mira attempted to use the inclusion relationship among squares, rectangles, and parallelograms while constructing definitions. However, as evident in her statement: "What I'm saying is that all properties of a square are also properties of a rectangle.... I'm saying that all properties of a rectangle and a square are also properties of a parallelogram", she failed to identify the opposing inclusion relationships correctly between these quadrilaterals. Sets of quadrilaterals with an inclusion relationship also have an inclusion relationship in the opposite direction between their critical attributes (Hershkowitz, 1990). Mira conceptualized subsets of a quadrilateral as special examples of the quadrilateral, as evident in her statement: "Every square is a rectangle, and every rectangle is a parallelogram". However, she later generalized that a definition appropriate for a quadrilateral could also serve as a definition for its subsets (i.e., a definition of a parallelogram could also be used as a definition of a rectangle). Her incorrect generalization about generating definitions of quadrilaterals was apparent when she stated, "I would say it is a quadrilateral whose opposite sides are parallel," when asked to construct another definition of a rectangle. Senior prospective teachers attempted to employ a hierarchical relationship strategy while defining. Unlike Mira, they used this strategy correctly.

Researcher: How would you define a rectangle?

Nuri: I would say it is a quadrilateral with opposite sides parallel and one interior angle of 90 degrees.

Researcher: How did you decide, Nuri?

Nuri: Like this, if it is a rectangle, it is already a parallelogram [Pointing to the diagram in Fig.4]. I used this information in the definition. Another property of a rectangle is that it has one angle of 90 degrees. I thought about what to add to a parallelogram to get to a rectangle. I used that in the definition as well. When we put it together, we got this definition.

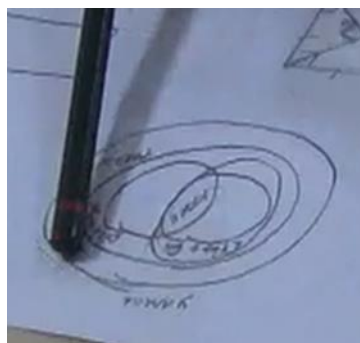


Figure 4. The diagram drawn by Nuri to show the inclusion relationships between quadrilaterals

- Researcher: Ok so, If I asked you to define a rhombus then, what would you say?
 Nuri: I would say that a rhombus is a quadrilateral with all sides equal.
 Researcher: How did you decide?
 Nuri: Because inside a parallelogram, it is a rhombus. The properties of rhombus come to mind, umm, that all sides are equal. It's an important property. That's why I would use it in the definition.
 Researcher: You mentioned important properties of quadrilaterals. What are they? Or why are they important?
 Nuri: The important property of each quadrilateral should be different so that such a shape can be separated. For example, for a square, umm, we say that all sides are equal and one interior angle is 90 degrees. This is the important property of a square. Each quadrilateral has its own properties. The important property of a rhombus is that all sides are equal. This should definitely be in the definition of a rhombus.
 Researcher: So, if I asked you to construct another definition for a rhombus, how would you define it?
 Nuri: For example, I can say that a square has an internal angle other than 90 degrees, umm no, that won't be correct! Actually, I can say that it is a parallelogram that has all sides equal.
 Researcher: So how did you decide that this definition is correct?
 Nuri: Because I already know that a rhombus is a parallelogram. If I make all sides equal, I will have a special parallelogram. We call it a rhombus.

As shown in the excerpt above, Nuri used this strategy correctly while constructing definitions of special quadrilaterals. Nuri conceptualized the inclusion relationships between parallelogram, rectangle, rhombus, and square. As evident in his statement: "I already know that a rhombus is a parallelogram. If I make all sides equal (pointing to the parallelogram), I will have a special parallelogram. We call it a rhombus.", Nuri comprehended that a subset of a quadrilateral is considered a special case of that quadrilateral (i.e., a rhombus is a special parallelogram) and that making some changes to the quadrilateral would result in the special case (i.e., making all sides of a parallelogram equal to have a rhombus). During the definition generation process, Nuri performed mental operations by asking what additional properties to add to a quadrilateral to make it more specific. Mahmut also employed the hierarchical relationship strategy while defining. When asked why the definition he generated should hold true, Mahmut constructed an argument using his mathematical referent knowledge, as shown in the excerpt below.

- Researcher: So, how can you define a kite?
 Mahmut: Kite is a quadrilateral that has at least one diagonal that bisects the other perpendicularly.
 Researcher: Can you tell me your strategy of generating this definition?
 Mahmut: The top of Kite is Quadrilateral, um, General Quadrilateral. It [Quadrilateral] does not have special properties but four sides, closed, etc. A quadrilateral should have perpendicular diagonals to be a kite. A special case of a kite is a rhombus. Rhombus should have two diagonals that bisect each other and are perpendicular to each other. So, kite is a quadrilateral in which one of its diagonals bisect the other at 90-degree angle.
 Researcher: How did you decide that this definition is correct?
 Mahmut: Let me draw it like this, umm, dot dot dot [Drawing two intersecting line segments in Fig. 5]. Since one bisects the other [Referring to the diagonals], they are equal [Referring that one diagonal bisects the other one and marking them to be same in length]. Let's draw one [Diagonal] longer so it doesn't look like a rhombus. Since it bisects this perpendicularly, it should be an isosceles triangle here. In an isosceles triangle when we draw a median from the top it should divide the base into two equal parts and be 90-degree angle here. The same thing should happen in this part [Marking the consecutive sides that are equal in length]. Then, these sides should be equal in length, and it proves that this shape is a kite.



Figure 5. Mahmut's argument to justify the kite definition

As shown in the excerpt above, Mahmut used the definition's given property to justify that two pairs of adjacent sides should be equal in length. In this study, the PSTs who employed the hierarchical relationship strategy generated various valid definitions of special quadrilaterals. Although the PSTs who used the hierarchical relationship strategy knew the critical properties of special quadrilaterals, as shown in Nuri's case, they primarily internalized the inclusion relationships between quadrilaterals while generating definitions. By contrast, PSTs who used the logical deduction strategy focused on the critical properties of each quadrilateral and addressed how to treat these properties differently in their definitions.

Ahmet: It [parallelogram] is a quadrilateral whose diagonals bisect each other.

Researcher: How did you decide on this definition?

Ahmet: Side angle side relation. We said that diagonals bisect each other like this [Drawing two-line segments that bisect each other and marking them the same to show that they are equal in length in Fig. 6]. These angles are the same, vertical angles [Marking vertical angles the same].

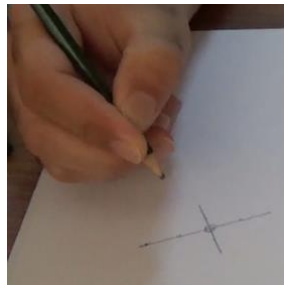


Figure 6. Ahmet's argument to justify the parallelogram definition

If we connect the points here [Referring to the vertices of the parallelogram], if we call this side to be b from an angle, this side will also be b [marking opposite sides of the parallelogram the same]. If we have two equal sides and one angle to be the same, so let's call this side t , this side will also be t [Marking the lengths of the opposite sides equal in Fig. 7].

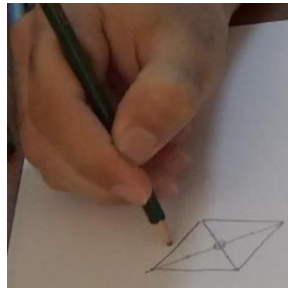


Figure 7. Ahmet's argument to justify that opposite sides are equal in a parallelogram

Ahmet generated definitions through logical deduction, as shown in the excerpt above. He knew that having diagonals that bisect each other would be enough to justify equal opposite sides. The PSTs who used the logical deduction strategy knew a specific quadrilateral could be defined by selecting one statement from a group of logically equivalent statements. Because any statement in the set of logically equivalent statements could serve as a valid definition for the quadrilateral, PSTs using the logical deduction strategy could easily construct multiple definitions. PSTs who used hierarchical relationship and logical deduction strategies correctly seemed to understand that each special quadrilateral could have more than one equivalent definition, and they generated multiple valid definitions for each quadrilateral. Next, we discuss these findings in light of the current literature.

Conclusion and Discussion

The results show that requiring prospective teachers to generate two different definitions for special quadrilaterals is a demanding task (see Table 3 and Table 4). When analyzing the results displayed in Tables 3 and 4, the participants, especially senior prospective teachers, were able to generate at least one appropriate definition for most of the quadrilaterals (except for an Isosceles Trapezoid). Thus, if participants were asked to generate one definition for each special quadrilateral, the results could have revealed a different picture. Yet, when asked to generate two different definitions for each quadrilateral, the definitions classified as “Necessary but not Sufficient” stood out. That is, the prospective teachers struggled to incorporate the critical properties of the defined quadrilaterals when generating various definitions of special quadrilaterals. Furthermore, among freshman prospective teachers, most generated definitions were categorized as “Neither Necessary nor Sufficient.” They either used imprecise terminology or incorporated properties that are neither necessary nor sufficient to define the quadrilaterals. To implement recommendations from current educational movements to communicate precisely and use clear definitions (CCSSM, 2010; NCTM, 2000) in mathematics classrooms at all levels, prospective teachers should know that every part of a definition matters. However, researchers have shown that learners struggle to recognize that all parts of a mathematical definition must be formulated succinctly and considered when generating examples/non-examples and/or arguments (Haj-Yahya, Hershkowitz & Dreyfus, 2023; Selden & Selden, 2008). The freshman prospective teachers in this study mostly generated definitions including terminology such as geometric structure, geometric shape, polygons or solids, which demonstrates the fact that they were unaware of the importance of each term used in a valid definition.

Researchers have attributed learners’ higher performance in generating definitions for some special quadrilaterals (i.e., parallelogram) to the linguistic structures of these quadrilaterals’ names (Avcu, 2023; Fujita & Jones, 2007). Similarly, both freshman and senior prospective teachers were more successful at generating definitions for parallelogram, rhombus, and rectangle in this study. Thus, the syntactic, semantic, and lexical structures of the Turkish names paralelkenar (parallelogram), eşkenar dörtgen (rhombus), and dikdörtgen (rectangle) may affect prospective teachers’ ability to generate appropriate definitions for these quadrilaterals. For instance, the Turkish name of a rhombus is “eşkenar dörtgen,” which is an endocentric compound, and the meaning of the quadrilateral could be construed by joining the adjective “eşkenar” (equilateral) to the noun “dörtgen” (quadrilateral). Similarly, because the Turkish name of a parallelogram (paralelkenar) is semantically and syntactically transparent, most participants defined it as having parallel (paralel) sides (kenar). Furthermore, participants commonly generated definitions such as “A quadrilateral

with right angles” or “A quadrilateral with 90-degree angles” for a rectangle. Given that the name of the quadrilateral (dikdörtgen (rectangle)) conveys the meaning of a perpendicular (dik) quadrilateral (dörtgen), such definitions should not be surprising.

Although the higher number of appropriate definitions of parallelogram, rectangle, and rhombus might have resulted from the fact that the names of these quadrilaterals are reminiscent of parallel sides, equal sides, or right angles as a critical property, the names of some special quadrilaterals might also serve as a barrier to generating correct definitions. For instance, most participants at all grade levels struggled to generate valid definitions of an isosceles trapezoid in this study. The name *ikizkenar yamuk* (isosceles trapezoid) could also function as an endocentric compound, and participants could infer the quadrilateral's meaning by combining the adjective *ikizkenar* (isosceles) with the noun *yamuk* (trapezoid). The transparency of the name of isosceles trapezoid could have explained the higher number of the generated definitions, such as “a trapezoid with a pair of opposite equal sides”, “a trapezoid with at least one pair of opposite sides equal in length” or “a trapezoid with equal side lengths” for an isosceles trapezoid. Because a parallelogram or rhombus could also fit these definitions, the quadrilateral's name served as a barrier to generating appropriate definitions. Researchers have also mentioned lexical ambiguity as a barrier to generating appropriate definitions for some special quadrilaterals, such as a trapezoid or a kite (Avcu, 2023; Türnüklü, Gündoğdu Alaylı & Akkaş, 2013). Thus, the results of this study, especially the number of appropriate/inappropriate definitions generated by the PSTs, could relate to the syntactic and semantic transparency and opacity of the names of these quadrilaterals.

In addition to the linguistic structures of the given names of special quadrilaterals and the semantic, syntactic, and lexical messages they conveyed to learners, understanding inclusion relationships between quadrilaterals also contributed to generating appropriate definitions for special quadrilaterals in this study. The PSTs who used the example strategy and the hierarchical relationship strategy were most influenced by their knowledge of inclusion relationships. For instance, while İlknur emphasized some properties to exclude the special cases of the defined quadrilateral during generating a definition as evident in her statement: “I wanted to exclude square from this definition, because I needed to write a definition for a rectangle”, Elvan, on the other hand, not only included special cases of a defined quadrilateral while defining but she also attempted to verify the generated definition for the special case as can be seen in her statement: “By changing the angles in this [the first example], for example, I can draw a shape like this [the second example], it [opposite sides] is still parallel, and its four sides are still equal”. Furthermore, the hierarchical relationship strategy for generating definitions of special quadrilaterals is built on the inclusion relationships between quadrilaterals. However, inclusion relationships contributed to PSTs who used this strategy correctly, while they became a barrier to generating appropriate definitions when PSTs used the strategy improperly. These findings revealed the complexity of understanding inclusion relationships between quadrilaterals. The literature has well documented the complexity of understanding inclusion relationships between quadrilaterals (Haj-Yahya, 2019; Fujita & Jones, 2007; Hershkowitz, 1990). Studies documenting learners’ difficulties generating valid definitions (e.g., Haj-Yahya, Mahameed & Haj-Yahya, 2024; Avcu, 2023; Fujita & Jones, 2007; Usiskin & Griffin, 2008) can be interpreted in light of the complexity of inclusion relationships.

Freudenthal (1973) distinguishes between two types of definitions: (1) constructive definitions (definitions of a concept constructed on an already familiar concept (i.e., a square is a rhombus with diagonals equal in length)), and (2) descriptive definitions (specifying the critical attributes of the defined concept (i.e., a square is a quadrilateral with all sides and

angles equal)). Because understanding a concept's constructive definition depends on understanding the appropriate inclusion relationship, PSTs mostly constructed these definitions using the hierarchical relationship strategy. By contrast, PSTs who used the logical deduction strategy focused on each concept's critical properties and questioned how to address them differently while generating equivalent definitions. Thus, the PSTs who employed the logical deduction strategy mostly generated descriptive definitions in this study. Beyond the types of definitions generated by PSTs using different strategies, PSTs also varied in how they verified whether the constructed definitions were valid. Haj-Yahya, Mahameed, and Haj-Yahya (2024) similarly conclude that using concept maps to teach inclusion relationships among special quadrilaterals affects students' ability to generate constructive definitions.

Harel (2008) argues that transformational reasoning involves goal-oriented operations on objects and anticipating the results of these operations. Among the PSTs who used the hierarchical relationship strategy, it was apparent that they performed goal-oriented mental operations on quadrilaterals and anticipated that these operations would yield the special cases of the quadrilaterals they were attempting to define. For instance, Nuri's statement of "I already know that a rhombus is a parallelogram. If I make all sides equal (pointing to the parallelogram), I will have a special parallelogram" demonstrates that he was performing mental operations on a parallelogram and anticipating that the results of this mental operation of making all its sides equal would result in having a rhombus. Harel (2008) identifies the essential characteristics of transformational reasoning as generality, operational thought, and logical inference (p. 491). He explains that operational thought manifests when an individual sets goals and tries to anticipate their outcomes, while logical inference manifests when an individual understands that mathematical justification must ultimately be based on logical inferences (Harel, 2008). The PSTs who employed the hierarchical relationship strategy demonstrated all these characteristics while generating definitions of special quadrilaterals. PSTs who used the logical deduction strategy also showed generality in the arguments they constructed to verify their definitions, as well as logical inference while generating definitions. Moreover, they tended to shape their reasoning based on foundational knowledge (i.e., theorems, definitions, etc.) while generating and/or evaluating definitions.

By contrast, PSTs who used the example strategy focused on the concept images of the quadrilateral being defined and the visual appearance of these examples while generating definitions. When these participants were asked to justify why these definitions hold true, they tended to form their justifications based on the constructed examples. For instance, as can be understood from Elvan's argument: "The definition here is also compatible with these examples I drew [pointing to the examples in Figure 2]. The opposite sides are parallel, and the sides are equal. It provides the first example. The second example is the same, so it provides the second example. I can't think of a counterexample right now. I think it is the same in all rhombi I can draw", she attempted to justify the generated definition by referencing the examples that she drew. Stylianides (2007) argues that the main difference between empirical arguments and deductive arguments lies in the mode of argument. That is, while empirical arguments provide insecure methods for validation, deductive arguments provide a secure method for validation that a mathematical statement holds true for all cases (Stylianides, 2007). PSTs who used the hierarchical relationship strategy and the logical deduction strategy justified their definitions by constructing a general argument, while PSTs who used the example strategy justified theirs through empirical reasoning. Thus, the main difference between PSTs who used the example strategy and those who used the hierarchical relationship and logical deduction strategies lies in their reasoning modes.

Future Directions

This study analyzed how prospective mathematics teachers, who enrolled in different years of a teacher education program at a public university in Turkey, generated definitions of special quadrilaterals. The results showed that PSTs at each grade level struggled to generate more than one definition of special quadrilaterals, highlighting the need for more experience with such activities in their mathematics and mathematics education courses (i.e., geometry, mathematics methods courses). Given the importance of constructing mathematical definitions in mathematics classrooms (CCSSO & NGA, 2010; NCTM, 2000), teacher educators should place greater emphasis on definition-generation processes in their instruction. Additionally, future research could explore which activities best foster prospective teachers' ability to evaluate mathematical definitions.

The results also showed that participants used various strategies to evaluate the presented definitions. Senior prospective teachers mostly used the hierarchical relationship strategy, while freshman prospective teachers used the example strategy. The study also documented the advantages and limitations of each strategy during the definition-generation process. Thus, future research could investigate which strategies different learner groups (i.e., middle/high school students or in-service teachers) could employ.

The results underlined the important role of language in construing meaning during the definition process. Language played a mediating role in generating appropriate definitions for some quadrilaterals (i.e., parallelogram or rhombus), while it played an obstacle role in generating appropriate definitions for others (i.e., isosceles trapezoids or kite). While focusing on the Turkish context, future research could explore how these findings might apply to other languages with similar compound mathematical terms.

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