

Secondary Preservice Teachers' Engagement in a Quadratic Functions Task: Reasoning and Content

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In this manuscript, we present an exploratory case study investigating one group of four preservice secondary mathematics teachers as they used GeoGebra to explore and describe how varying a in $y = ax^2 + bx + c$ affects the position of the vertex of the associated parabola. We examined their reasoning and identified four main reasoning sequences that helped them progress towards their final conjecture. Throughout these sequences, they established (1) the path is linear, (2) the slope depends on b , (3) the slope is not dependent on a and not dependent on c , and (4) the equation that describes the path of the vertex is $y = \frac{b}{2}x + c$. We further described and classified preservice teachers' reasoning using Lithner's (2008) framework for creative and imitative mathematical reasoning. We identified which mathematical concepts anchored the preservice teachers' arguments and were deepened through their reasoning. Our results indicate that the preservice teachers engaged in creative mathematical reasoning during each reasoning sequence. Finally, our analysis suggests that the participating preservice teachers appeared to deepen their knowledge of the concept of dependence, particularly in the context of linear functions.

Introduction

Many mathematics education professional associations and government education agencies have described reasoning and sense making as crucial components of PreK–12 mathematics learning (Australian Curriculum, Assessment and Reporting Authority, 2010; National Centre for Excellence in the Teaching of Mathematics, 2024; National Curriculum in England, 2021; National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics [NCTM], 2000, 2009, 2014, 2018). These reports echo research that illustrates connections between reasoning and learning (Jonsson et al., 2020; Hiebert & Wearne, 1993; Hiebert & Grouws, 2007; Lithner, 2008, 2017; Stein & Lane, 1996). Reasoning that leads to learning is driven by tasks that elicit struggle (Lithner, 2008, 2017; Thompson, 1996; Yackel & Hanna, 2003). Thus, we are interested in tasks that elicit struggle to engage students in reasoning that leads to learning. Likewise, we are interested in the quality of reasoning because it can affect the quality of the learning and learning outcomes.

Lithner (2008) introduced a framework that enables researchers to distinguish between reasoning that involves less or more sense-making (imitative reasoning, IR; or creative, mathematically founded reasoning, CMR, respectively). CMR is reasoning that occurs during productive struggle while solving tasks (Lithner, 2008). Teaching that emphasizes CMR results in better learning outcomes (Jonsson et al., 2020; Jonsson et al., 2022; Jonsson et al., 2014). Unfortunately, this type of reasoning is rare in school mathematics (Lithner, 2004).

Several researchers (e.g., Hollebrands, 2007; Sacristán et al., 2009) have suggested that technology-based activities may afford students opportunities to engage in reasoning and focus on mathematical relationships. Mathematical action technologies (MAT), that is,

“those that can perform mathematical tasks and/or respond to the user’s actions in mathematically defined ways” (Dick & Hollebrands, 2011, p. xii) and their mathematical action technology uses (MATUs; Cullen & Hertel, 2023) can support students’ reasoning and sense making. Feedback from MATs, such as GeoGebra, can contribute to CMR (Granberg & Olsson, 2015). For this reason, MAT-based tasks that elicit CMR can enhance the learning experiences of both school students and preservice secondary teachers (PSTs). PSTs, specifically, must experience rich tasks from a learner’s perspective, so they can envision and later implement comparable tasks in their classrooms (Conference Board of the Mathematical Sciences, 2012; Loong et al., 2017; Stylianides & Stylianides, 2006).

NCTM (2000) described one such task (hereafter “the quadratics task”) and argued that students could “explore the effects of changes in parameter as a means of better understanding classes of functions. For example, explorations with functions of the form $y = ax^2 + bx + c$ led to some interesting results” (p. 299). NCTM claimed that the effects of parameters a and c on the graph are relatively easy to observe. Authors of practitioner-focused articles on the affordances of technology (Edwards & Özgün-Koca, 2009; Fallon & Luck, 2010) have examined students’ discovery of the locus of the path traced by the vertex of the general quadratic, but only for varying b . Other authors described the equations of the paths traced by the vertex as each parameter varies (Cullen et al., 2020; Norman, 1988) or extended the task to higher-degree polynomials (Cullen et al., 2013; Norman, 1988).

Based on their use of the quadratics task, Cullen et al. (2020) suggested that PSTs are likely to progress through a sequence of increasingly accurate conjectures about the path of the vertex as a is varied (i.e., it will not move, it moves along a linear path, it moves along the linear path $y = \frac{b}{2}x + c$). Ultimately, Cullen et al. (2020) recommended this task because it engages PSTs in reasoning (NCTM, 2000), connects to content in PSTs’ future teaching assignments (parametric equations), and can highlight two MATUs (i.e., promoting cycles of proof and presenting multiple representations). These claims make this task a compelling candidate for further investigation, especially given that Cullen et al. (2020) focused on presenting a framework rather than describing the task’s implementation. Thus, describing PSTs’ reasoning as they generate increasingly accurate conjectures about the path and documenting the mathematics learned could determine the aptness of the quadratic task for PSTs.

Moreover, the task is compelling because quadratic functions are central to secondary mathematics curricula, and researchers have shown that students struggle to understand how parameters affect them. For example, Chazan (2006) and Ellis and Grinstead (2008) found that PSTs and secondary students, respectively, harbor misconceptions, with many assuming that a indicates the slope of the parabola. Ellis and Grinstead reported that students who used graphing calculators to explore the quadratics task were unable to identify the effect of each parameter on the vertex or any potential interactions among the parameters. Specifically, they wrote, “Although the students indicated that a influenced the shape of the parabola, they did not demonstrate a belief that changing the value of a would change the location of its vertex” (p. 286). They concluded that classroom factors led students to overgeneralize the role of linear equation parameters in linear equations. Mutambara et al. (2020) confirmed this result by having PSTs explore the quadratics task in a non-technological context and found that few PSTs were able to describe the effects of parameters on the vertex.

For these reasons, the quadratics task in an MAT context might elicit CMR, potentially leading to a deeper understanding of the role of parameters in a general quadratic function or related mathematical ideas. In other words, we explore whether the quadratics task addresses the first purpose of what Stylianides and Stylianides (2006) called a teaching-

related mathematics task (i.e., a task that can foster learning that is important for mathematics teaching).

In this paper, we focus on the work of one four-person group within a class of PSTs investigating the quadratics task using GeoGebra. Two research questions guided our investigation.

As they explore the quadratics task:

1. How can PSTs' reasoning be classified as they focus on the effect of the quadratic term parameter?
2. What do PSTs learn about the mathematics that anchors their reasoning?

By focusing on PSTs' reasoning and mathematical engagement in a GeoGebra-based exploration of the quadratics task, we aim to provide stronger evidence for teacher educators to determine whether the quadratics task, recommended by several authors, fulfills its promise for PSTs. As Granberg and Olsson (2015) recommended, our findings may be especially valuable because we document the outcomes of an extended task conducted in an everyday classroom.

Theoretical Perspectives

In their development of a conceptual model of mathematical reasoning, Jeannotte and Kieran (2017) noted that there is a “conceptual blur” (p. 6) of descriptions of reasoning in the literature. To clarify this blur, Jeannotte and Kieran synthesized research and classified it into two main groups: (a) research focused on structural aspects of arguments (e.g., claim, data, warrant) or steps that describe the transitions between structural components (e.g., inductive step, deductive step) and (b) research focused on process aspects (e.g., identifying a pattern, generalizing, validating, proving). Within this model, Jeannotte and Kieran identified Lithner's (2008) conception of reasoning as focused on the production of new knowledge. Thus, Lithner's framework provides the more specific lens we need to assess the quality of learning-related reasoning.

Imitative and Creative Reasoning

Lithner's (2008) framework was designed as a tool for basic research, specifically for task solving, and to support questions related to mathematics teaching. Lithner (2008) used the framework for classifying reasoning by contrasting a student's different solution processes. Later, Lithner (2015) used his framework to determine how teaching emphases relate to learning outcomes and to describe how tasks should be structured to improve learning (Lithner, 2017). Because of its dual purpose, Lithner's framework supports our goals of assessing the quality of reasoning elicited by a task whose potential has not been closely examined and of assessing the task's targeted potential for an audience of PSTs.

Lithner (2008) defined reasoning as “the line of thought adopted to produce assertions and reach conclusions in task solving” (p. 257). Lithner (2008) clarified that a line of thought need not be formal or even correct; however, the reasoner must perceive that they have some sensible reasons supporting their approach. For Lithner (2008), the nature of the backing helps classify the type of reasoning as IR or CMR as students engage in task solving. According to Lithner (2008), solving a task is a four-step process:

1. A (sub) task is met, which is denoted a problematic situation if it is not obvious how to proceed.
2. A strategy choice is made, where ‘strategy’ ranges from local procedures to general approaches and ‘choice’ is seen in a wide sense (choose, recall, construct, discover,

- guess, etc.). It can be supported by predictive argumentation: Why will the strategy solve the task?
3. The strategy is implemented, which can be supported by verificative argumentation: Why did the strategy solve the task?
 4. A conclusion is obtained. (p. 257)

Further, Lithner (2008) noted that task solving involves changes to students' knowledge states during the solution process, always motivated by reasons, even when those reasons are unclear or poorly supported (Lithner, 2008). The nature of the strategy choice and the associated predictive and verificative arguments determine the nature of the reasoning (Lithner, 2000), whether it is IR or CMR (Lithner, 2008). IR can take various forms (e.g., reproducing a complete solution from memory or identifying and implementing a learned algorithm [AR]). If a student is not engaged in IR—either memorized or AR—then they are engaged in CMR.

By looking at distinctions between IR and non-IR student task solutions, Lithner (2008, 2015) identified three criteria that characterize CMR. Those criteria were later phrased as:

- a) Creativity: The learner creates a reasoning sequence not experienced previously or re-creates a forgotten one.
- b) Plausibility: There are predictive arguments supporting the strategy choice and arguments for verification, explaining why the implementations and conclusions are true or plausible.
- c) Anchoring: The arguments are anchored in the intrinsic mathematical properties of the components of the reasoning. (Lithner, 2017, pp. 939–940)

Creative Feedback as a Driver of Reasoning

Because MATs react to input in mathematically defined ways (Dick & Hollebrands, 2011), learners must interpret the feedback that the tool provides in response to their actions. Dynamic geometry environments (DGEs) such as GeoGebra—a type of MAT—can support engagement in cycles of reasoning, one use of MATs (Cullen & Hertel, 2023; Cullen et al., 2020). As Laborde (2007) wrote with respect to MATs, in general, and DGEs, specifically, the “interplay of conjectures and checks, of certainty and uncertainty, [is] made possible by the exploratory power and checking facilities offered by the environment” (p. 83). When students work on a task within a DGE, the need for validation often arises from observing unexpected properties. Thus, proof or validation explains phenomena observed on the screen (Laborde et al., 2006).

While discussing similar ideas regarding GeoGebra, Granberg and Olsson (2015) observed, “unlike a textbook, GeoGebra did not provide the correct answers, and, unlike guidance from a teacher, GeoGebra did not offer clues on how to proceed” (p. 60). In this way, GeoGebra can contribute to students' engagement in CMR by providing data and supporting validation processes, including during collaborative problem-solving (Granberg & Olsson, 2015). Granberg and Olsson described this output as *creative feedback* because it provides users with “a visualization of their created or implemented solving strategies without presenting correct answers or further guidance” (p. 60). Thus, as in collaborative work, GeoGebra provides the feedback needed in an adidactical situation (Brousseau, 1997) because students must determine the adequacy of their solution. Like Granberg and Olsson, we note that Lithner's framework supports investigating reasoning, particularly CMR, in a GeoGebra environment.

Methods

Setting and Participants

We conducted an exploratory case study (Merriam, 1998) focused on one group of PSTs (the case) in their penultimate semester, having completed many collegiate mathematics courses, including at least three semesters of calculus, linear algebra, discrete mathematics, statistics, Euclidean and non-Euclidean geometry, and abstract algebra. The PSTs were concurrently enrolled in a pedagogy course and a capstone course connecting collegiate and high school mathematics. The study took place in an in-person course at a medium-sized university in the Midwestern United States. The course focused on problem-solving through tasks that highlight the role of technology in mathematics teaching and learning (Cullen & Hertel, 2023; Cullen et al., 2020). Throughout the course, the PSTs used a variety of MATs (Dick & Hollebrands, 2011), including GeoGebra. The class met once a week for 2 hours and 50 minutes.

The Task and Procedure

As a warmup, the instructor asked PSTs to imagine the effects of varying a , b , and c (one at a time) on the graph of $y = ax^2 + bx + c$. They were instructed to work independently, then in groups, without the use of technology. They made two correct claims: (a) varying a makes the parabola open up or down and appear wider or narrower, and (b) varying c shifts the graph up or down. Their initial claim for b : Varying b shifts the graph left or right, was not correct. Using this non-technology warm-up enabled us to observe that PSTs' initial claims focused on the graph holistically rather than on the parameter's effects on the vertex, consistent with prior research (Ellis & Grinstead, 2008; Mutambara et al., 2020).

PSTs were then directed to discuss the effects of varying a , b , or c on the vertex of $y = ax^2 + bx + c$ (i.e., the quadratics task; Cullen et al., 2020). The instructor demonstrated how to enter the equation into GeoGebra (version 5), which generated sliders for each parameter. Then, PSTs were encouraged to work in groups to develop a conjecture for the specific path followed by the vertex as each parameter varied individually. A file that emulates the task setup is available [here](#). For the reader's convenience, we included checkboxes to hide or show the vertex and to enable tracing, as well as a button to erase traces. PSTs did not have a file with these checkboxes. After graphing the function, which generated sliders for the parameters, they had to use native GeoGebra features to add the vertex and enable the trace to generate data as the parameters varied.

The quadratics task was implemented during two class meetings early in the semester. All data in this study were obtained from the first class (2 hours and 10 minutes of video) and the first 30 minutes of the second class. PSTs were required to submit initial conjectures by the end of the first class. After the second session, each PST group submitted proofs for their final conjectures.

Data Collection and the Case

After an initial review of the data collected for this study, we used a post-hoc purposeful sampling approach, which resulted in the selection of the case (focus group). The focus group consisted of four PSTs: Joelle, Marlina, Nancy, and Jared, all pseudonyms¹. This group was

¹ While conducting this research, we complied with all ethical standards required by the Institutional Review Board at which the research was conducted, including obtaining informed written consent, and securing the

selected for deeper analysis because it proved to be an information-rich case. Their dialogue, especially that between Jared and Joelle, contained explicit arguments that allowed us to access their reasoning. Although each PST in the group had a laptop computer, they were encouraged to work together, with two PSTs per laptop: Joelle with Jared and Marlina with Nancy. At times, however, group members worked individually or all together. Periodically, the instructor invited groups to share ideas or approaches with the class.

Three stationary cameras captured full-class interactions, with each camera focused on a single group. Cameras were placed behind PSTs to capture audio and screen displays, including dragging, GeoGebra feedback, and occasional handwritten scratchwork. The camera for our case was between Jared and Joelle.

Data Selection and Analysis

Of the three parameters, the reasoning used to generate the conjecture for varying a was the most complex. For this reason, and because of the previously documented misconceptions about a (Ellis & Grinstead, 2008; Mutambara et al., 2020), we examined the PSTs' reasoning regarding the vertex path resulting from varying a .

Lithner's (2008) framework is used to determine if the reasoning sequence that students produce while engaged in task (or subtask) solving is IR or CMR. With this in mind, we identified the subtasks and associated reasoning sequences that participants generated while engaged in the quadratics task. In this way, these reasoning sequences became the units of analysis for our study. We consider the steps of each reasoning sequence a unit of observation (Morin et al., 2021).

The first two authors iteratively viewed video recordings to identify subtasks of the quadratics task. The subtasks were not necessarily self-evident from the task itself, because multiple solution paths were possible. Disagreements were handled as part and parcel of the collaborative, iterative process of viewing, summarizing, mapping to reasoning steps, and identifying the type of reasoning inherent in the verbalizations associated with each reasoning sequence. The third author served as a reliability check, viewing videos, reading the analysis, and sharing disagreements. Any disagreements were resolved by reviewing the videos and conducting a written analysis by all three authors until consensus was reached.

The PSTs had completed a subtask when they made a verified (to them) conjecture that represented a significant advance in their work toward generating an explicit description of the vertex path obtained by varying the value of the parameter a .

The reasoning during a subtask constituted a reasoning sequence (Lithner, 2008), and we identified the steps of each sequence through multiple rounds of video viewing as described above. In a step, PSTs had a clear focus on the action they took. That is, PSTs either generated or discussed evidence during their exploration. Their discussions involved several types of key verbalizations, including making *observations* about evidence on the screen or on paper, sharing their *wonderings* in which they asked why a strategy they were using would (predictive argument) or did (verificative argument) support their conjecture, and making *conjectures* about the evidence. We distinguished observations from conjectures by determining whether PSTs attempted to verify the statement during the reasoning sequence. If they engaged in verification attempts, we characterized the statement as a conjecture. When evident, we also noted changes in their knowledge state (Lithner, 2008). We identified knowledge states at moments when PSTs explored one of their own key

data on password-protected devices approved for this purpose. To provide anonymity to the extent possible, we use pseudonyms throughout the manuscript.

verbalizations. We reviewed and verified our step selection by summarizing actions, feedback, dialogue, knowledge states, and screen images to ensure that, taken together, they formed a coherent unit.

To address the first research question, we examined the quality of their reasoning about CMR. To do so, we considered whether the strategies for verifying or refuting conjectures and the associated predictive or verificative arguments were consistent with IR or CMR (e.g., the nature of the strategy), using Lithner's (2008) framework to justify our classification.

For the second research question, we identified mathematical properties that anchored PSTs' arguments across all reasoning sequences, as well as the developing mathematics ideas embedded in the successive knowledge states within each reasoning sequence. Each step in task solving provided an opportunity for their understanding of properties to evolve. Likewise, we described how these properties were evident in PSTs' final proof product. We considered only properties at or above the level of introductory algebra. Based on our results, we narrowed our focus to one central concept.

Results

Creative Mathematical Reasoning in the Reasoning Sequences

Based on the PSTs' progress through the task, we identified four subtasks, each corresponding to a reasoning sequence (denoted RS1, RS2, RS3, and RS4, respectively) related to the path traversed by the vertex as a is varied:

RS1: Path is linear, but not $y = bx + c$.

RS2: Slope of the path is dependent on b .

RS3: Slope of the path is not dependent on a and is not dependent on c .

RS4: Path of the vertex is $y = \frac{b}{2}x + c$.

Each reasoning sequence section of the results includes: (a) an overview of the sequence, (b) supportive evidence such as dialogue and screen images, and (c) a discussion of whether and how the reasoning sequence constitutes CMR. After presenting evidence from each connected reasoning sequence, we summarize them in a table that describes key aspects of the steps that led to the main conclusion.

Although we parsed the reasoning sequences in steps (each denoted by a hyphenated number, such as RS1-1), these steps did not constitute a linear progression for the PSTs. For example, PSTs verbalized an observation or a question in one reasoning sequence that they were unable to interpret until a later sequence. In other instances, PSTs proposed conjectures that contradicted those that they had previously supported. In this way, the task in this study led to a progression of connected reasoning sequences that elicited students' struggles with the underlying concepts and, eventually, sense-making in relation to the larger task.

Reasoning Sequence 1: Path is Linear, but not $y = bx + c$.

Within the two steps of RS1, the PSTs made one observation (O1) and one conjecture (C1). The correct observation was immediately followed by an incorrect conjecture that was ultimately rejected. At the end of this reasoning sequence, the PSTs had ascertained that the path was linear but did not match their initial guess.

RS1-1: Joelle dragged the a slider. GeoGebra feedback indicated that the vertex traces were linearly aligned. Joelle claimed that dragging the a slider results in a linear path (O1; Figure 1). Jared quickly claimed that the line is $y = bx + c$ (C1).

PST dialog	Screen Image
Joelle: It's a line, when you move a, it traces a line. Jared: Oh, does it, it traces the line when...$[y =]bx + c$? Is that it?	

Figure 1. Joelle's Claim of Linearity

RS1-2: Joelle crafted a strategy for testing Jared's guess. They used that strategy to determine that the path traces they had generated did not match $y = bx + c$ and dismissed C1 (Figure 2).

PST dialog	Screen Image
Joelle moved the slider for a to 0 so that she and Jared could compare the line $y = bx + c$ to the traces generated in RS1-1. Jared: No, I'm wrong.	

Figure 2. Jared's Suggestion, Then Rejection of a Path Equation Claim

In summary, the PSTs discovered the path's linearity by dragging the a slider and observing the traces of the vertex. After making a seemingly arbitrary guess that the equation of the path was $y = bx + c$, they implemented a strategy for testing their conjecture. The plan they enacted (supported by an implicit predictive argument) involved creating a representation of $y = bx + c$ by dragging the a slider to zero so that the parabola was transformed into the line. When the traces forming the linear path did not match the

representation of $y = bx + c$, the PSTs interpreted the feedback as a contradiction to their conjecture. They concluded that the conjecture was incorrect (verificative argument).

Although one might claim that PSTs' initial guess about the path equation was not part of any reasoning process (IR or CMR) that Lithner (2008) described, the process that PSTs used to investigate and reject their arbitrary guess did require the use of a plausible strategy with an associated implicit predictive argument. If the representation of $y = bx + c$ could be compared to the traces, there would be visual evidence (not proof) of whether the traces were likely coincident with that linear equation. This strategy of comparing a known line to a line formed by the traces of the vertex was mathematically founded. When the paths did not match, the PSTs reported that they "were wrong" (implicit verificative argument). This dismissal of a conjecture based on a counterexample was also anchored in intrinsic mathematical properties of invalid claims (they do not match the evidence). Because the reasoning in RS1 was creative and the strategy was plausible and anchored in mathematics, we classify this reasoning as CMR.

Reasoning Sequence 2: Path Slope is Dependent on b .

Within the five steps of RS2, the PSTs made one observation (O2), two wonderings (W1, W2), and one conjecture (C2). PSTs were able to use an intentional strategy to generate evidence for the conjecture.

RS2-1: The PSTs had varied b and c and observed the feedback. This process left traces that could be modeled by $y = -ax^2 + c$ (parabola opening down) and $x = \frac{-b}{2a}$ (vertical line). Without erasing these traces, Jared began to drag the slider for a . Joelle interrupted to ask about c again, so Jared stopped dragging the slider for a and started dragging the slider for c . Then Jared dragged the a slider again. This left traces on the screen that appeared to form a pair of parallel lines connected by a vertical transversal. Jared wondered aloud (W1) about what might be causing GeoGebra's feedback (Figure 3).

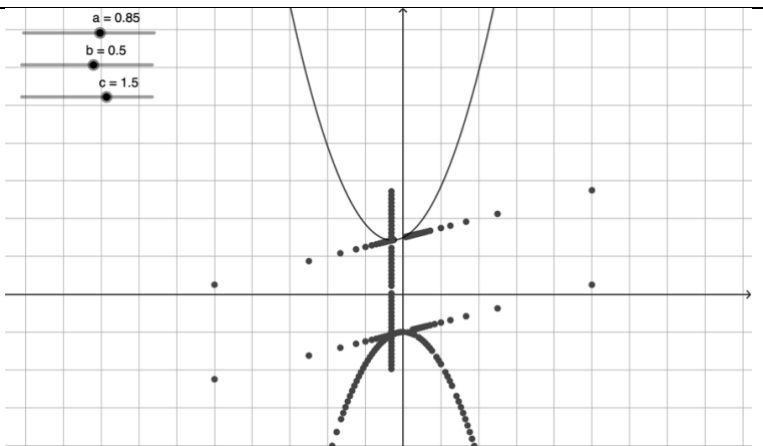
PST dialog	Screen Image
Jared: I am just wondering why is that slope...like that?	

Figure 3. Jared's Observation of Parallel Traces From Varying c and Leaving b Constant

RS2-2: Jared used the line tool to superimpose a line over the traces from the topmost parallel line (Figure 3). Jared erased the previous traces, leaving the constructed line. Jared dragged the b slider and then the a slider. This left the appearance of two intersecting lines (Figure 4). He noted that the path had changed (C3). The

GeoGebra feedback illustrated the slope's dependence on b , but Jared did not verbalize this yet.

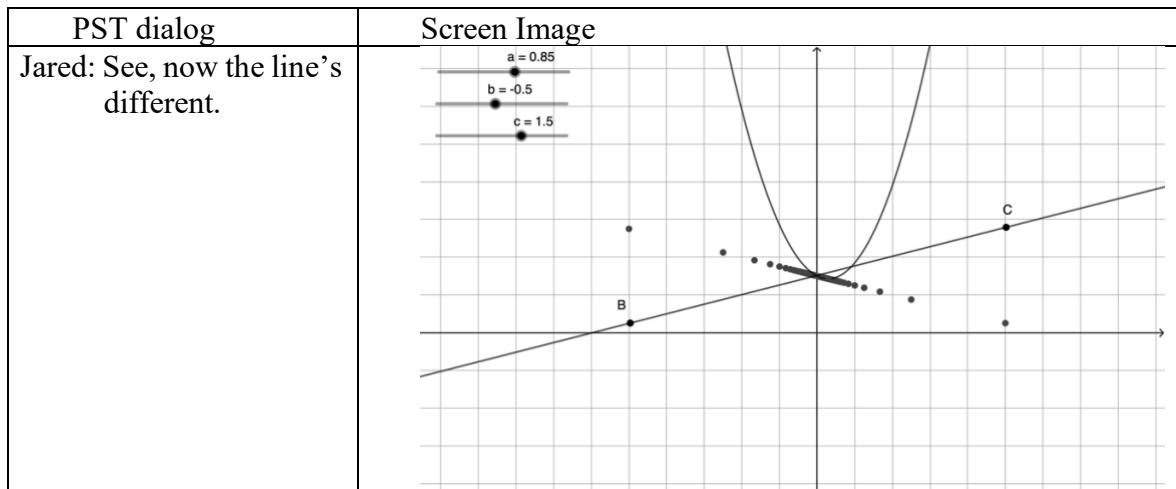


Figure 4. Jared's Observation of a Changed Linear Path After Changing the Value of b

RS2-3: Jared dragged the c slider, then dragged the a slider again. GeoGebra's feedback showed two sets of parallel traces from dragging the a slider and one set of vertical traces from dragging the c slider. As shown in Figure 3, Figure 5 depicts these traces as forming a pair of parallel lines connected by a vertical transversal. Again, Jared wondered why this pattern was forming (W2).

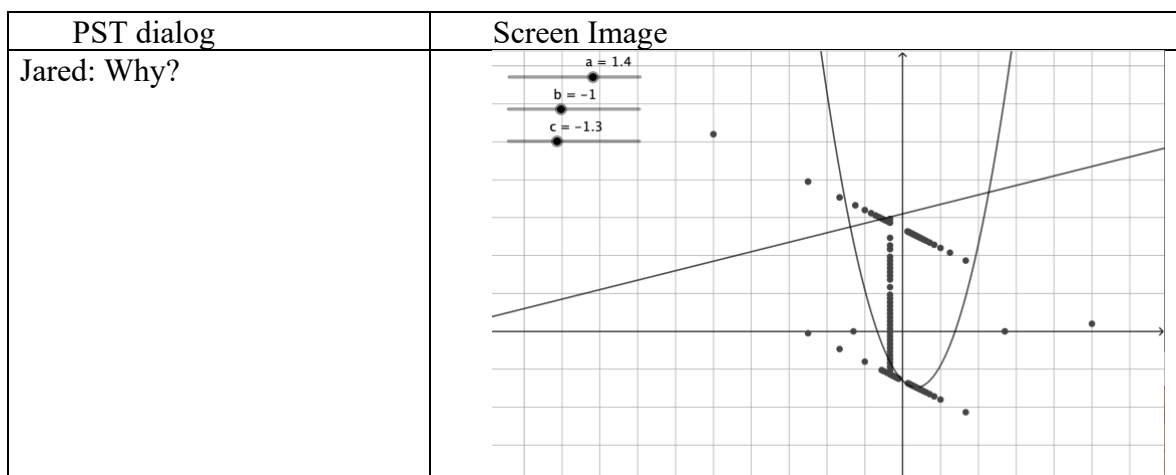


Figure 5. Jared's Observation of Parallel Traces from Varying c and Leaving b Constant Again

RS2-4: Jared seemed to enact an intentional plan (strategy supported by a predictive argument) to vary sliders to illustrate the effect of b . Jared varied b while articulating his plan to change location, then varied a again. GeoGebra produced a path of traces with a different slope than the previous path (Figure 6). This appeared to be a eureka moment for Jared as he declared that the slope depends on b (C2).

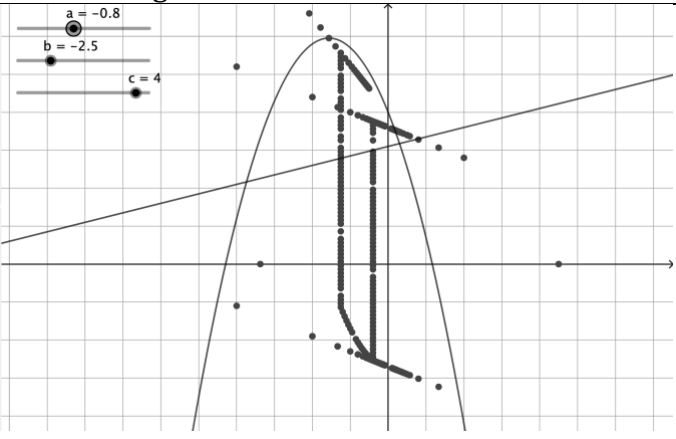
PST dialog	Screen Image
Jared dragged the b slider. Jared: Ok, and then, let's go ahead and go back up. [Dragging the slider for c to shift the parabola up, then dragged the a slider again.] Jared: Oh! Ok, so the slope of the line that it traces is dependent on b; [The slope depends on] where b's at.	

Figure 6. Jared's Evidence and Claim That Slope Depends on b

RS2-5: Jared erased the traces and implemented a strategy of repeating the sequence of varying a , varying b , then varying a (strategy supportable by a predictive argument). This demonstration was meant to convince (verificative argument) Joelle of C2 (Figure 7). Joelle immediately pressed for more specificity about the equation.

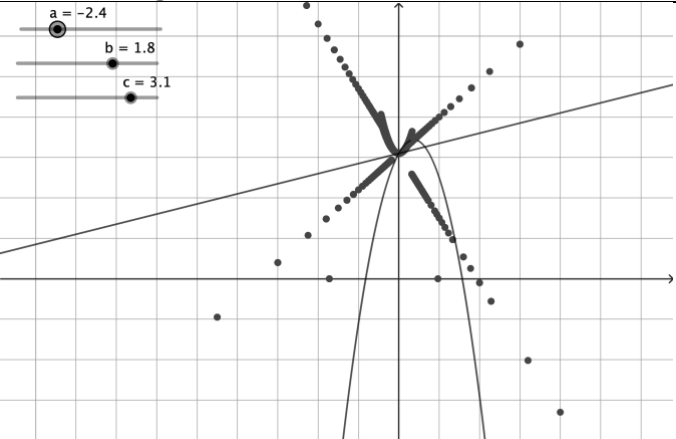
PST dialog	Screen Image
Jared: So here. [Erases prior traces.] So we'll do one here. Jared: So you get that line, right? [Gesturing to the line with a negative slope.] Joelle: Right. Jared: Now if we go...over here [adjusting the b slider to the right.] and we get [varying the a slider to produce the linear path with the positive slope] look at that. Joelle: Oh. What is that line though?	

Figure 7. Jared's Demonstration to Joelle That Slope Depends on b

In summary, the PSTs appeared first to determine the slope's dependence on b by unsystematically changing parameter values (slider dragging), and then by systematically varying one parameter while holding the other parameters constant. They appeared motivated to use a systematic approach to parameter variation (slider dragging) when they faced evidence that the path's slope was invariant under certain conditions. However, they could not explain the causes of GeoGebra's feedback. By systematically varying the parameters, the PSTs isolated the effect of parameter b ; thus, their strategy was supported by predictive and verificative arguments. In addition to the evidence for the effect of b , the GeoGebra feedback could have been interpreted as indicating that the slope does not depend on c . However, PSTs did not appear to recognize this evidence.

PSTs' sequence of steps was not consistent with the IR that Lithner (2008) described. Jared did not recall a fully formed solution (memorized reasoning) or employ an AR strategy. Thus, RS2 appeared to be creative for Jared because he expressed surprise with his observations ("See, now the line is different") and asked himself "why" that was happening. The strategy was plausible because GeoGebra's feedback provided implicit predictive and verificative arguments to support adjusting parameter sliders one at a time to reveal evidence of dependence. Jared predicted the outcome when he replicated the process for Joelle. This action also constituted a verificative argument. Finally, the arguments were anchored in the intrinsic mathematical property of dependence. Because the reasoning in RS2 was creative and the strategy was plausible and anchored in mathematics, we classify this reasoning as CMR.

Reasoning Sequence 3: The Slope is Not Dependent on a and Not Dependent on c

During this reasoning sequence, PSTs clarified to themselves that b was the only parameter influencing the slope. This also provided an opportunity for the PSTs to repeat the strategy they had developed in the previous reasoning sequence by applying it to investigate the effects of other parameters. Within the five steps of RS3, the PSTs made two conjectures (C3, C4) and one wondering (W3).

RS3-1: With the vertex plotted and the trace feature on, Jared varied slider a (while keeping other parameter sliders constant) to create a linear collection of traced points (Figure 8a).

RS3-2: Jared varied b through its entire range $[-5,5]$, creating a parabolic path of traces, then varied c along the entire range $[-5,5]$, leaving b at -5 and c at 5 (Figure 8b).

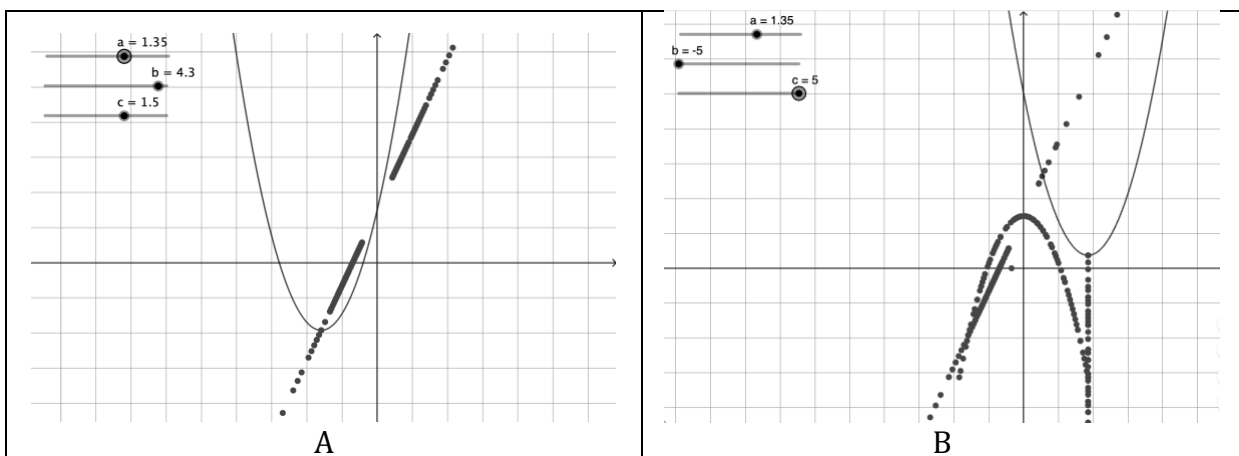


Figure 8. Jared's Investigation of Whether Slope Depends on a

RS3-3: Jared erased the previously generated traces, varied a again, while considering whether, and eventually concluding that, slope does not depend on a (C3). Marlina challenged the conjecture by asking “why not” (W3; Figure 9).

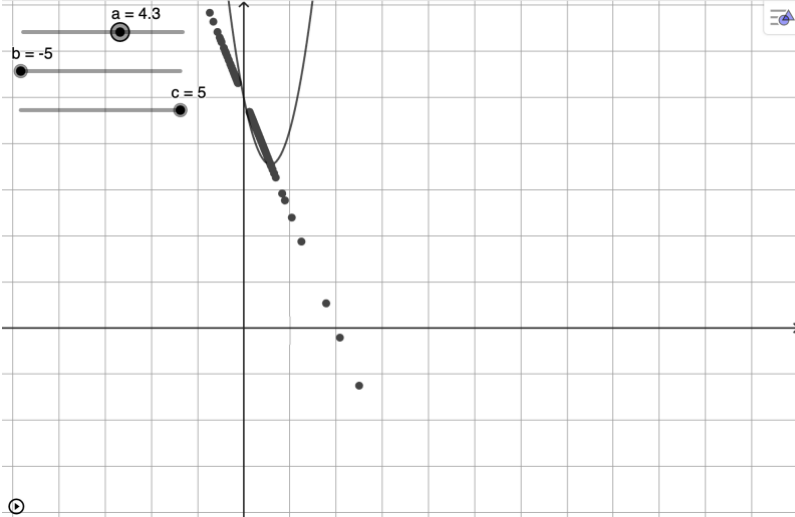
PST Dialog	Screen Image
Jared: [After dragging the slider for a, then for b, then c, then a again]. Okay, the slope does not depend on a. Well... it only depends on a... oh wait maybe a is in the numerator... because... wait... I guess the slope does not depend on a. Marlina: Why not? Jared: Because— Marlina: The slope of this line?!	

Figure 9. Jared and Marlina’s Discussion of Whether the Slope Depends on a

RS3-4: Jared interpreted the feedback on the screen with respect to parameter effect (verificative argument). Although varying a changed the location of the vertex, the vertex’s position was always constrained to the same linear path. Joelle emphasized that the slope (of that linear path) is invariant, verifying C3.

Jared: [Jared moved the slider for a and explained] $a \dots a$ is -20 , and it’s the same line, and now a is 20 , and it’s the same line.

Marlina: I am confused because this whole line that we traced exists only because of a . Like if you...

Joelle: Yeah, but the slope exists because of a , but it’s not cha... if a is big, the slope of the line isn’t changing—

Marlina: Oh, okay

Joelle: It’s still the same. So, the slope isn’t dependent—

Marlina: (interrupting) Ohhhh yeah!

Joelle: —on a .

RS3-5: Joelle appeared to have varied the sliders on the screen, but the PSTs blocked the screen. As they observed Joelle varying sliders, Marlina and Joelle commented as they tried to interpret GeoGebra’s feedback. Jared watched the screen intently but did not say anything. Then, Jared pointed to the screen and repeated that slope depends on b , and stated that slope does not depend on c (C4) as an

interpretation of the meaning of the parallel lines resulting from a strategy of varying a , varying c , then varying a again (verificative argument).

Marlina: When a is positive, it's going...down...it's going to negative infinity. But when a is negative, it's going to positive infinity.

Joelle: Is this th...is this the line of like...solutions...? No.

Jared: Okay, the slope of the line is dependent on b , and then...but it's not dependent on c . Because if you shift c , it's [the slope] the same...it's [the linear path is] parallel, it's parallel. So, it's b , ultimately.

Jared's justification that the vertex remains on the same line despite a being -20 or 20 was not initially clear to him. It seemed likely that he, like Marlina, struggled to differentiate the idea that the individual values of the x - and y -coordinates of the vertex are dependent on a (because both x and y are defined in terms of a) from the notion that the slope in the linear relationship between x and y is independent of a .

In summary, the PSTs—led by Jared—made progress in this reasoning sequence by further developing their strategy for eliciting evidence to determine which parameters affect the slope of the linear path. Not only did the PSTs refine their strategy of holding some parameters constant and systematically varying others, but they confirmed (to themselves) that varying a leads the vertex to traverse a linear path whose slope depends only on b , and not on a or c . Although this sequence began with Jared varying only one parameter at a time over its visible range (based on slider settings), the PSTs used less unstructured parameter variation in solving this subtask. Instead of simply noticing variant or invariant slopes and seeking an explanation for their observations, as they initially did in RS2, Jared intentionally applied the same strategy to investigate the effects of parameters a and c . As in RS2, this strategy for obtaining evidence for dependence is supportable by (implicit or explicit) predictive or verificative arguments.

Although Jared appeared to have identified a strategy for determining the effects of parameters on slope, the group's collective struggle to clarify and interpret the results of this strategy indicated that the PSTs were not using IR; instead, their reasoning indicated CMR. Jared's strategy was plausible because GeoGebra's feedback provided implicit arguments supporting his strategy for determining dependence by considering whether the slope was invariant for distinct values of a . Joelle provided a verificative argument to convince Marlina. Similarly, Jared interpreted the meaning of the parallel traces for different values of c to verify that the slope is independent of c . Their arguments were anchored in the mathematical concept of dependence and its relation to variant or invariant properties. Because the reasoning was creative, plausible, and anchored in mathematics, the PSTs' reasoning in RS3 constituted CMR.

Reasoning Sequence 4: Path of the Vertex is $y = \frac{b}{2}x + c$

This reasoning sequence occurred at the beginning of class 2. At the end of class 1, the PSTs recorded the path traversed by the vertex when varying a : $y = \square x + \square$, with $\square$ as an empty box. At some point, the slope box was replaced by kb . Thus, they began RS4 with the representation $y = kbx + \square$. Despite earlier discussions, the PSTs had to convince themselves of this relationship again, then determine a value for k . In so doing, PSTs used an algebraic strategy and repeated a verification strategy from RS1. In RS4, the PSTs made two observations (O3, O4), two wonderings (W4, W5), and two conjectures (C5, C6).

RS4-1: Because the slope depends on b , $y = (kb)x + \square$, with k as a constant. [Referred to the claim recorded on paper, which was an algebraic representation of the slope that depends on b , their ending point from the previous day.] While adjusting the sliders, Jared and Joelle began discussing the feedback they observed when the b slider was set to 0 (Figure 10). They noted that “it [the path equation] would just be $y = c$ ” when b is 0 (O3), but after commenting that there is no line when b is 0, they moved on without clarifying whether the y -intercept is always c in the path equation.

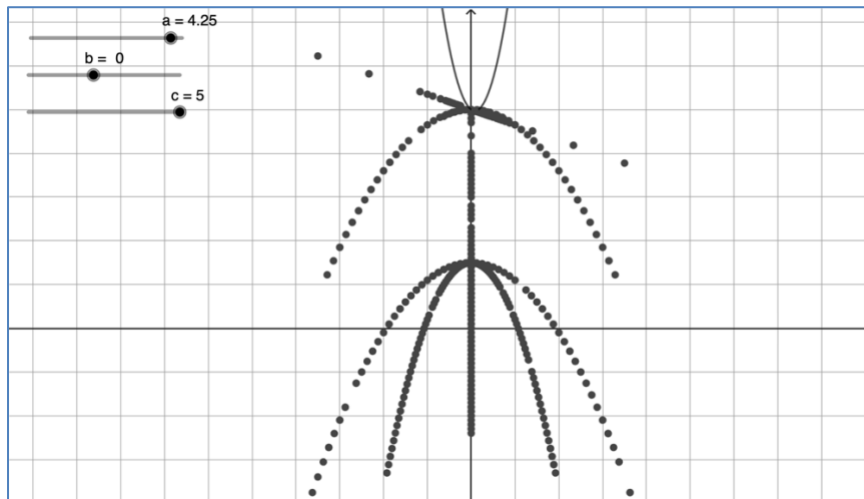


Figure 10. Feedback from Setting b Slider to 0

RS4-2: Jared wondered aloud whether the slope could be $\frac{c}{b}$ (C5). Based on his verbalizations and a partial view of the screen, we believe that Jared attempted to verify his conjecture by graphing $y = \left(\frac{c}{b}\right)x$, even though the group had already determined that the slope is independent of c . After the graph appeared and did not match the previous traces, Jared dismissed that conjecture with a disappointed “Oh.”

RS4-3: Jared considered whether the sign of b was related to the sign of the slope (O4) and whether the slope could be inversely related to b (i.e., the slope is $\frac{k}{b}$ instead of kb). Jared dismissed these options and returned to trying to find a specific value of k in the expression kb .

Jared dragged the b slider to a positive value, then dragged the a slider back and forth across its range to generate a linear path. Then Jared moved the slider for b to a negative value, then dragged the slider for a again to generate a linear path. Looking at the slope indicated on the screen, Jared and Joelle discussed GeoGebra’s feedback.

Jared: It’s a positive slope whenever b is positive and then... a negative slope when b is negative (O4).

Joelle: Which makes sense, because that’s part of this [pointing to kb on the page where they recorded their incomplete conjecture that the path is $y = kb x + \square$]. But we just need to figure out what this is [pointing to empty box for y -intercept].

Jared: [Returning to slope] Okay, so it's... some constant over b . Is it over b ? I think it's over b cuz ... or, no, no, no, no, no, no, it's not over b it's just some constant times b . $2b$. Is it just $2bx$?

RS4-4: When a set of traces resulting from varying a was on the screen, Jared used the GeoGebra line tool to construct a line that coincided with the traced ordered pairs. Then, Jared used GeoGebra to determine the slope of his superimposed line (C6; Figure 11).

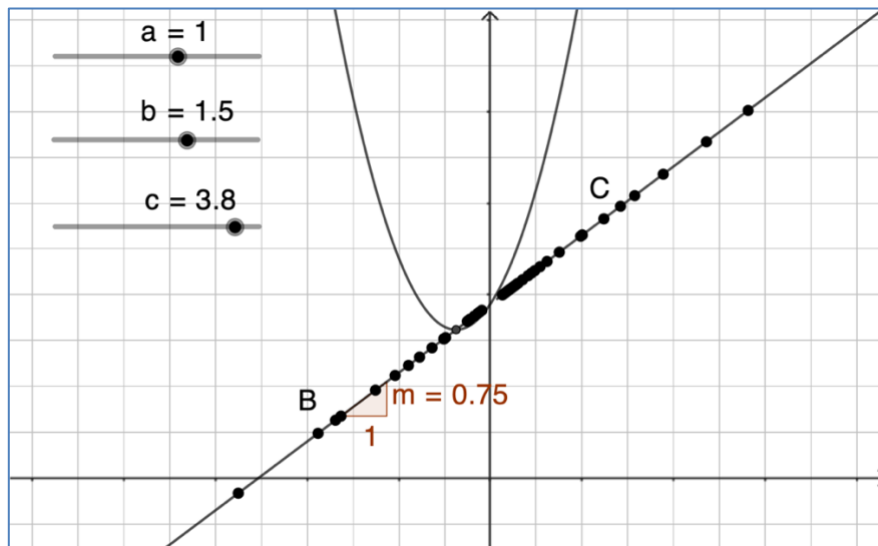


Figure 11. Jared Determined Slope Using a Specific Case

Note. The figure illustrates a superimposed line over traces from varying a . Jared used GeoGebra to measure the slope associated with the parameter values shown by the sliders.

RS4-5: Looking at the screen, Jared read, " $y = \frac{3}{4}x + 3.8$ ". Then, Jared recorded pencil-and-paper calculations as he and Joelle encouraged each other. While Jared worked on paper, Joelle focused on the value of c . Jared's calculations and our interpretations of each line are in Figure 12. The calculations led to C6 (predictive argument).

Joelle: We just have to find c . Or not c . Plus whatever [referring to the y -intercept].

Jared: I will solve this!

Joelle: We can write the equation of that line.

Jared: I'm emotionally invested. Yes. So, $y = 3.8$. [Jared recorded work in his notebook.]

Joelle: Is it always plus c no matter what the graph is?

Jared: Um. It could be minus.

Joelle: Right, but minus c ?

Jared: I don't know. [He continued his work with c as the y -intercept anyway.]

Calculation	Interpretation
$y = \frac{3}{4}x + 3.8$	Substitute the measured slope and value for c into a general linear equation of the form $y = mx + b$
$kb = \frac{3}{4}$	Substitute the slope value from the equation into $m = kb$
$k\left(\frac{3}{2}\right) = \frac{3}{4}$	Substitute the value of b on the screen (1.5)
$k = \frac{3}{4} \cdot \frac{2}{3}$	Solve for k
$k = \frac{1}{2}$	Simplify
$y = \frac{1}{2}bx + c$	Substitute k into $y = kb + c$

Figure 12. Interpreting Jared's Calculations for Determining Slope

RS4-6: Jared attempted to confirm the value of k that he had obtained using the specific case. He typed $\frac{1}{2}bx + c$ in the input line in GeoGebra to graph $y = \frac{1}{2}bx + c$. He and Joelle tested C6 (verificative argument) by dragging sliders and reported success, as the traces remained on the line $y = \frac{1}{2}bx + c$. Although they were convinced of the equation after their test, both wondered why the equation defined the path (W5, W6).

Joelle: Now, let's try a different one. And so we can move...

Jared: Ok, so, vary a ... [varied the slider for b , then for a . The vertex of the parabola traveled along the line $y = \frac{1}{2}bx + c$, as they varied the values of a .] changes with...

Jared: Yeah! [Again, the vertex of the parabola continued to travel along the line Jared entered as they varied the a slider.]

Joelle: Right. That's the equation!

Jared: Yeah, but I don't know why! Like I mean...

Joelle: I don't know why either.

Jared: [Dragging the slider for a one last time.] Oooh, baby, yeah... That's it!

During RS4, PSTs derived and tested the scalar value in the slope expression. Before enacting their strategy, PSTs briefly suggested that the slope might be $\frac{c}{b}$, which contradicted their conclusion in RS3 that the slope does not depend on c . They tested this conjecture by graphing a line with this slope and varying the value of a ; the traces did not match the slope of the test line. The strategy for testing their conjecture was supported by predictive arguments, even though the conjecture itself was not grounded in a reasoned approach. As in RS1, guessing a value or an equation was not productive for the PSTs.

Their ultimate strategy appeared to be motivated by the observation that the sign of the slope matched the sign of the b parameter. This brought their attention back to the idea that slope depends only on b and a multiplicative scalar. After Jared quickly guessed a few slope

expressions involving only a constant and b , he dismissed them just as quickly, without any discussion or GeoGebra investigation. The PSTs began their systematic approach to determining the constant scalar by using GeoGebra to superimpose a line over the traces and measure its slope. Using the slope and the b -value of the slider, Jared algebraically solved for the scalar, k . After determining the scalar value for the specific case, the PSTs surmised that k would remain constant and tested this by graphing a line controlled by the sliders. They varied the values of the remaining parameters, and the GeoGebra feedback indicated that the vertex-path traces coincided with their line. Predictive or verificative arguments supported a multi-step predictive strategy. We note that before Jared used his algebraic strategy to generate and test their final equation, the PSTs questioned the value of c as a y -intercept. However, once they verified their equation, they did not return to fully interrogate or resolve their own questions about c .

Although the PSTs applied prior algebraic knowledge as part of their strategy (i.e., how to solve for an unknown), their overall multi-step strategy required reasoning that went beyond IR. The reasoning appeared creative because Jared and Joelle developed or adapted a method to illustrate that the slope depends on b . There was no direct route to determining kb as they also considered b , $\frac{c}{b}$, $\frac{k}{b}$, and $2b$. Their overall approach (both the GeoGebra and algebraic components) was plausible because there are predictive arguments that both strategies would work. Additionally, both Jared and Joelle were able to interpret the verification process by graphing $y = \frac{b}{2}x + c$ and varying the sliders. The PSTs' strategies were anchored in the mathematics of direct variation (a form of dependence) and in solving equations. Because their reasoning was creative, plausible, and anchored in mathematics, CMR was evident in RS4.

Because we classified PSTs' reasoning as CMR in four reasoning sequences that refer to the four subtasks of the larger task, we conclude that PSTs used CMR for the overall task. In other words, we determined that a sequence of CMR reasonings for all subtasks of an extended task constituted CMR for the larger task.

To further reflect on the four connected reasoning sequences, we encapsulated the PSTs' journey through the entire task in Table 1. The table is organized chronologically, flowing through all steps of each reasoning sequence. We include the overall strategy for the primary argument of each reasoning sequence, and, for each step, the actions, feedback (GeoGebra or hand calculations), any key verbalizations (i.e., observations, conjectures, wonderings), the nature of the explicit or implicit arguments associated with the strategy, and any distinct knowledge state that seemed to be associated with the step. In the next section, we examine the evolution of these knowledge steps in our results, as PSTs learned the mathematics that underpinned that reasoning.

Table 1
Summary of PSTs' Journey Through the Reasoning Sequences

Step	Action	Feedback (Evidence)	Key Verbalizations	Strategy Support	Knowledge States
RS1-1	Varying a	Traces appear linear	O1: Path is linear C1: Path is $y = bx + c$		Varying a results in a linear path.
RS1-2	Making $a = 0$	Graph of $y = bx + c$ does not match traces	C1: Path is $y = bx + c$	P/V	Varying a does not result in the vertex traveling along the path $y = bx + c$.
Overall strategies for RS1: Drag slider to explore effects on vertex; test a path equation by comparing graphed line to traces.					

Step	Action	Feedback (Evidence)	Key Verbalizations	Strategy Support	Knowledge States
RS2-2	Graphing a line over the traced points	Line and traced path are not parallel	O2: Now the path is different		The parameters sometimes affect the slope and sometimes don't.
RS2-3	Varying c , then varying a	Traces appear parallel	W2: Why?		
RS2-4	Varying b , then varying a	Traces are not parallel	C2: Slope depends on b	P	The slope of the line created when varying a is dependent on parameter b .
RS2-5	Erasing traces, varying a , then varying b , then varying a		C2: Slope depends on b	P/V	The slope of the line created when varying a is dependent on parameter b .
Overall strategy for RS2:					
Systematically vary a parameter, while holding others constant, to determine effect of the parameter.					

Step	Action	Feedback (Evidence)	Key Verbalizations	Strategy Support	Knowledge States
RS3-2	Varying b , then varying c	Traces appear parabolic; traces appear vertical			
RS3-3	Erasing traces, then varying a	Slope is constant for different values of a	C3: Slope not dependent on a W3: Why not?		The slope of the line created when varying a is not dependent on parameter a .
RS3-4	Varying a	Slope is constant for different values of a	C3: Slope not dependent on a	V	Although varying a creates the line, the slope of that line is not dependent on a .
RS3-5	(Screen not recorded)	Traces appear parallel	C4: Slope not dependent on c	V	The slope of the line created when varying a is not dependent on parameter c .
Overall strategy for RS3:					
Systematically vary a parameter, while holding others constant, to determine effect of the parameter (twice; echo of RS2 strategy).					

Step	Action	Feedback (Evidence)	Key Verbalizations	Strategy Support	Knowledge States
RS4-2	(Looking at claim recorded on paper)	Slope of the graph of $y = \frac{c}{b}x$ does not match the traces	C5: Slope is $\frac{c}{b}$	V	The slope of the path of the vertex as a is varied is not $\frac{c}{b}$.
RS4-3	Varying b	Sign of the slope matches the sign of b	O4: Slope is positive when b is positive and negative when b is negative		If the slope of the path of the vertex as a is varied is kb , then k is positive.
RS4-4	Constructing a line over the traced points and measure slope	Slope for one case is $kb = 0.75$			
RS4-5	Substituting specific values for slope and y-intercept ($y = \frac{3}{4}x + 3.8$), then, focusing on solving for k , given specific value of b in $kb = \frac{3}{4}$	$k = \frac{1}{2}$	C6: Path is $y = \frac{1}{2}bx + c$	P	Result for a specific case can be generalized to form conjecture about the slope coefficient that applies to all cases (solving).
RS4-6	Graphing the line $y = \frac{1}{2}bx + c$	Graph of $y = \frac{1}{2}bx + c$ matches traces	C6: Path is $y = \frac{1}{2}bx + c$ W4: ...but I don't know why? W5: I don't know why either.	V	The path of the vertex as a is varied can be described as $y = \frac{1}{2}bx + c$.

Overall strategies for RS4:

Test a path equation by comparing graphed line to traces (twice; echo of RS1 strategy);
Use a specific case to generate value of slope coefficient.

Notes: RS1, RS2, RS3, and RS4 represent Reasoning Sequences 1, 2, 3, and 4, respectively. The hyphenated number suffixes represent steps in the reasoning sequence. Key verbalizations included observations (O), wonderings (W), and conjectures (C), numbered consecutively throughout the task. Strategy supports were linked to predictive (P) or verificative (V) arguments.

Mathematical Properties and Learning

Across the four reasoning sequences, the PSTs progressed through 12 distinct, yet often related, knowledge states. Looking at this progression of knowledge states, we see the PSTs' reasoning was anchored in a developing understanding of the property of dependence. In RS1, they recognized that the path is linear and eliminated a specific conjectured path. In this first reasoning sequence, there were only implicit connections to properties of dependence. During RS2 and RS3, their focus shifted as they formulated the remaining subtasks to address dependence explicitly. Evidence of dependence was obtained by intentionally varying parameter values while holding others constant and noting the effects of these changes on the path's slope. This focus on dependence was evident in the language they used in their conjectures (e.g., "the slope depends on b " and "the slope does not depend on c "). Finally, in RS4, Jared represented dependence symbolically by recording the slope as kb . He then used an algebraic strategy to determine that $k = \frac{1}{2}$ which led to the final knowledge state of this task: the path of the vertex as a is varied can be described as $y = \frac{1}{2}bx + c$.

PSTs cycled through making correct and incorrect claims about the slope of the path based on evidence of dependence in graphical, numerical, and algebraic relationships. PST struggles were evident when they transitioned to systematic dragging, as evidenced by their wonderings (why questions) about the significance of the creative feedback generated during unsystematic dragging, and by their fluctuation between contradictory claims. That struggle illustrated a tenuous grasp on dependence early in the process and a more substantial understanding that coincided with refinement of their strategies. Jared's summary of C4 ("Okay, the slope of the line is dependent on b , and then...but it's not dependent on c because if you shift c , it's [the slope] the same...it's [the linear path is] parallel, it's parallel. So, it's b , ultimately.") showed commitment to the claim with more confidence than was evident earlier when he wondered how to interpret GeoGebra feedback with respect to dependence relationships. Those indicators provide evidence that learning resulted from task engagement.

The decision to use an algebraic representation of the dependence relationship showed another dimension to the PSTs' thinking about dependence. Although they represented the slope's dependence on b as kb , they revisited that decision and discussed some alternatives. Jared briefly conjectured, then rejected, the possibility that b was inversely proportional to the slope of the line (RS4-2). Jared also considered how the signs of the slope and b compared, when he stated, "it's a positive slope whenever b is positive and then... a negative slope when b is negative."

Other algebraic representations of dependence were illustrated in the PSTs' written proof of their final conjecture (Figure 13), in which they used two arbitrary points to determine the slope. The coordinates of the vertex were represented parametrically in the ordered-pair notation. In the second-to-last line of the proof, PSTs expressed the slope between two points on the vertex's path in terms of all three parameters. However, the simplified form (the last line of the proof) provided evidence that this slope depended only on b and was therefore constant when b was held constant. This allowed them to confirm their conjecture that the path must be linear.

$y = ax^2 + bx + c, \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right) \right)$ $y = a \left(\frac{-b}{2a} \right)^2 + b \left(\frac{-b}{2a} \right) + c$ $\dots$ $y = \frac{-b^2}{4a} + c$ $\text{Point 1: } \left(\frac{-b}{2a}, \frac{-b^2}{4a} + c \right)$	$y = ax^2 + bx + c, \left(\frac{-b}{2(a+k)}, f\left(\frac{-b}{2(a+k)}\right) \right)$ $y = a \left(\frac{-b}{2(a+k)} \right)^2 + b \left(\frac{-b}{2(a+k)} \right) + c$ $\dots$ $y = \frac{-b^2}{4(a+k)} + c$ $\text{Point 2: } \left(\frac{-b}{2(a+k)}, \frac{-b^2}{4(a+k)} + c \right)$
$m = \frac{y_2 - y_1}{x_2 - x_1}$ $m = \frac{\left(\frac{-b^2}{4(a+k)} + c \right) - \left(\frac{-b^2}{4a} + c \right)}{\left(\frac{-b}{2(a+k)} \right) - \left(\frac{-b}{2a} \right)}$ $m = \frac{1}{2} b \leftarrow \text{slope!}$	

Figure 13. Key Algebraic Steps in Joelle, Jared, Marlina, and Nancy's Proof

In response to our second research question, we note that PSTs struggled with and learned about mathematical dependence across representations primarily in connection to linear functions. Dependence was evident in their consideration of which parameters were involved in the slope, how those parameters related to the slope, in determining the value of k in kb , and in their selection of vertex coordinates. Although PSTs drew upon other teaching-related content while solving the quadratics task (e.g., linear function representations, distinguishing values of a variable from rates of change of variable values), they either struggled less with those concepts or they were not central to their progress through the reasoning sequence. Thus, we only claim that dependence was a focus of learning during their reasoning about the quadratic task.

5 Discussion

From our analyses, we determined that the PSTs engaged in CMR while exploring the quadratics task. The PSTs' primary struggle and learning were tied to the concept of mathematical dependence.

PSTs Growth Through Struggle

As Ellis and Grinstead (2008) found with high school students, our results indicated that when PSTs encounter the quadratics task, they are likely to report that parameters affect the parabola's overall shape. Likewise, as Mutambara et al. (2020) and Ellis and Grinstead (2008) found, PSTs did not anticipate that parameters had any specific effects on the vertex, either alone or in concert. We did not observe evidence of Chazan's (2006) and Ellis and Grinstead's findings that PSTs and secondary students interpreted a as the slope (gradient) of the parabola. Perhaps this is due to their immediate engagement with technology.

Ellis and Grinstead (2008) surmised that classroom factors may lead students to overgeneralize the role of the parameters of linear equations. Although we did not observe

this phenomenon, perhaps other classroom factors, such as the availability of MAT, helped PSTs overcome the effects of any “negative met-befores” (i.e., ideas constructed during prior experiences that cause obstacles to new learning) that might otherwise lead to overgeneralization and limit engagement in CMR (Jukić Matić, 2015).

To analyze links between learning and classroom experiences, we found Lithner’s (2008) framework useful because it helped us identify engagement in productive struggle throughout an extended reasoning sequence, by examining the properties that anchor strategies and noting changes in conviction as participants oscillated between observations and both true and false claims. Additionally, by documenting the knowledge states throughout the reasoning sequences, we demonstrated the PST’s development of the concept of dependence. Documenting the specific knowledge gained during the task could help teacher educators or researchers who consider using this task understand its potential outcomes under similar conditions. Likewise, users could explore the effects of different design decisions on learning outcomes. This would allow the design to align with the desired knowledge to be learned, as suggested by Brousseau (1997) in his theory of didactical situations.

To apply Lithner’s (2008) framework to a longer task with multiple reasoning sequences containing multiple steps in each, we focused on the groups’ reasoning sequences to extract information about CMR, and we looked across several steps and considered a collection of strategies until we focused on the strategy that led to the final claim.

Although we found Lithner’s framework (2008) sufficient for our needs, our focus may have overlooked additional nuances in our PSTs’ reasoning. We note two alternatives that may provide additional detail on the reasoning generated during the quadratics exploration task, addressing a different research question. First, Birkeland (2019) found that when PSTs engaged in collaborative problem solving on number-theory tasks, some of their reasoning was neither IR nor CMR but rather somewhat flexible. Second, Rohati et al. (2023) developed a rubric for reasoning behavior during task solving that spanned four levels (imitative, algorithmic, semi-creative, and creative) across several cognitive, metacognitive, and affective domains. We did not apply this framework because Rohati et al.’s revision was designed for individual, rather than collaborative, task solving. Thus, the usefulness of Birkeland’s and Rohati et al.’s considerations of additional levels and dimensions of reasoning may provide more opportunities for investigation.

Dependence, Variable, and Covariation

In our study, we classified PSTs’ reasoning as CMR when they explored the path traversed by the vertex as a varies in the general quadratic $y = ax^2 + bx + c$. We described how PSTs created or recreated strategies for reasoning about claims that were primarily connected to the concept of mathematical dependence. By reviewing the PSTs’ reasoning about dependence, we suggest that they connected their knowledge of dependence to the concepts of variable and covariation. Our results lend support to Jonsson et al.’s (2022) argument that providing students with an opportunity to struggle by constructing their own solution methods using CMR enhances conceptual understanding.

The task, as posed, contains five letters (a , b , c , x , and y) and the PSTs needed to make sense of the different roles of those letters across different moments in the task. Given the importance and ubiquity of variables in high school mathematics (Schoenfeld & Arcavi, 1988), the quadratics task can be considered a teaching-related mathematics task (Stylianides & Stylianides, 2006) or one that fosters learning important for mathematics teaching. As the PSTs reasoned through this task they treated the letters in the problem differently. Their

claim that the path is linear, and the slope is $\frac{b}{2}$, is evidence that, although they varied b throughout their exploration to generate data, they at least implicitly treated b as a value that is a constant, because linear functions have constant slope.

While engaged in CMR, PSTs were learning about the concepts of variable and covariation because the arguments that anchored their task-solving strategy choices hinged on those concepts. Covariational reasoning is defined as “the cognitive activities involved in coordinating two varying quantities while attending to how they change in relation to each other” (Carlson et. al., 2002, p. 354). Using this definition, the quadratics task could be interpreted as about covariation between x and y , or, perhaps more closely to how the PSTs in this study explored the task, as covariation among a , b , c , and the slope of the vertex's path. In either case, the task is not presented in a way that makes covariation explicit. The latency of covariation in the quadratics task, compared to the overt attention to the two quantities of interest (i.e., height and amount of water) in the bottle task (Carlson et. al., 2002), adds a layer of complexity, which does not seem to be addressed in the covariational reasoning framework. We suggest that further research could refine the covariational reasoning framework by clarifying what is necessary for learners to first identify and articulate a problematic situation as covariational.

By engaging in an extended, problem-based exploration, PSTs had the potential to develop a profound understanding of fundamental content (Ma, 1999). Such depth of knowledge is essential for teachers who require knowledge beyond the individual skills often developed in secondary schools (e.g., calculating the slope of a line, associating a constant slope with a linear function; Stump, 1999). Likewise, it is important to engage teachers in big-picture thinking about curricular topics (Chazan et al., 2008). We suggest that mathematical dependence may be one of those important themes, and we note the potential of this version of the quadratics task to elicit such thinking.

Related to dependence, Vinner and Dreyfus (1989) noted that few (about 25%) of their respondents (college students in a variety of majors and junior high school mathematics teachers) connected the concept of function to a dependence relation (e.g., one factor depends on the other). In contrast, about 86% of mathematics majors, like our PSTs, were familiar with this idea. However, our research did not indicate that the PSTs possessed as nuanced an understanding of dependence across multiple representations at the beginning of the exploration as they did by the end. Understanding mathematical dependence is essential for teachers so that they can distinguish between necessary and sufficient conditions in functions and make sense of the concepts of association and causation.

The Role of Technology

The use of DGE technology may have afforded PSTs plausible strategies for novel tasks that they might not have had with pencil and paper. In a paper-and-pencil context, students must reason only abstractly, using symbolized relationships among quantities. The parabola's vertex coordinates can be thought of as parametrically defined relationships in which x depends on b and a , and y depends on a , b , and c . Describing the relationship between x and y as one parameter varies requires a sophisticated understanding of parametric relationships. Perhaps, then, it was unsurprising that Mutambara et al. (2020) found that Zimbabwean PSTs struggled to describe a plan for investigating the effects of a , b , and c on the vertex of $y = ax^2 + bx + c$ when it was presented in a paper-and-pencil context.

In using GeoGebra for the quadratics tasks, the PSTs had access to tools and strategies that extended beyond the analysis of symbolic relationships. Dragging is a low-risk activity that encourages multiple cycles of systematic strategies, refinements, and reflection until the

actor can confidently predict how the environment will respond to each action. This approach, using tools such as GeoGebra, can “creatively increase cognitive processing” during problem solving (Kuzle, 2016, p. 55). The omnipresent visual, symbolic, and numerical representations not only make less-accessible tasks more accessible (Olsson, 2018) but also provide opportunities to deepen knowledge and overcome initial misconceptions, as documented by Chazan (2006) and Ellis and Grinstead (2008).

Although arbitrarily varying sliders may not seem like an approach that demands much reasoning, interpreting GeoGebra’s surprising and creative feedback (Granberg & Olsson, 2015) required explaining what was observed on the screen (Fuller et al., 2011; Laborde et al., 2006). The surprise engenders the intellectual need for certainty (Harel, 2013). This need may drive what Pea (1987) called the *purpose perspective* on cognitive technologies, which posits that such technologies can affect a student’s choice to think mathematically. Moreover, this is a choice, because the use of technology does not guarantee that students will successfully solve tasks (Olsson, 2018). In our study, the PSTs’ enthusiasm and perseverance seemed to reflect their choice. In this way, we believe that the technology did not provide extrinsic motivation but rather engendered intrinsic motivation arising from agency and ownership of the problem (Pea, 1987).

Conclusion

Stepping back from the two research questions, we found evidence that the quadratics task (Cullen et al., 2020) using technology (GeoGebra) can elicit struggle, engage PSTs in CMR, and thus be more likely to promote the generation of new knowledge (Lithner, 2008). We anticipate that other researchers may obtain similar results with the quadratic tasks using PSTs under similarly supported conditions. However, because our findings were based on an information-rich sample and the affordances of the classroom we studied, they illustrate what is possible with the quadratics tasks for PSTs, rather than what may occur in other situations.

Further, although we agree with Cullen et al. (2020) that the quadratics task is a good fit for PSTs because of the mathematical content addressed, we suggest that the content is more directly connected to mathematical dependence than to parametric equations. Regardless, the new knowledge produced has the potential to enhance subject-specific content knowledge (Shulman, 1986) related to content (mathematical dependence or parametric equations) that PSTs would teach in their future classrooms. We recognize that we focused on only one group of PSTs in a specific class context and suggest that further analysis of this task with other secondary PSTs in other contexts would be warranted.

From a practical standpoint, considering both research questions together may help mathematics teacher educators determine whether a task should be included within the limited time available in a preservice teacher education program. Verifying that a task requires CMR increases the likelihood that the task will lead to mathematics learning (Jonsson et al., 2020; Jonsson et al., 2022; Jonsson et al., 2014). Likewise, examining the mathematics used to anchor CMR arguments may help determine whether that content is likely to be connected to the PSTs’ future classroom teaching assignments. Using both criteria can help mathematics educators focus on selecting tasks that are more likely to develop PSTs’ profound understanding of fundamental mathematics (Ma, 1999).

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