Measurement Approach to Teaching Fractions: A Design Experiment in a Pre-service Course for Elementary Teachers

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In this paper, we present a design experiment in a "Teaching Mathematics" course for prospective elementary teachers where we sought to develop a *measurement approach* to fractions. We focus on the conceptualization of the mathematical content of the approach. We attribute our progress in the conceptualization to our efforts to overcome the challenges we had to face in bringing our students – prospective teachers – to thinking about fractions in a theoretical way. We describe some of these challenges in the paper. The approach was inspired by an approach under the same name, but addressed to children, developed by the psychologist V.V. Davydov and described in the paper by Davydov and Tsvetkovich (1991). Davydov's measurement approach proposes that, in order to develop a concept of fraction with sources in reality, children's attention should be directed to multiplicative relationships between quantities defined in terms of concrete units (such as kilos, inches or cups) rather than to indeterminate objects such as pizzas or cakes. Our original contribution is a systemic study of these relationships and operations on them, as a theoretical system, using definitions derived from measurement situations, mathematical reasoning based on these definitions and generalizations of observed patterns. It is intended to support a gradual process of abstraction of the notion of fraction as an abstract number that represents a measure of the relationship between two quantities. Furthermore, our proposal for conceptualizing fractions in this way is addressed to teachers, not to children. By its focus on relational reasoning about quantities and gradual construction of a theoretical system, the approach both requires and is expected to foster the development of quantitative reasoning and theoretical thinking.

Introduction

The aim of the paper is to describe a measurement approach to fractions for prospective elementary teachers that we developed in the course of a *design experiment* in the sense of (Cobb, diSessa, Lehrer, & Schauble, 2003; Cobb, Zhao, & Dean, 2009). Three iterations of the experiment were conducted, in the winter terms of 2013 (Year I), 2014 (Year II) and 2015 (Year III), with three cohorts of undergraduate students taking a "Teaching Mathematics" course offered as part of an Elementary Education program at a North-American university. The experiment took place within the institutional frame of this 13-week course, as the 8-week unit on fractions[1]. In the course, the second author was the instructor in all three iterations; the first author was present in all classes on fractions in Years I and II as an observer, taking notes and interacting with individual students during small group activities. Both authors collaborated in the conceptualization of the measurement approach, inspired by an approach under the same name, but addressed to children, described in (Davydov & Tsvetkovich, 1991).

The paper focuses on this work of conceptualization. It is the part of a design experiment where "the research team… specif[ies] the significant disciplinary ideas and forms of reasoning that constitute the prospective goals or endpoints for student learning

[1] The remaining 5 weeks of the course were devoted to geometry.

[or] propose an alternative conception of the domain" (Cobb, diSessa, Lehrer, & Schauble, 2003, p. 10). The domain of which we propose an alternative conception is fractions knowledge for future elementary teachers.

The conception rests on the assumption, shared by many mathematics educators, that future elementary teachers' understanding of fractions is often made of two disconnected parts: the *material conception* or the visually-based idea of fraction of something, and the *formal conception* or the formal calculus on expressions of the form $\frac{a}{b}$ with a and b being whole numbers. We share with other mathematics educators the conviction that it is important to help prospective teachers to connect the material and the formal parts of their conceptions (Parker & Baldridge, 2003; Van de Walle & Lovin, 2006; Reys, Lindquist, Lambdin, Smith & Colgan, 2010; Sowder, Sowder & Nickerson, 2011; Lamon, 2012). But we propose to do it differently: by means of a *theory of fractions of quantities*.

Fraction of a quantity is a mathematical theorization of the visual and intuitive idea of fraction of something. In the material conception, one thing (pizza, cake, shaded part of a diagram) is the fraction $\frac{a}{b}$ of something (called a "whole" or a "unit"), when the thing is made of a equal parts and the whole is made of b such parts[2]. When we make explicit the quality of the objects we are comparing in this situation (length, area, volume, weight, or number of elements), and use an explicit unit to measure this quality, then we are not comparing objects but *quantities* (Thompson, 1994). When we replace the *equal parts* into which the objects are divided by the units in which their chosen quality is measured, we obtain the notion of fraction of a quantity. This notion, defined as follows, constitutes the foundation of our proposed theory:

> Quantity A is the fraction $\frac{a}{b}$ of quantity B if there is a common unit u such that A measures a units u and B measures b units u.

The plan of the course was as follows. We would start by questioning the common textbook fraction exercises such as "How much pizza has been eaten?" accompanied by drawings of circular pizzas cut into slices. We would ask, how is the amount of the pizza measured? Is it the weight? The surface area? The number of slices? The number of calories? We would bring a real pizza and kitchen scales to the classroom. We would weigh the whole pizza and each slice in grams or ounces, and we would find that each slice is a different fraction of the weight of the whole pizza, because a real pizza is usually not a perfect circle and the slicing is also quite rough. For example, if the whole pizza weighs 548 g, and has been cut into 8 slices, one slice may weigh 72 g, and another – 68 g. Then the weight of the former slice is an "ugly" fraction – 72/548 – of the weight of the whole pizza (or 18/137, if we use 4 g as the common unit of measure). For 72 g to be the ideal one-eighth of the weight of the whole pizza, the latter would have to be 576 g, not 548 g. Based on this and other similar experiences (e.g., comparing the volumes of an apple and of its parts), students would already have some idea of the notion of fraction of a quantity as we intended it.

Our notion of fraction of quantity is even more primitive than the common *material conception* described above (part-whole relationship) because there is no intermediary abstraction implicitly made, as it is the case when one looks at a diagram or a picture of a chocolate chip cookie cut into parts, and *assumes* the parts to be "equal" in some implicit sense. That children do not spontaneously make these idealizations is evident in how they

[2] We assume that a and b represent whole numbers, and b is non-zero.

would argue that a cookie cut "in half" is not fairly shared if one of the sharers ends up with more chocolate chips (thus the object at stake has not only area, but also number of chocolate chips, as measurable qualities of interest).

Fraction of quantity is thus closer to the concrete, but it is also amenable to doing authentic mathematics, we argue, via two mechanisms: *quantification*, which is, essentially, looking at objects for their quantifiable aspects, and *theorization*, or stabilizing meanings via definitions and reasoning within a system, based on agreed upon rules. Both capture ways of inquiry into reality that mathematics, as a human practice, has always been advancing.

The definition of fraction of a quantity is thus used to derive – by means of *theoretical thinking* (Sierpinska, Bobos, & Pruncut, 2011) and *quantitative reasoning* (Thompson, 1993; 1994) – answers to problems about relations between quantities. Generalizations of these solutions serve, later on, to motivate the rules of operations on abstract fractions – the object of the aforementioned *formal conception* of fraction – by showing reasons why these operations have been defined the way they were.

The process of transition from fractions of quantities to abstract fractions is supported by problems of comparison of fractions of quantities, similar to some of the problems that are associated with the "ratio sub-construct" of fractions in the "five sub-constructs" tradition[3]. For example, given that there is so much water in one bottle (e.g., 300 ml of water in a 750 mL bottle) and so much water in another (e.g., 105 mL in a quarter-litre bottle), which bottle is fuller? Or, which mix of water and syrup is sweeter? Or, which sea water has higher salinity? Or – a problem about environmental consciousness such as this one:

> If suburb A counts 2574 households and $\frac{2}{3}$ of them compost their organic waste, while suburb B counts 3878 households and $\frac{1}{2}$ of them compost their organic waste, which one is more environmentally conscious?

In this context, fractions acquire the meaning of measures of qualities such as fullness, sweetness, salinity, etc., or, in general, measures of *how-much-ness* of one quality relative to another quality of the same kind. They can be thought of as single numbers on a par with whole numbers, which, on their part, can be conceived of as measures of qualities such as two-ness, three-ness, etc., of sets of objects. Then, one can think of developing a theory of those numbers independently of quantities they could be fractions of: defining their equivalence, order, operations, properties of those operations, etc. A theory of abstract fractions is thus obtained[4].

Thus, what we call *measurement approach to fractions* for prospective teachers, is this process of using the theory of fractions of quantities to go from the idea of fraction of something to a theory of abstract fractions. In the paper, we describe the approach in some detail and present some of the challenges of its implementation.

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[3] For example, Noelting's (1978) problem about the "orangeiness" of a juice was classified as requiring the "ratio" sub-construct in (Charalambous & Pitta-Pantazi, 2007).

[4] Mathematics education literature to teaching fractions in elementary school sometimes claims that their objective is to teach rational numbers or "rational number concepts". But this is misleading, because the aim is never to teach rational numbers in the sense that this term is understood in theoretical mathematics, that is, as a theoretical system constructed algebraically (as the quotient field on the ring of integers) or analytically as a by-product of more general issues such as conceptualizing rates as numbers rather than as pairs of numbers (Thompson & Saldanha, 2003, pp. 8-10). This is why we prefer saying "abstract fractions" or "fractions as abstract numbers" to saying "rational numbers".

We start, in section 2, by making explicit our theoretical perspective and the basic concepts underlying our approach. In section 3, we describe the methodology of our research. In section 4, we begin describing the measurement approach by contrasting it with some other approaches known in the literature. Section 5 contains more details on how we taught the courses. In the last section we share some reflections on our design after three years of experimentation.

A caveat is in order. The aim of this paper is not to show that the measurement approach to fractions for teachers *works*; that all or most students understood fractions better after the treatment than before it. This is not a pre-test/post-test type of research. In general, the purpose of design experiments in mathematics education is not necessarily *to determine what works* or to prove that some particular approach improves students' learning of a mathematical topic or area but rather "to develop a class of theories about both the process of learning and the means that are designed to support that learning…." (Cobb, diSessa, Lehrer, & Schauble, 2003, pp. 9-10). We hope to show, using analytical arguments and reflections on observed future teachers' behaviour, that the measurement approach to fractions makes sense. We also describe some of the difficulties that prospective teachers are likely to experience with the approach. Difficulties are to be expected and we see them as playing a positive role. Difficulties are not only inevitable – fractions are a complex notion and no approach is going to make them easier to understand, to learn and to teach – but are constitutive of learning. They must be revealed, understood and overcome for learning to occur.

Theoretical Framework

Our theoretical framework is made of 1) a Vygotskian perspective on the purposes of education and content development; 2) a model of theoretical thinking and, 3) a concept of quantitative reasoning.

A Vygotskian Perspective on Education and Content Development

Constructivist approaches to teaching, based on Piaget's views of the relations between cognitive development and education, advocate adjustment of the mathematical content of teaching to the stage of the child's development (Piaget, 1969/1990)[5], or, more generally, "following the student's lead" (Arcavi & Schoenfeld, 1992, p. 323; see also Confrey, 1990 and Wood, 1995). In Davydov's curriculum, on the other hand, content is chosen, organized and taught based on the idea, inspired by Vygotsky's work (Vygotsky, 1987), that the role of school instruction is to *further* the development by engaging the child in a process of conscious theorization of the historically shaped social practices assumed to stand at the source of the target knowledge to be taught (Davydov, 1991b). Children do not go to school to learn how to speak grammatically, or just to calculate and solve problems more efficiently. They go to school to become aware of what they do as they speak and calculate: that's what it means to learn grammar and arithmetic.[6]

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[5] « En un mot, le problème central de l'enseignement des mathématiques est celui de l'ajustement réciproque des structures opératoires spontanées propres à l'intelligence et du programme ou des méthodes relatifs aux domaines mathématiques enseignés. » (Piaget, 1969/1990, p. 66)

[6] "The preschool child possesses all the basic grammatical and syntactic forms. He does not acquire fundamentally new grammatical or syntactic structures in school instruction. From this perspective, instruction in grammar is indeed a useless undertaking. What the child does learn in school, however, is conscious awareness of what he does." (Vygotsky, 1987, p. 206)

Aims of the measurement approach to fractions for prospective teachers. In our research, we adopted a similar view with regard to the goals of instruction in mathematics courses for prospective elementary teachers. We did not aim at re-teaching them a course on fractions for children, only better than they have been taught themselves when they were children. We addressed our course to adults capable of solving at least some *simple* problems about fractions of things (see explanation below) and of performing formal calculations on fractions without necessarily being aware of the mathematical-theoretical underpinnings of what they were doing. We aimed at helping them to develop this awareness.

What we mean by simple problems on fractions of things are problems where conceptualization of the things as quantities is not necessary and performing numerical operations is sufficient to obtain an answer that is at least numerically correct. Here is an example. A questionnaire administered in the first lesson of Year I experiment revealed that most (27 out of 35) students – the prospective teachers – could correctly solve this simple problem about fractions of things:

> Mary used ¾ of a 500 mL bottle of water. Jane used ½ of a 750 mL bottle of water. Which one used more water? Justify your answer.

Students mostly ignored the millilitres and obtained the answer by performing numerical operations such as dividing the number 500 by 4 and multiplying the result by 3. They considered writing down these operations in their answers as a sufficient justification of their conclusion that the amounts are equal. On the other hand, the next question was not such simple problem: conceptualizing the thing as a quantity was necessary. The question was:

> Mary ate ¾ of an 8-inch pizza. Jane ate ½ of a 10-inch pizza with the same filling. Which one had less pizza? Justify your answer.

Most students (all but 4 out of the 35) applied the same numerical strategy as in the previous problem, dividing 8 by 4 and multiplying the result by 3, obtaining 6, then comparing this with 5 (10 divided by 2), and concluding that Jane had less pizza than Mary.[7] They understood the formula for taking a fraction of a quantity *numerically*, not *quantitatively*, as Thompson & Saldanha (2003) would say.

Reflection on object sources of mathematical concepts as a basis of curriculum development. According to Davydov, developing an approach to teaching a mathematical concept should start by a reflection on the social practices whose conscious theorization has led, in the history of mathematics, to the development of that concept. These social practices, in Davydov's and his collaborators' papers are called the "object source of origin" for the mathematical concept (Davydov & Tsvetkovich, 1991) or "object content" of a concept (Davydov, 1991a).

The idea of object source is similar to Brousseau's notion of fundamental situation for a mathematical concept (Brousseau, 1997; Brousseau, Brousseau, & Warfield, 2014) in that both refer to problem situations for the solution of which the mathematical concept is a useful (or even optimal) tool.

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[7] More details about students' responses to these problems can be found in (Sierpinska, 2016).

Measurement as object source for the concept of number. Both Brousseau and Davydov consider measurement as the source of the notion of number in general and fractions in particular. Number is conceived as an answer to the question, "How many given units does this quantity measure?" For example, if a certain length of material measures 3 cubits, the answer to such question is: 3. This number 3 tells us how many *times* the length is greater than the chosen unit; the number 3 is the ratio of the length of the material to the unit in which it has been measured.

This notion of number as a ratio was the basis of long-term instructional experiments in the first three primary grades conducted by Davydov and his collaborators in the 1960s (Lompscher, 1994; Fridman, 1991; Schmittau & Morris, 2004; Schmittau, 2005). Children were first introduced to numbers in the context of direct comparison (by juxtaposition, superposition) of objects relative to qualities such as length, width, height, weight, etc. They were taught to record the results of their comparisons using symbols such as +, <, >. Next, the situations were changed to make the direct comparison of objects difficult or impossible; this forced the children to measure each object using sticks, pieces of string, and other devices and compare the measurements, not objects. The relationship between the quantity they were measuring and the unit of measure was represented using expressions such as "A/c = 5", interpreted as: the unit c is contained 5 times in the quantity A.

Change of unit as an object source of the concept of fraction. Number thus arises in the context of a multiplicative quantitative operation. The object source of the operation of multiplication is considered to be the practice of change of unit in measurement (Davydov, 1991a) and, since a fraction represents a multiplicative relationship between two quantities of the same kind, change of unit is also an essential element of the object source of fraction. If I want to measure a quantity A with some unit B, more often than not neither B is contained in A a whole number of times nor A is contained in B a whole number of times, but there is a remainder. If there is a smaller unit u that fits a whole number n of times into A and a whole number m of times into B, then the ratio of A to B is n to m. In this case, there is a convention to treat this ratio as a new kind of number, called "fraction", denote it by the symbol $\frac{n}{m}$ and say that A measures $\frac{n}{m}$ units B.

A top-down conceptualization of fractions for children. In this approach, fractions are defined as those numbers that can be represented as ratios of whole numbers. It is a top-down conceptualization of fractions: the idea of *number in general* is introduced first; fraction is then introduced as a *special kind of number*. In Davydov's curriculum, the notion of number is built so that children do not associate it only with counting discrete sets and therefore with whole numbers. For those children, mathematics begins with measuring all kinds of objects with different kinds of units.

A bottom-up conceptualization of fractions for prospective teachers. Our students in the experiment – adults preparing to become elementary teachers – had not followed Davydov's curriculum as children, however. They were brought up in a tradition where "mathematics begins with counting" (Parker & Baldridge, 2003, p. 1). In this tradition, building the idea of fraction starts from counting as well: counting the slices of pizza eaten and the slices into which the whole pizza was cut and writing these two whole numbers one on top of the other. Other activities and problems are used to help children conceive of this pair of numbers as a number in its own right. This is a bottom-up approach.

We tried to acquaint our students with Davydov's top-down approach in the first year (Year I) of the experiment, but, as described in (Sierpinska, 2016), we were not satisfied with the conceptions of fractions they displayed by the end of the course. The students extended their repertoires of problems about fractions by problems about fractions of quantities measured in standard units but the material (pieces of pizza), the formal (calculus on expressions a/b) and quantitative (fractions of quantities) conceptions of fractions were not integrated in the problems they created or chose for the Problem Book they had to write as the final assignment in the course (Sierpinska, 2016); they were treated in separate problems.

Reflecting on the possible reasons of such outcome, our first impulse was to blame the foreignness of the top-down approach for the students. But then we saw another, perhaps more important reason. The approach passes too quickly from measuring quantities to abstract numbers, from the quantities A and c, to their ratio A/c, bypassing and leaving unaddressed the students' basic fraction-of-something conception of fraction. This visual conception is not given a chance to evolve into a more theoretical concept that could be conceptually linked with the formal conception of fraction. As a result, the idea of fraction-of-something stays in its primitive, intuitive state and functions as an obstacle to the construction of a systemically connected knowledge about fractions. This realization led us to the idea of delaying the passage from measurement of quantities to abstract numbers by a substantial period of time devoted to working on the theorization of the idea of fraction of something in the form of the theory of fractions of quantities. This version of the measurement approach was developed in the summer and fall after the Year I experiment and tried for the first time in the Year II experiment and then, after slight modifications, in Year III.

Our design of the bottom-up measurement approach to teaching fractions to prospective elementary teachers developed together with the construction of a *theoretical system*: a theory of fractions of quantities. We envisioned elementary teachers' "mathematical horizon" (Ball & Bass, 2009) as a vertically structured system[8] where, in the first stage, the arithmetic of fractions of quantities builds on the arithmetic of whole numbers and whole numbers of quantities[9], and, in the second stage, the arithmetic of abstract fractions builds on the arithmetic of fractions of quantities. Fraction of quantity, the central concept of our measurement approach, is the missing link between the arithmetic of whole numbers and whole numbers of units, which is derived rather spontaneously from the concrete reality, and the arithmetic of abstract fractions, which is not nearly as natural to learn. The passage, we argue, requires theoretical thinking and quantitative reasoning. We explain below.

[8] We allude here to Bernstein's distinction between vertical and horizontal discourses (Bernstein, 1999); see also (Lerman, 2010).

[9] We are referring here to operations on whole numbers that produce whole numbers (e.g., 2 + 3, 3 – 2, 2 × 3, 6 ÷ 2), and to operations of addition and subtraction of whole numbers of units (e.g., 2 kg + 3 kg), multiplication of whole numbers of units by whole numbers (e.g., 2 × 3 kg) and division of whole numbers of units by whole numbers if producing whole numbers of units (e.g., 6 kg ÷ 2).

Theoretical Thinking

Our aim in the courses, in accordance with the Vygotskian perspective, and Davydov & Tsvetkovich's views on the purposes of teaching fractions, was to bring future teachers to *think theoretically* about fractions (Davydov & Tsvetkovich, 1991, p. 107).

The term *theoretical thinking* is used in the sense of the model described in our previous research (Sierpinska, Bobos, & Pruncut, 2011). We summarize it briefly here. Theoretical thinking is defined by three features: it is thinking that is reflective, systemic and analytic. There is evidence of *reflective thinking* in a student if he or she reflects back on their solutions, checking if the answer makes sense or if there is maybe a simpler solution, or if the problem is related to a previous one, etc. *Systemic thinking* refers to thinking within a conceptual system, where (a) meanings are established by definitions, rather than only by metaphors, images, or usage in a context (*systemic-definitional thinking*); (b) the validity of statements is derived from knowledge already established within the system and not by ad-hoc arguments from outside the system (*systemic-based on proofs*); and (c) there is an awareness that knowledge constructed this way is hypothetical rather than true in some absolute sense; all claims are made on some assumptions and it is important to be aware of making these assumptions, and also to question them: are they all necessary?; what if they were replaced by other assumptions? (*systemic-hypothetical*). *Analytic thinking* refers to being attentive (a) to notational conventions, and using them consistently and, (b) to logical relations among the different parts of statements or arguments. Accordingly, in Years II and III, we expected the future teachers to *prove* such statements as, e.g., "A quantity which is $\frac{a}{b}$ of $\frac{c}{d}$ of a quantity A is the fraction $\frac{a\times b}{c\times d}$ of the quantity *A*", which is a theorem in the theory of fractions of quantities.

This proving task would not be given to students as such ("prove that…"), but would be embedded in a problem of representing a quantity given as a concrete fraction of a concrete fraction of some quantity as a single fraction of that given quantity; for example,

The quantity $\frac{3}{4}$ of $\frac{5}{7}$ of $1\ kg$ is what fraction of $1\ kg$?

The numbers would be chosen so as not to make the answer obvious. Moreover, the denominator of the first fraction and the numerator of the second are relatively prime (have no common factors beside 1), so the relations between the measures of the quantities involved cannot be simplified. These choices make it easier for students to see the general in the particular, and serve as the ground for a *generic example proof* (Yopp, Ely, & Johnson-Leung, 2015) of the target theorem. Students would be asked to reason out the answer based on the definition of fraction of a quantity, and not only by illustrating it using diagrams or manipulatives such as strips of paper. Diagrams could be drawn to visualize the relations in the situation, and get a rough estimate of the quantity, but they were deemed insufficient as a justification of their answer. Representing quantities as line segments (*Figure 1*) was often used by the teacher in outlining the relationships between quantities and students adopted this representation as well in thinking about problems.

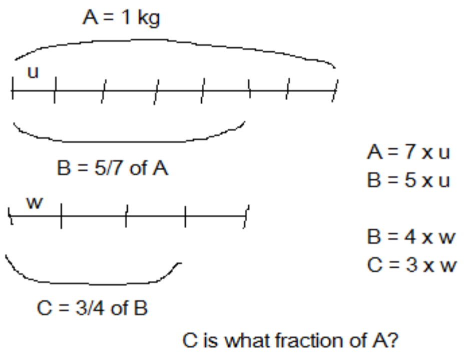

Figure 1. Diagram representing the problem, 3/4 of 5/7 of 1 kg is what fraction of the kilo?

Based on a rough diagram such as the one in *Figure 1*, one can guess that three-quarters of five-sevenths of a kilo is a little bit more than half a kilo. But a more precise answer requires analytic work. In the figure, some of this analytic work has already been done: the quantity $\frac{3}{4}$ of $\frac{5}{7}$ of 1 kg has been broken down into three quantities: $A = 1\ kg$, $B = \frac{5}{7}$ of A, and $C = \frac{3}{4}$ of B. Also, the definition of fraction of a quantity (given in the Introduction to this paper) has been used to claim that for some unit u, A measures 7 units u and B measures 5 units u, and that for some w, B measures 4 units w and C measures 3 units w. The question is, C is what fraction of A? This is the main part of the reasoning and it is not shown in the diagram. It would go like this: quantities C and A cannot be compared directly because they are expressed in different units: C is in w's and A is in u's. A change of unit is necessary. This change of unit must respect the equality $5 \times u = 4 \times w$, since both are measures of B. If we take a smaller unit v such that, for example, $w = 5 \times v$ (we divide the unit w into 5 smaller units), then $5 \times u = 4 \times 5 \times v$; so, by proportionality, $u = 4 \times v$. Converting A and C into units v, we get $A = 7 \times 4 \times v$ and $C = 3 \times 5 \times v$. Using the definition of the fraction of a quantity again, we conclude that C is the fraction $\frac{3\times5}{7\times4}$ of A. Students would be tempted to replace 7×4 by 28 and 3×5 by 15, but we would advise them to delay doing this until after they had the time to see the pattern and generalize their reasoning.

Reasoning within a system may seem like narrow-mindedness, empty formalism, and other such things, strongly criticized in post-modern mathematics education. We argue, however, that forcing oneself to think and reason within a conceptual system is opening one's mind to different points of view and considering them in a non-judgmental way, without taking sides. It prepares us for the effort of understanding how other people – for example, children – think. We may be more willing to try to identify their assumptions, the things they take for granted, the rules they use to draw conclusions, and, on that basis, imagine what they expect and what they might do in various situations.

Thinking theoretically about fractions of quantities might therefore be very useful in preparing to teach this concept to elementary school children. First of all, it is not far from thinking the way a pre-fractions elementary school child might think. In the bottom-up version of the measurement approach, as we started teaching it from Year II, the concept of fraction of a quantity is built from only such mathematical knowledge that a pre-fractions elementary school child (who has not been following Davydov's curriculum) is likely to be familiar with: whole numbers used as multipliers (words and symbols responding to the question "how many?") and quantities measured in whole numbers of standard units (e.g., 3 kg, 24 cents). An adult who suspends his or her knowledge of rational numbers, algebra and good or bad procedural habits of dealing with fractions, and forces him or herself to reason based on this knowledge alone and uses only those concepts about fractions of quantities that have already been built, is learning to experience learning mathematics form the point of view of a pre-fractions elementary school child.

Theoretical thinking also includes reflective thinking – not just solving a problem for the sake of getting the answer expected by the teacher and obtaining a good grade but reflecting back on one's solution, generalizing it, seeking alternative solutions, extending the problem, modifying it, considering the consequences of adopting a different set of assumptions, etc. All these activities characterize a reflective teacher as much as a reflective researcher in theoretical mathematics and are most desirable in the education of future elementary teachers.

Teachers must also nurture their linguistic and logical sensitivities and be careful in representing problems, concepts, and numerical and quantitative relations so as to avoid notational ambiguities and logical inconsistencies – these are sensitivities associated with the analytical thinking dimension of theoretical thinking.

Thus the Measurement Approach has a potential to prepare future teachers to teaching fractions by allowing them to re-learn fractions on the basis of only such knowledge as a pre-fractions child would possess. But they must be open to this experience and understand its benefits. Achieving such openness on the part of undergraduate students studying to become teachers of young children is one of the challenges of this approach.

Quantitative Reasoning

The theory we constructed is a theory of fractions of quantities. Its objects are quantities and quantitative relationships; the focus is not on the numerical values of the quantities or on operations on these numbers but on the quantities as conceptual entities in and of themselves and operations on the quantities. Reasoning within this theory requires, therefore, quantitative reasoning. We refer here to the notions of quantity and quantitative reasoning as proposed by P. Thompson:

> Quantities are conceptual entities. They exist in people's conceptions of situations. A person is thinking of a quantity when he or she conceives a quality of an object in such a way that this conception entails the quality's measurability. A quantity is schematic: It is composed of an object, a quality of the object, an appropriate unit or dimension, and a process by which to assign a numerical value to the quality. (Thompson, 1994, p. 184)

We understand quantitative reasoning as reasoning which involves performing connected sequences of quantitative operations, i.e., "mental operations by which one conceives a new quantity in relation to one or more already-conceived quantities" (Thompson, 1994, p. 185). For example, taking a fraction of a quantity is a quantitative operation by which a new quantity is conceived. Changing the unit for measuring a quantity is a quantitative operation. Consciously using these operations to justify why three-quarters of two-thirds of a litre is one-half of a litre is making a quantitative reasoning. In this reasoning, the concrete numerical values of the fractions are not as important as the structure of the relationships among the quantities at play: one quantity is a fraction of another quantity which is itself a fraction of a third quantity, and we want to know what fraction of that third quantity is the first one. As Thompson was saying in another paper:

> Quantitative reasoning is the analysis of a situation into a quantitative structure – a network of quantities and quantitative relationships…. A prominent characteristic of reasoning quantitatively is that numbers and numeric relationships are of secondary importance, and do not enter into the primary analysis of a situation. What is important is relationships among quantities. (Thompson, 1993, p. 165)

These features of quantitative reasoning support the processes of generalization, leading to formulating the *theorems* in the theory of fractions of quantities.

Methodology

As any design experiment, the research evolved through, first, a phase of reflection on the sources of meaning of the target mathematical content (the concept of fractions appropriate for future elementary school teachers) in interaction with preliminary trials (in winter 2013, Year I). Based on this initial experience, a complete 8-weeks course based on the Measurement Approach was designed (summer-fall 2013) and implemented in the winter of 2014 (Year II). After a reflective analysis on the results of this first implementation, a revised version of the design was drawn and implemented in the winter 2015 class of the course (Year III).

An account of our initial conceptualization of the measurement approach to fractions and Year I trial can be found in (Sierpinska, 2016). In the present paper, we speak mainly about the revised versions of the measurement approach tried in Years II and III. We focus on the theoretical conceptualization of the approach, but we also give some information about the prospective teachers' responses to the approach, because these responses played an important role in the process of the conceptualization. A response mattered for us, whether it was widespread among the students or exhibited by only a few, if it made us realize an unsuspected or unintended meaning of what we were saying in class or writing in the formulations of problems. The research was not quantitative.

In this section, we first describe, in general terms, the design experiment methodology as we applied it in our research, and then give more details about our research procedures.

Design Experiment

Design experiment is an ambiguous term since it refers to both *a product* of practicing research in mathematics education as a design science (Wittman, 1995), and to *a research methodology* that outlines the process of conceiving and evaluating educational interventions aimed at developing a particular product, such as a curriculum, a sequence of lessons, or a piece of software. Both meanings, as product and method, cohabit in such design-based research approaches in mathematics education as didactic engineering (Artigue, 1992; 2009; 2014) or design experiment (Cobb, diSessa, Lehrer, & Schauble, 2003; Cobb, Zhao, & Dean, 2009). The methodology of our research is closer to Cobb et al.'s understanding of design experiment than to didactic engineering: we did not use the theory of didactic situations as our theoretical framework. However, in our re-thinking of the mathematical content of the planned teaching – a phase of research present in both didactic engineering and design experiment – we paid special attention to object sources of the concept of fraction. As already mentioned before, object sources – a Davydovian concept – is close to the notion of fundamental situation, characteristic of didactic engineering.

Davydov and his collaborators' papers do not dwell on the methodology of the research they were conducting. But their research can be classified as instructional design experimentation, with its usual cycle of several iterations of design, implementation, reflection and revision. Their papers on the teaching of multiplication, fractions and other mathematical concepts are usually made of four parts: presentation of the common ("traditional") approach to teaching a concept; critical assessment of this approach; presentation and justification of a new approach, and a detailed description of a *model teaching* of the concept using the new approach. In the last part, the teaching is presented as successful; it is not an account of a trial that has had its shortcomings and is in need of improvement. It is a finished product. These descriptions may appear somewhat idealized, cartoonish, with children not behaving as we know children to behave, but they are not meant to represent a concrete real class of children. They only illustrate the details of the approach; the teacher's essential interventions are highlighted and explained. The critical assessment of common approaches and justification of the proposed approach are built from the perspective of the assumed object sources of the meaning of the mathematical concept to be taught and learned.

In this paper, we also start by presenting what we assume to be the object sources of the concept of *fraction of a quantity* – the basic concept in our approach – and describe how the development of the content of the course is built on these object sources. But rather than presenting a model realization of the design, we outline the main challenges in its realization[10]. It is in these obstacles, in particular, that we claim the generalizability of our results – they are the same as the "concrete universals" one seeks to find in linguistics research (Erickson, 1986).

Sources of Data

The experiment was conducted in a regular, 13-week long undergraduate course in a program leading to a teaching certificate at the elementary level. While the program was preparing teachers of all subjects (not only mathematics) at this level, the course in which the experiment took place was "Teaching Mathematics II". In Teaching Mathematics I, the focus was on whole numbers; Teaching Mathematics II was on fractions and geometry. Admission into the program was based on candidates' average grade in secondary school and a two-year college; more than a passing grade in a basic level high school mathematics course or prior teaching experience were not required. Most students were quite young (20-25), but in each year there were a few people in the 30-40 age range. There were 38 students enrolled in the course in Year II and 34 in Year III. Most were female; the number of male students did not exceed 4 in each year.

Each week the class would meet two times: one 2-hour class where new material was introduced and a 1¼-hour lab, devoted to problem solving in small groups and discussing problematic solutions or interpretations of homework assignments.

The data collected in this research consist of:

Documents produced by the researchers: course outlines, printed Course Notes (used as texts for students); instructor's notes from classes, labs, office hours and emailed

[10] We do not present a model realization not because we don't have it yet, but because we don't think such one universal ideal model exists. Each realization depends on the personalities, cognitive propensities and academic pasts of both the teacher and the students. Much depends on the questions students ask in the class and the teacher's ability to use them as opportunities to clarify the concepts.

exchanges with students; the field observer's notes (Year II only); Power Point lecture presentations.

Copies of all students' written work for the courses: homework assignments (10-12 in each year) and examinations (Midterm and Final).

Ways of Analysing the Data

In the descriptions of students' solutions we will occasionally use simple descriptive statistics but our analyses of the data are essentially conceptual and interpretive, and we make no quantitative claims. We are interested in the nature of students' difficulties, not in their distribution in the population. As Cobb, McClain and Gravemeijer (2003, p. 4) contend, rather than implying homogeneity of all students' learning, we make "empirically grounded claims about the conditions for the possibility of all students' learning".

Any general statements made here are, however, not the result of analysing isolated pieces of data, but rather of systematic reading of the entire data corpus and using Glaser and Strauss's method of producing categories grounded in the data through an iterative process of conjecturing, refining, or refuting emergent rubrics (Glaser & Strauss, 1967), using data triangulation. But these categories were not built on the data alone; we used any concepts and categories we were used to thinking with, a process which is included in the more recent characterizations of interpretive methodology of research (Erickson, 1986; Eisenhart, 2009).

Students' solutions to problems given as homework assignments, class tests, or invented by themselves were first analysed one by one. We first tried to understand how each student interpreted the problem – what problem he or she was, in fact, solving – and how he or she could have been reasoning, based on the written traces of said reasoning. We transcribed (or inserted screen clippings of a scan of the student's work) in one column of a two-column table, and wrote our hypotheses as to what the student was thinking in the second column.

For example, in the Midterm test given in Year III of the experiment, one of the questions was:

> (Year III, Midterm, Version A, Question 2a) A jar can be filled to capacity with either 6 tall glasses or 9 short glasses of water. What fraction of the capacity of the tall glass is the capacity of the short glass? Derive your answer from one of the definitions of a fraction of a quantity.

Some explanation of the context is in order here. In Year III of the experiment, beside the common-unit definition of fraction of quantity mentioned before (a theorization of the idea of fraction-of-something), another characterization of fraction of quantity was added, based on the idea of commensurability and derived from the social practice of barter transactions (more about it later). It was also called a *definition*. So, there were two definitions of a fraction of a quantity in use. One, labelled DoF-I, was saying that quantity A is the fraction $\frac{a}{b}$ of the quantity B if $b \times A = a \times B$ (the "barter definition"). The other, labeled DoF-II, said that the quantity A is the fraction $\frac{a}{b}$ of quantity B if there is a common unit u such that A measures a units u and B measures b units u. In both definitions it is assumed that a and b are whole numbers and that the quantity B is non-zero.

One student produced the solution in *Figure 2*.

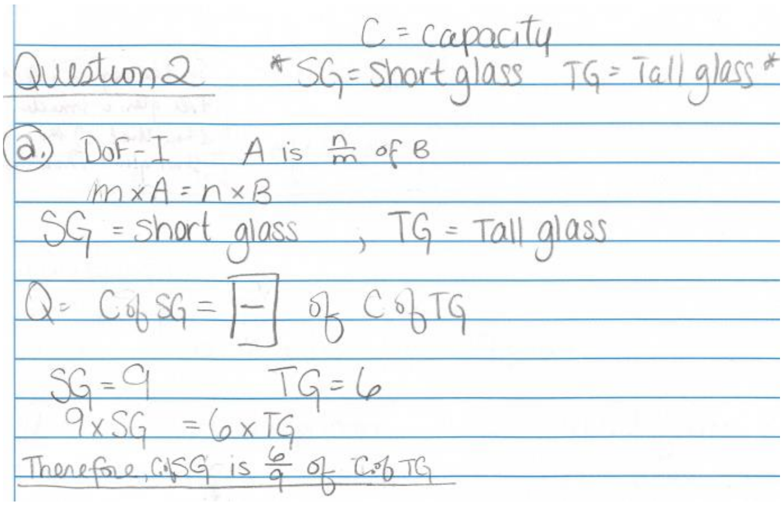

Figure 2. A student's solution to Question 2a of the Midterm exam in Year III of the experiment.

This solution was transcribed and divided into meaningful elements ("semantic units") or steps of the solution. Each element was then put into one row of the left column of a two-column table (Table 1). In the right column of the row we put our interpretation of this element of the solution. The bottom row of the table contained a summary evaluation of the solution regarding evidence of quantitative reasoning and theoretical thinking. We then discussed our interpretations and agreed on a common interpretation of the student's solution.

As we were thus analysing the consecutive students' work, we were noticing commonalities and differences, and those led to grouping students' interpretations of the problems and ways of reasoning and thinking into larger categories. Eventually, we had a set of aspects that would characterize students' solutions to a given problem in the form of questions to which the answer could be "yes" or "no". These questions were then put as titles of columns in a spreadsheet file and students' names were put in the rows. Answer "yes" for a given question would be coded 1 and "no" – 0, which facilitated counting the numbers of students displaying each particular way of thinking about the problem.

Table 1.
Analysis of student SIII.1's solution to Question 2a of the Midterm exam in Year III of the experiment.

Transcript: Year III, Midterm version A, Q2a, SIII.1	Comments
C = capacity SG = short glass, TG = tall glass	Identifies the main objects in the problem – tall glass and short glass – and the quality of these objects – capacity – on which the problem is focused. Assigns abbreviations to the objects and the quality.
DoF-I A is n/m of B m × A = n × B	Quotes DoF-I without logical connectors.

[Question is:] C of SG = ?/? of C of TG	Identifies the two quantities to compare: capacity of the short glass and capacity of the tall glass. Interprets the question as intended: capacity of the short glass is what fraction of the capacity of the tall glass?
SG = 9 TG = 6	Uses the abbreviations "SG" and "TG" in a different sense now: number of short glasses and number of tall glasses, resp.
9 x SG = 6 x TG	Re-uses the abbreviations "SG" and "TG" in yet a different sense: capacity of the short glass and capacity of the tall glass, resp. Writes the relation between the two capacities, as assumed in the problem.
Therefore C of SG = 6/9 of C of TG	Applies DoF-I to obtain the final answer. Returns to the initial meanings of the abbreviations.

Summary. Problem interpreted as intended. Evidence of quantitative reasoning. Theoretical thinking: evidence of systemic-definitional thinking (applies a definition correctly); analytical sensitivity requires development: omits logical connectives in quoting a definition; uses letters more as shorthand than as variables.

Measurement Approach to Fractions in the Context of Other Approaches

In this section, we present certain aspects of our approach to fractions by contrasting it with some other approaches, known in the literature.

Quantitative Approaches to Fractions

In Niemi's (1996) review of literature on the teaching and learning of fractions some papers are grouped together as representing approaches focused on "quantitative models" of fractions. This group includes the classic "Psychology of number" of MacLellan & Dewey (1900), and Davydov & Tsvetkovich's exposition of their measurement approach to fractions (1991) , which are indeed very similar in their conceptualization of number as ratio arising from measurement and fraction as an abstraction from the context of fractions of measured quantities. Ohlsson's analysis of "mathematical and applicational meanings of fractions" (Ohlsson, 1988) is cited in this group as proposing to conceptualize fractions' applicational meaning similarly to McLellan & Dewey's definition of fraction[11]. Let us quote one of the formulations of McLellan & Dewey's definition from the original:

> In its primary conception a fraction may be considered as a number in which the unit of measure is expressly defined. In the quantities 4 dimes, 5 inches, 9 ounces, the units of measure are not explicitly defined; their value is, however, implied or else there is not a definite conception of the

[11] Ohlsson (1988) defines the mathematical meaning of fractions as an ordered triple (x, y, r), where x and y are integers, y is non-zero and r is a number which, multiplied by y, gives x.

> quantity. In $8\frac{4}{10}$, $\frac{5}{12}$ foot, $\frac{9}{16}$ pound, the units of measure are explicitly defined; and each of these expressions denotes four things:
>
> 1. The unity or standard of reference from which the actual unit of measure is derived.
> 2. How this unit is derived from the unity of reference.
> 3. The absolute number of these derived units in the quantity.
> 4. This number is the ratio of the given quantity to the unity of reference.
>
> For example, in $\frac{9}{16}$ pound, the unity of reference is one pound; it is divided into 16 equal parts to give the direct measuring unit; the number of these units in the given quantity is 9; the ratio of the given quantity to the reference unity is $\frac{9}{16}$. (McLellan & Dewey, 1900, pp. 243-4)

The conditions 1, 2 and 3 of the above definition define a fraction of a quantity in a similar way we do it in our common unit definition (

Figure 3). In the bottom-up version of our measurement approach, the 4th condition – abstraction of the fraction as an abstract number – is purposefully delayed until later in the course.

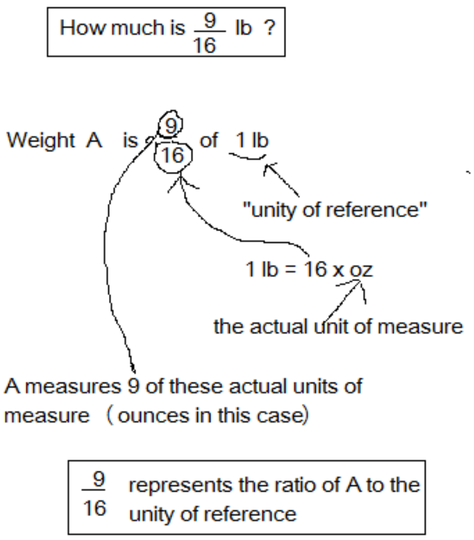

Figure 3. McLellan and Dewey's definition of fraction contains a definition of the expression “quantity A is such and such fraction of quantity B”.

Niemi seems to suggest that papers written from the perspective of the subconstructs model of fractions (Kieren, 1976; 1992; 1993; Behr, Lesh, Post, & Silver, 1983; Behr, Harel, Post, & Lesh, 1993), and focused on the Measure subconstruct qualify as promoting a quantitative approach to fractions. The Measure subconstruct is often associated with problems consisting in identifying fractions on a number line (Bright, Behr, Post, & Wachsmuth, 1988). But such problems do not feature prominently in our measurement approach, despite shared terminology. There is more on the differences between our conceptualization of fractions and the subconstructs model in section 4.4.

Conceptual Schemes at Play in Reasoning about Fractions

There have been efforts to conceptualize and/or experiment with consistently "quantitative" or "measurement" approaches to fractions beside those of Davydov's group. On the North-American scene, P. Thompson has long criticized the prevailing approaches in dealing with multiplicative relationships in mathematics education literature and teaching practice and advocated a more conscious and consistent distinction between numerical and quantitative mental operations in reasoning (Thompson, 1993; 1994; Thompson & Saldanha, 2003).

When Thompson and Saldanha set to analyse "what it means to understand fractions well" (2003, p. 95), they did so in terms not of their sources in historically developed forms of human activity or of problem-situations for which fractions are an optimal solution, but in terms of conceptual schemes (Vergnaud, 1994) that a cognitive subject must develop in order to engage in coherent reasoning about problems involving fractions. The results of their analyses, expectedly, do overlap with Davydov's in identifying the essential mental operations. For example, their description of the "measurement scheme" encompasses thinking of a measure of a quantity as the ratio of the quantity to the unit in which it has been measured and being aware that change of unit does not change the magnitude of a quantity. The measurement scheme entails understanding that,

> … if m is the measure of quantity B in units of quantity A (i.e. B is m times as large as A), then nm is the measure of quantity B in units of $1/n^{\text{th}}$ of A. Conversely, if m is the measure of B in units of A then m/n is the measure of B in units of nA." (Thompson & Saldanha, 2003, p. 101)

Thus, the operation of change of unit – the object source of multiplication and therefore also of fractions for Davydov – is essential also in "conceptualizing reciprocal relationships of relative size" (p. 107) which, for Thompson and Saldanha, is the foundation of fraction schemes.

As mentioned before, in our approach to teaching fractions to prospective teachers, we chose to build a connection between the visual and the formal conception of fraction based on a theory of fractions of quantities. This theory starts from a definition of the phrase "quantity A is the fraction $\frac{a}{b}$ of quantity B". Thompson & Saldanha argue against using "of" in such phrases and propose to replace it by "as large as"; e.g., "A is 6/5 as large as B" (p. 107). Their argument is that "of" may suggest that A is a part of B so statements such as "A is 6/5 of B", where the numerator is larger than the denominator make no sense for students who think of fraction primarily as a part of a whole. They admit, however, that statements "A is a/b of B" are interpreted correctly by those who conceive of A and B as "separate amounts" (p. 107).

If we chose not to take their advice, it is because the phrase containing "of" was intended to provide a quantitative theorization of the students' intuitive notion of fraction *of* something and it was sufficiently close to phrases they would spontaneously use when connecting their intuitive conception and the proposed definition. We did, however, take note of the fact that the part-of-a-whole conception of fraction is very strong. To counter the interpretation against which Thompson & Saldanha are warning in their paper, we devoted much attention to the distinction between objects and quantities in all activities throughout the course. In one of the first activities in the course, an apple would be brought to class, just like in the most traditional approaches to fractions (Colburn, 1884). But we did not just cut the apple in two more or less equal parts and expect the students to say that each part is "a half" of the whole apple. The apple was weighed on a kitchen scale, and its volume was measured by immersing the apple in a measuring cup filled with enough water. A single object – an apple – but two qualities, weight and volume, and two quantities, the measure of the weight in grams and the measure of the volume in millilitres. The apple was then cut in two parts. The parts were again weighed and their volumes were measured. Then we asked questions such as, *the weight (volume) of this part is what fraction of the weight (volume) of the whole apple?* Then the answers were not the pretty fraction $\frac{1}{2}$. The weight of a part of the apple was found, for example, to be $\frac{90}{165}$ of the weight of the whole apple. The fraction was obtained directly from the measures of the two quantities in grams. When students suggested to reduce the fraction to $\frac{18}{33}$, our response was, *yes, but what does this reduction mean in terms of the measures of the two quantities?* The part of the apple measures 18 units; the whole apple measures 33 units; what are those units? The students were led to see reduction of fractions as a result of the quantitative operation of change of unit.

Brousseau's Measurement Approach to Fractions

In the 1970s, in France, Guy Brousseau conducted long term design-based research on teaching fractions in elementary school (not in teacher education) using measurement situations. This research was first reported by one of Brousseau's doctoral students (Ratsimba-Rajohn, 1982), in French. A detailed account of this research is now available in English (Brousseau, Brousseau, & Warfield, 2014). In this research, the design of didactic situations for teaching fractions was based on a thorough epistemological study of the notion of fraction, resulting in a theoretical model called "a fundamental situation" for fractions, that is, a set of characteristics of problem situations in the solution of which fractions appear as a useful, if not optimal tool. As mentioned before, there is a definite kinship between the notion of "fundamental situation" in Brousseau's *Theory of Situations* (Brousseau, 1997) and Davydov's notion of "object source" of a mathematical concept.

Ratsimba-Rajohn (1982) derived the meaning of fraction of a quantity from two practices. One is the practice of obtaining a more precise measurement by division of a unit into smaller units (*fractionnement de l'unité,* p. 74), which corresponds to our *common unit definition*. The other (*commensuration*, p. 74-5) consists in evaluating the relative values of two quantities by equating their multiples. Ratsimba-Rajohn gives two examples of situations where this practice may occur: 1) evaluating the weight of one sheet of paper knowing that 540 such sheets weigh 64 grams; 2) reasoning in the context of barter exchanges, for example, evaluating the relative market values of some kind of fabric and of tea knowing that 20 meters of the fabric can be exchanged for 10 pounds of tea[12]. This practice leads to defining the meaning of the phrase "quantity A is the fraction $\frac{a}{b}$ of quantity B" by the condition $b \times A = a \times B$. We will refer to this definition as *barter definition*, as this name is shorter than "commensuration definition".

The research conducted by Ratsimba-Rajohn aimed at verifying the hypothesis that children are more likely to use strategies aligned with the barter definition in situations of measuring one quantity with another where a *small* multiple of the one can be easily found

[12] Ratsimba-Rajohn cites Karl Marx' *The Capital* for this particular example of barter transaction.

to be equal to a small multiple of the other, but will move to finding a common unit for the two quantities when those multiples would have to be very big for the quantities to be equal to each other. The research did not clearly confirm the hypothesis; children were often stuck with strategies aligned with the barter definition. Even after discovering the common unit strategy, children continued using the barter strategy. Sometimes they used the latter strategy to justify results obtained with the former (Ratsimba-Rajohn, 1982, p. 109).

Thus, it was the barter definition of fraction of a quantity that was at the basis of Brousseau's "thickness of a sheet of paper" situation by means of which children were to be first introduced to fractions (Brousseau, Brousseau, & Warfield, 2014). For children, the barter meaning was to be constructed as an optimal measurement strategy in situations of measuring something so thin that previously known techniques of direct measurement could not be applied. The problem situation designed and experimented with children involved measuring and comparing the thicknesses of various types of paper: the goal was to code the thickness by a pair of whole numbers (e.g., 20 sheets; 3 mm) and learn to know when these pairs represent the same thickness and how to tell if one such pair represents a thickness greater than another. Thus, from the outset, the abstract notion of fraction was aimed at.

As mentioned before, our experience in Year I advised us against passing directly from measurement activities to abstract fractions. In Years II and III, we decided to precede the abstraction of fraction as a number by a long period of working with fractions of quantities. This is one difference between Brousseau's approach and ours. Another difference is the relation between the two meanings of fraction of a quantity. In Year II of the experiment, we used only the common unit meaning in defining the notion of fraction of a quantity. In Year III, both the barter and the common unit definitions were introduced; the former was labeled DoF-I, the latter – DoF-II. We treated them as equivalent characterizations of the concept of fraction of a quantity; they had, for us, the same epistemological value as tools of reasoning. It can be proved that if A is $\frac{a}{b}$ of B in the sense of DoF-I, then A is also $\frac{a}{b}$ of B in the sense of DoF-II, and vice versa. In the course, students convinced themselves of this equivalence on generic examples (Yopp, Ely, & Johnson-Leung, 2015). In Brousseau's design, however, the barter and common unit meanings were treated as measurement strategies, not definitions, and it was assumed that the barter strategy was the more primitive one and special didactic situations were to be designed to help children move to the common unit strategy.

The Five-subconstructs Model of Fractions

By far the most widely known and applied conceptualization of fractions is the "five-constructs model" (Kieren, 1976; 1993; Behr, Lesh, Post, & Silver, 1983; Behr, Harel, Post, & Lesh, 1993; Niemi, 1996; Lamon, 2001; 2012; Charalambous & Pitta-Pantazi, 2007; Wong & Evans, 2008; Pantziara & Philippou, 2012). The measurement approach that we developed in our research takes into account all five aspects, but the classification of these aspects into five distinct subconstructs is not justified in the approach. In the research literature, each of the subconstructs is often associated with being able to answer certain questions (Charalambous & Pitta-Pantazi, 2007). For example, the Measure subconstruct is associated with questions about representation of fractions on a number line and finding fractions between two given ones. From the point of view of our approach, questions associated with a given construct would usually not be considered as belonging to the same type: our taxonomy of problems relies rather on the quantitative operation at stake in a given problem. Moreover, even the formulation of the problem would be different.

In our approach, especially in the bottom-up version, we emphasize the distinction between abstract fractions and fractions of quantities. Fractions of quantities are quantities. For example, $\frac{3}{4}\ L$ – or $\frac{3}{4}\ of\ 1\ L$ – is a quantity: it is the volume measuring three units such that four of these units make one litre (in this case the unit is $250\ mL$). Fractions as such, on the other hand, are abstract numbers; $\frac{3}{4}$ is the name of a number that multiplied by 4 gives 3. But in problems associated with the subconstructs model, a symbol of the form $\frac{a}{b}$ may mean a quantity or an abstract number depending on the context. In fact, being able to guess which meaning is intended in a given problem, and, in case a quantity is intended, guessing what is the referent quantity, are often considered evidence of understanding fractions (Charalambous & Pitta-Pantazi, 2007; Wong & Evans, 2008).

In our approach, the kind of quantity at play (whether it be number of objects, length, area, weight, volume, etc.) is explicit or clearly implied. There are no "wholes" and "parts"; there are two measured quantities. We would not consider problems such as the one in *Figure 4* as appropriate because the reference quantity is not specified and the formulation gives the learner permission to identify an abstract number ($\frac{2}{3}$) with an object (e.g., a partly shaded diagram) or an unspecified quantity (perhaps the area of the shaded part, or the number of the shaded parts, or the non-shaded parts, or something else yet).

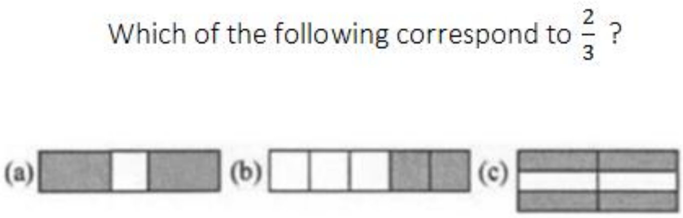

Figure 4. A question representing the Part-of-the-whole subconstruct in (Charalambous & Pitta-Pantazi, 2007).

The other question representing the *Part of the whole* construct in (Charalambous & Pitta-Pantazi, 2007) is: "If [drawing of 4 dots configured as on dice] represents $\frac{2}{3}$ of a set of marbles, draw the whole set of marbles". This question qualifies as representing the *Part of a whole* construct, because the marbles represented by the 4 dots can be thought of as part of the whole set of marbles and the fraction representing the relation is less than 1. In our approach, we would think not of objects but of their measures – in this case, two cardinalities, one being 4 and the other unknown – which are separate abstract entities, so the notion of a fraction being a part of a whole does not make sense.

In the style of the problems in our courses, we would rather ask, for example:

> Jack and Jill were playing a game of marbles. Jack lost 4 marbles to Jill and realized that he thus lost $\frac{2}{3}$ of the number of marbles he came with. How many marbles he came with?

This problem would be classified as a problem of the type "Given quantity is given fraction of WHAT QUANTITY? A problem such as the following would also fall into this type, although the compared sets of marbles are physically separate and the given fraction is greater than 1:

> Jack and Jill were playing a game of marbles. Before the start of the game, Jack had 12 marbles which was $\frac{6}{5}$ of the number of Jill's marbles. How many marbles did Jill bring to the game?

The point we want to make here is that the basis of classification of problems into types in our approach is quite different from the way problems are brought together from the point of view of the subconstructs model.

We distinguish three types of basic multiplicative problems about fractions of quantities, depending on which of the three variables in the statement "A is $\frac{a}{b}$ of B" is unknown:

Type "A": WHAT QUANTITY is the given fraction of the given quantity?
Type "a/b": Given quantity is WHAT FRACTION of another quantity?
Type "B": Given quantity is the given fraction of WHAT QUANTITY?

These problems qualify as multiplicative because, if we define the product of a fraction $\frac{a}{b}$ by a quantity B as the quantity A which is the fraction $\frac{a}{b}$ of B, then type "A" problems require multiplication of a fraction by a quantity; type "a/b" problems require division[13] of a quantity by a quantity, and type "B" problems require division of a quantity by a fraction.

Besides these basic multiplicative problems, the theory of fractions of quantities treats basic additive problems, where two fractions of the same quantity are added or subtracted to produce a fraction of that quantity, several types of problems about ordering fractions of quantities, and problems where two fractions of a unit of length are multiplied to produce a fraction of a unit of area. More complex problems require coordinated solving of several such basic types of problems.

For example, a mildly complex problem we gave as homework in the second week of Year III involved two type "A" problems, besides using the two equivalent characterizations of fraction of quantity, DoF-I and DoF-II. The situation in the problem was:

> Jack bought a tablet of dark chocolate: pure cocoa was 4/5 of the weight of the whole tablet.
>
> Jill bought a tablet of bitter-sweet chocolate. The weights of the tablets were the same: 100 g.
>
> To consume the same amount of pure cocoa, Jack would have to eat 3 of his tablets, and Jill - 4 of hers.

Two of the questions in the problem were:

> The amount of cocoa in the bitter-sweet chocolate tablet is what fraction of amount of cocoa in the dark chocolate tablet?
>
> If Jill ate a whole tablet of the bitter-sweet chocolate, how much cocoa has she consumed, in grams?

If we denote by Cd the weight of pure cocoa in the dark chocolate tablet and by Cb – the weight of pure cocoa in the bitter-sweet chocolate tablet, then the information in the problem tells us that $3 \times Cd = 4 \times Cb$. By the barter definition, this implies that Cb is

[13] Qualifying this operation as division follows from understanding division by something as the operation inverse to multiplication by this something.

$\frac{3}{4}$ of Cd. On the other hand, we are told that Cd is $\frac{4}{5}$ of the weight of the tablet, that is, of 100 g. [Problem of type "A"]. This means that if 100 g is 5 units then Cd is 4 of these units. Since one such unit is 100 g divided by 5, or 20 g, then Cd = 4 × 20 g = 80 g. So now we can find Cb, because we know that Cb is $\frac{3}{4}$ of 80 g. [Another problem of type "A"] Reasoning based on the common unit definition leads to Cb = (80 g : 4) × 3 = 60 g.

One would expect the Measure subconstruct to be the closest to our measurement approach. But here too, there is little similarity. This construct seems to be associated with representing fractions on the number line and various problems about ordering abstract fractions.

From the point of view of our measurement approach the problems quoted in (Charalambous & Pitta-Pantazi, 2007) as requiring this construct are quite heterogeneous. In Problem 1 a certain distance on the number line is given as representing the fraction $\frac{5}{9}$ of an unknown distance. This is a problem of type "B", just like the problem about marbles quoted above. For us, the two problems would be grouped together. In the subconstructs approach, they belong to different subconstructs.

The second problem (Problem 2) associated with the Measure subconstruct in (Charalambous & Pitta-Pantazi, 2007) is: "Name one fraction that appears between $\frac{1}{8}$ and $\frac{1}{9}$ ". For us, this problem belongs to the theory of abstract fractions where order is defined formally: $\frac{a}{b} < \frac{c}{d} \Leftrightarrow a \times d < b \times c$. In our approach, we would not place Problems 1 and 2 in the same group. They belong to different theories. Problem 1 belongs to the theory of fractions of quantities; Problem 2 – to the theory of abstract fractions.

In the theory of fractions of quantities questions about order would be embedded in a context of quantities, such as e.g., this problem, given in the final examination in Year II:

> A retailer purchased 9 gallons of canola oil and wants to put it in smaller cans for sale. He knows his customers will NOT be interested in buying less than $\frac{1}{5}$ gallon or more than $\frac{2}{5}$ of a gallon of oil at a time. If he decides to put the oil in $\frac{2}{5}$ - gallon cans, how many of them will he need? Show how you calculated the answer, and explain what operations on fractions of quantities you have performed.

The subconstruct of Ratio has something in common with our approach, because it is described as focused on "comparison between two quantities" (Charalambous & Pitta-Pantazi, 2007, p. 297) and is customarily associated with problems where the two quantities are themselves fractions of different quantities (e.g., problems comparing the taste of water solutions, their sweetness, salinity, acidity, "oranginess", etc.). In the bottom-up version of our measurement approach, introduced in Year II, such problems play an important role in that phase of the development of the theory of fractions of quantities where we work on the passage from fractions of quantities to abstract fractions.

In Year I of the experiment, at the beginning of the course, we used the word "ratio", without explicitly defining it, to define number (as ratio of a quantity to the unit in which it was measured) and fraction as a particular kind of number. As mentioned before, we were not satisfied with the effect of this approach on students' conceptions of fractions. So, in Year II, fraction of quantity was defined without reference to the idea of ratio. The notion of ratio was introduced by the end of the course, after the theory of fractions of quantities, and after a brief expository lecture about the mathematical theory of rational, irrational and real numbers. We needed the notion of real number to define ratio, because we wanted to highlight the fact that, given two quantities of the same kind, it may happen that neither is a fraction of the other, but we can always speak of their ratio. For example, the circumference of a circle is not a fraction of its diameter, but the ratio of these two lengths exists: it is the real number called π. In the textbook that we produced for the course, we defined what it means for one quantity to be in such and such ratio to another as follows (Sierpinska, 2015, p. 24):

> The ratio of one quantity to another can be expressed as an ordered pair of real numbers, or as a single real number.
>
> Let A and B be two quantities of the same kind, and let a and b be positive real numbers.
>
> DEFINITION OF RATIO AS A PAIR OF NUMBERS (DoR-I) We say that the ratio of A to B is $a : b$ if $b \times A = a \times B$.
>
> DEFINITION OF RATIO AS A SINGLE NUMBER (DoR-II) We say that the ratio of A to B is the number r, if $A = r \times B$.

These definitions were preceded and followed by examples of usage of the phrases "the ratio of … to … is…" and "is the fraction … of…" and how one can sometimes translate one into the other and sometimes not, and when it is more convenient to use a single number to represent the ratio and when a pair of numbers is more meaningful.

Teaching the Measurement Approach to Fractions to Prospective Elementary Teachers

We did not expect students to get accustomed to the measurement approach quickly. Especially, we did not expect them to find it easy to conduct reasonings based on definitions. Even mathematics majors learn it only at the university, sometimes in special "Introduction to mathematical thinking" courses. So we took the time to nurture their understanding by offering many problems to solve and then discussing their solutions with them. There were weekly assignments composed of up to 10 short quiz-type exercises (graded electronically) and 1-2 longer essay-type questions (graded manually, with individual comments explaining any mistakes); classroom discussions in interactive lectures (2 hours weekly) and labs (1 ¼ h weekly). There was a mid-term exam whose purpose was formative, rather than evaluative. There was also a final examination, whose purpose was evaluative but its weight in the course was only 40%. The whole unit on fractions lasted 8 weeks each year. A subset of the exercises and problems used in the experiments can be gleaned from the texts used for the courses (Sierpinska & Bobos, 2015; Sierpinska, 2015). Examples have also been given in the previous sections of this paper.

A brief outline of the mathematical organization[14] of the content of the courses in Years II and III of our experiment has been given in the Introduction. Some details of the didactic organization were given in the examples that we used to contrast our approach with other approaches in section 4. Here, we offer more information on the didactic organization of the beginning of the courses, focusing on the introduction of the definitions of fraction of a quantity. We also give examples of students' responses to the approach, focusing on issues related to students' understanding of the definitions.

[14] We use the terms *mathematical* and *didactic organization* of content in a course in the sense of the Anthropological Theory of the Didactic (Chevallard, 1999; Barbé, Bosch, Espinoza, & Gascón, 2005).

Definitions of the Phrase, "Quantity A is the fraction a/b of quantity B"

The courses started by posing the question: What does it mean for one quantity to be a fraction of another quantity? This required clarifying what the instructor meant by quantity and was an opportunity to distinguish between quantities and abstract numbers. In Year II, hands-on activities of measuring quantities (weights / volumes of an apple and its parts, weights of a pizza and its slices) led to distinguishing quantities from objects and rephrasing sentences about fractions *of something* as sentences about fractions *of quantities*. The meaning of these sentences was then stated, by the instructor, in the form of the common unit definition:

> Let $Q1$ and $Q2$ be quantities of the same kind. Let a and b be two whole numbers, with $b \neq 0$. We say that **the quantity $\boldsymbol{Q1}$ is the fraction $\boldsymbol{\frac{a}{b}}$ of the quantity $\boldsymbol{Q2}$** if there exists a common unit $\boldsymbol{u}$ such that $Q1$ measures a units $\boldsymbol{u}$ and $Q2$ measures b units $\boldsymbol{u}$. (Year II definition of fraction of a quantity; emphasis in the original)

In Year III, additionally, the barter meaning was introduced, based on, first, a story of a transaction whereby 5 sheep were exchanged for 3 cows, and then by a hands-on activity of measuring the length of a coffee stirrer with the length of a toothpick. The definitions, as already mentioned, were labelled DoF-I (barter) and DoF-II (common unit).

> DoF-I: If two non-zero quantities, A and B, are related so that some whole multiple of one of them is equal to some whole multiple of the other, i.e., $m \times A = n \times B$ where n and m are whole numbers, then we say that A is the fraction $\frac{n}{m}$ of B.

> DoF-II: We say that a quantity A is a fraction of a non-zero quantity B, if we can find a common unit $\boldsymbol{u}$ such that both quantities measure whole numbers of such units. If A measures n units $\boldsymbol{u}$ and B measures m units $\boldsymbol{u}$, where n and m are whole numbers, then we say that A is the fraction $\frac{n}{m}$ of B.

In Year III, in line with our pursuit of fostering systemic thinking, we convinced the students that the two definitions are equivalent: if A is $\frac{a}{b}$ of B in the sense of one definition then A is also $\frac{a}{b}$ of B in the sense of the other. This was done, in class, by means of questions such those below:

> 1. Given two quantities A and B such that $2 \times A = 3 \times B$, explain why there must be a unit u such that A measures 3 units u and B measures 2 units u.
>
> 2. Given two quantities A and B. For some unit u, A measures 4 units u and B measures 5 units u. Explain why the quantities $5 \times A$ and $4 \times B$ must be equal.

Students first worked on these examples in small groups at their desks and then a few were asked to present their reasoning to the whole class. The teacher's role in these presentations was to turn these examples into generic example proofs (Yopp, Ely, & Johnson-Leung, 2015), by drawing students' attention away from these particular values as such, and pointing to the generalizable elements of the reasoning. Line segment diagrams were used as a visual support (Figure 5).

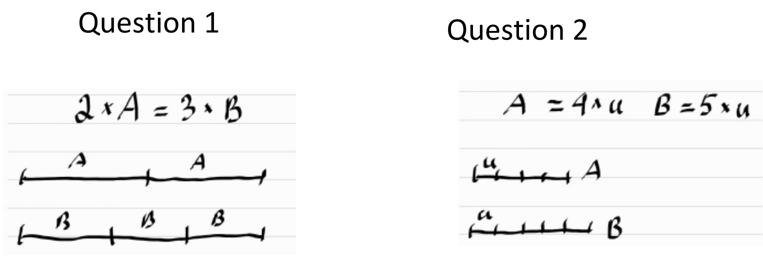

Figure 5. Hand drawn line segment diagrams used to represent the given relations in questions aimed at demonstrating the equivalence of the common unit and the barter definitions of fraction of a quantity.

Identifying the Quantities Involved in the Fractional Relation

An important issue in the visual approach are mistakes related to not asking oneself “What is the unit?” in a problem about fractions (Lamon, 2012, p. 98). The problem is exacerbated by the fact that, in the visual approach, the reference quantity is often implicit and/or ambiguous. But this problem exists also in the measurement approach. In spite of efforts to make quantities explicit and to avoid ambiguity, unintended interpretations do occur. Hence the need to confront students with situations similar to those they know from learning fractions by the visual approach where their old and new understandings of fractions would compete in identifying the quantities considered in a fractional relation.

Below we give examples of tasks that proved effective in revealing these competing understandings in students in our experiment.

Equal Sharing Tasks – Probing Students’ Understanding of the Common Unit Definition

Equal sharing problems seemed quite likely to provoke discussions about what quantities are being compared in a problem. After a sharing situation was described, two questions were asked: one where each person’s share was to be expressed as a fraction of the total amount shared, and one where each person’s share was to be expressed as a fraction of the unit in which the total amount was measured. For example,

> Five people are sharing, equally, 8 identical small thin-crust pizzas, topped with a “Hearty Marinara” sauce and cheese. Assume we measure the quantity of pizza by its weight. Each pizza weighs 355 g. Justify your answers.
>
> a. What fraction of the 8 pizzas will each person get?
>
> b. What fraction of 1 pizza will each person get?

When first confronted with a problem of this type in Year II, quite a few students produced the same fraction in both questions: one-fifth. Here is how one of them reasoned: Each pizza will be cut into 5 slices; a rough drawing is made (*Figure 6*).

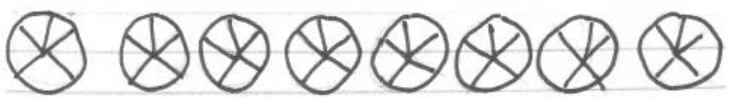

Figure 6. A student's representation of 8 pizzas cut into 5 slices each.

Then the student wrote:

> 355 g per pizza a total of 2840 g of pizza. If 1 pizza is split 5 ways each piece is 71 g. One person will get 71 g of pizza 8 times = 568 g in total. 568 g is what fraction of 2840 g?

The student's answer to that question is shown in *Figure 7* . The student swiftly switched from fractions of quantities to formal operations on abstract fractions.

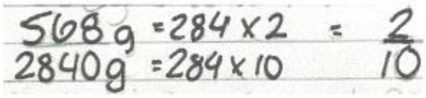

Figure 7. The student's answer to "568 g is what fraction of 2840 g?"

The student's final answer to question (a) was: "therefore 1 person would get $\frac{2}{10}$ of 8 pizzas."

This student's response to question (b) was based on the assumption of the particular way of slicing the pizzas and sharing them: each person takes one slice from each pizza that has been cut into 5 slices. Since one slice weighs 71 g and the whole pizza weighs 355 g, one slice is two-tenths of the whole pizza. This last fraction is obtained, again, by formal fraction reduction. Hence each person gets $\frac{2}{10}$ of one pizza. (*Figure 8*)

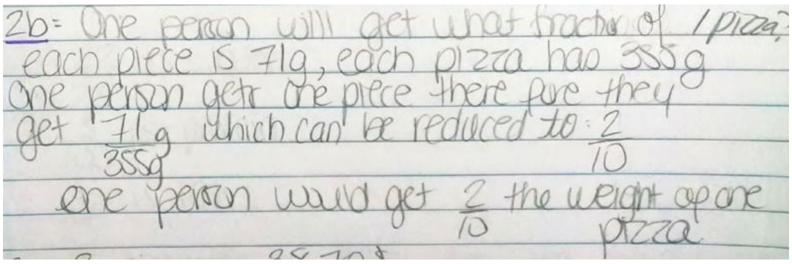

Figure 8. The student's response to question (b).

These problems and students' solutions were discussed in the labs with them.

Probing Students' Understanding of the Barter Definition

The following problem, which appears to be an easy application of the barter definition, produced nevertheless many incorrect responses. Many students did not correctly distinguish between the quantities and the multipliers in an implicitly described relation between the capacities of tall and short glasses:

> (Year-III-Midterm-Question 2) **a**. A jar can be filled to capacity with either 6 tall glasses or 9 short glasses of water. What fraction of the capacity of the tall glass is the capacity of the short glass? Derive your answer from one of the definitions of a fraction of a quantity. **b.** A jar can be filled to capacity with either 6 tall glasses or 9 short glasses of water. If the capacity of the short glass is 30 mL, what must be the capacity of the tall glass? Show your reasoning. **c.** A jar can be filled to capacity with either $6\frac{3}{5}$ tall glasses or $9\frac{3}{10}$ short glasses of water. What fraction of the capacity of the tall glass is the capacity of the short glass? Show your reasoning.

In this question, the quantities to compare – capacity of the short glass, and capacity of the tall glass – are explicitly mentioned, but the capacities are not specified in terms of a standard unit (e.g., mL), but by their relations with a third quantity, the capacity of a jar. The equality 6 × capacity of tall glass = 9 × capacity of sort glass must be deduced from the text of the problem; it is not explicitly given. Moreover, the order of the quantities is reversed in the question compared to the phrase "capacity of the short glass is $\frac{6}{9}$ (or $\frac{2}{3}$) of the capacity of the tall glass" which is expected as an answer.

The question 2b is much simpler and it is possible to solve it without using fractions. The objective of this question was to provide students with an opportunity to verify their answers to 2a. If they obtained an incorrect answer in 2a and used it to calculate the answer to 2b, they were likely to obtain an obviously absurd answer (the tall glass having smaller capacity than the short glass). This might motivate them to go back to Question 2a and revise their solution; this would be taken as evidence of *reflective thinking*.

Question 2c has the same structure as 2a but it is more difficult because the multipliers are not whole numbers. The notion of fraction of a quantity is less easily avoided. The given information can be written as $6\frac{3}{5} \times \text{capacity of tall glass} = 9\frac{3}{10} \times \text{capacity of short glass}$, but one cannot directly apply the barter definition to obtain the answer as it was possible in Q2a. This information has to be processed using the notion of fraction of a quantity so that an equality of whole number multiples of the capacities of the tall and short glasses is obtained. Therefore this question was a sharper diagnostic instrument to test students' understanding of fractions of quantities.

But even the question 2a was not straightforward and of the 34 students writing the test, only 13 (38%) obtained a numerically correct answer (the fraction 6/9 or 2/3), and the numerically correct answer was accompanied by a satisfactory quantitative interpretation in only 10 of these cases. The solution of one of the 3 students whose answer was numerically but not quantitatively satisfactory is transcribed in *Figure 9*.

6 tall short is / of tall
9 short short is 9/6

tall glass = 9/6 of short glass

$\boxed{\text{short glass = 6/9 of tall glass}}$ or $\boxed{2/3}$

A = n/m of B

short glass is n/m of tall glass

6 is n/m of 9

6 x m = 9 x n

n/m = 6/9 = 2/3

Figure 9. Transcript of a student's solution to Question 2a (Year III, Midterm).

The student appears to interpret the numbers 6 and 9 as representing the quantities, not the multipliers: the arrows in her solution (Figure 9) connect "short glass" with the abstract number "6" and "tall glass" with "9." In the problem as stated, 6 and 9 are multipliers and the quantities are the capacities of the tall and short glasses. What this student writes does make sense, however, because the relationship "9 × capacity of short glass = 6 × capacity of tall glass" indeed implies that if the capacity of the short glass is 6 units then the capacity of the tall glass is 9 units. Had she made this explicit, her reasoning would be quantitatively satisfactory. But she did not represent this equality in any way in writing. We can't say if she conceived of the 6 and 9 as being numbers of some units of capacity and she was only unable to put it in words or if she merely had a vague sense that a short glass cannot be a fraction greater than 1 of the tall glass, which is a mixture of thinking of fractions as abstract numbers and a quantitative intuition. If we interpret what she wrote literally, she might have been solving the numerical problem: "6 is WHAT FRACTION of 9?" Or she may have just looked at the two numbers in the problem, 6 and 9, and knew that the answer should be $\frac{9}{6}$ or $\frac{6}{9}$; she chose the latter because it made more sense in the statement, "short glass is $\frac{6}{9}$ of tall glass." She wrote this answer first; it may be that the rest of her solution is just a formal discourse that she believed the teacher expected from her by way of justification based on a definition of fraction of a quantity.[15] She did not derive her answer from a definition. Similar behaviour could be observed in the other two students who obtained numerically correct answers in Question 2a, without evidence of satisfactory quantitative reasoning.

In total, there was evidence that 8 (24%) students understood the problem as asking: "the number of short glasses is WHAT FRACTION of the number of tall glasses", and 3 as "the number of tall glasses is WHAT FRACTION of the number of short glasses."

Question 2b was generally well done, with 24 (71%) of solutions correct. But 17 of these solutions (50% of all solutions) did not use fractions. Of those 17, 13 solutions were correct. In the correct solutions, quantitative reasoning on whole numbers of millilitres and numbers of glasses was used: the capacity of the jar was calculated as 30 mL × 9 = 270 mL, and then this quantity was divided by the number of tall glasses: 270 mL ÷ 6 = 45 mL to obtain the capacity of the tall glass.

Among the four incorrect solutions that did not use fractions, three *contradicted quantitative common sense*: the capacity of the tall glass was less than the capacity of the short glass. Such lack of sensitivity to quantitative contradictions appeared also in 5 students who did use fractions to solve Q2b. Only two of these 8 students appeared to notice the contradictions and proceeded (albeit unsuccessfully) to revise their solutions, thus displaying some *reflective thinking*. Other students were satisfied with producing a number, any number, and moved on to the next question.

Among students who used fractions in Question 2b (17 or 50%), there were 5 students who used the incorrect fraction they obtained in 2a to solve 2b. Their results also contradicted quantitative sense – the capacity of the tall glass was less than the capacity of the short glass – yet this did not entice them to revise their solutions to Q2a. Again – thus exhibiting a lack of *reflective thinking*.

Question 2c was, predictably, the least well done. While 13 students (38%) obtained numerically correct answers in 2a, only 7 students (21%) did so in 2c. Six of these 7 had also numerically correct answers in 2a. The student who obtained an incorrect answer in

[15] This behaviour is similar to one known in research on undergraduate students' difficulties in Linear Algebra as symptomatic of the "obstacle of formalism" (Dorier, Robert, Robinet, & Rogalski, 2000).

2a but a numerically correct answer in 2c, was not reasoning quantitatively in Q2c: she was solving the numerical problem "$6\frac{3}{5}$ is WHAT FRACTION of $9\frac{3}{10}$?"

Ten of the numerically correct solutions in Q2a were based on (or justified with) a satisfactory quantitative reasoning, but even this did not guarantee such reasoning in Q2c: there were only 4 such solutions. Most students were solving the problem "One of the given numbers is what fraction of the other given number?", confusing quantities and multipliers. They obtained the answers by dividing one of the given multipliers by the other, formally, as abstract fractions, with most not stopping to reflect on the quantitative sensibility of their results. The reflection did come later, however, when the tests were returned: this problem was thoroughly discussed in the following lab.

Verifying if a Given Quantity is Indeed a Given Fraction of Another Given Quantity

Another type of questions that served specifically to help students understand the definition(s) of a fraction of a quantity was to verify if a given quantity A is both a given fraction $\frac{a}{b}$ and another given fraction $\frac{c}{d}$ of another given quantity B. The fractions $\frac{a}{b}$ and $\frac{c}{d}$ were chosen to be equivalent as abstract fractions but the first quantity was not necessarily this particular fraction of the second. For example, the statement S to verify could be:

$$1\frac{1}{8}\ lbs \text{ is both } \frac{3}{7} \text{ and } \frac{24}{56} \text{ of } 2\frac{3}{4}\ lbs \qquad (S)$$

S is a conjunction of two statements, $S1$ and $S2$, both of which have to be true for the statement to be true:

$$\underbrace{\left[1\frac{1}{8}\mathrm{l}b \;\; is \;\; \frac{3}{7} \;\; of \;\; 2\frac{3}{4}lb\right]}_{S1} \; AND \; \underbrace{\left[1\frac{1}{8}lb \;\; is \;\; \frac{24}{56} \;\; of \;\; 2\frac{3}{4}lb\right]}_{S2}$$

For $S1$ to be true, by the common unit definition, there should be a unit u such that $1\frac{1}{8}\ lb = 3 \times u$ and $2\frac{3}{4}\ lb = 7 \times u$. But this is impossible because the unit which satisfies the first condition ($\frac{3}{8}lb$ or $6\ oz$) does not satisfy the second condition. So $S1$ is false and this is enough to conclude that S is false. If $S1$ were true, then it would be necessary to check if $S2$ is also true. But if $S1$ was true, then $S2$ would also be true because 24 and 56 are 8 times larger than 3 and 7, respectively; so it would be enough to take a common unit 8 times smaller than the one responsible for the validity of $S1$.

If the barter definition is available, then it is enough to verify if 7 times the first quantity is the same amount as 3 times the second quantity (for $S1$), and if 56 times the first quantity is the same amount as 24 times the second quantity (for $S2$). Again, if $S1$ is true then $S2$ is true and vice versa.

Such questions were discussed in class, in the labs, given as homework and were included in the midterm and final examinations. We give a brief account of Year II students' solutions to a problem of this type in their final examination, which took place 5 weeks after the end of the unit on fractions in the course. The question was:

> (YEAR II – FINAL – Question 1b) Use the definition of a fraction of a quantity to justify the statement: $1\frac{1}{8}\ lb$ is both $\frac{3}{7}$ and $\frac{24}{56}$ of $2\frac{3}{4}\ lb$.

Recall that the barter definition was not available in Year II. So "the definition" refers to the common unit definition of fraction of a quantity.

The task is "to justify", but the statement is false. For students who tend to ignore the quantitative context of problems and focus on numerical values alone, the fact that $\frac{3}{7}$ and $\frac{24}{56}$ are equivalent as abstract numbers could easily create the illusion that the statement is true and it is not necessary to check the other elements of it.

Of the 38 students who wrote the final examination in Year II, only 5 (13%) did not fall into the trap and found that neither $S1$ nor $S2$ are true. These five students considered the concrete values of the given quantities to establish the relation between them and check it against the given fractions – before discussing the equivalence of the two abstract fractions. For these students, the definition acquired full operational power, leading them to disprove the statement when one of the main definitional conditions was not satisfied. Each solution could be discussed as an interesting case study with regard to the issue of transfer of knowledge disseminated in class: despite the fact that the question was far from being open-ended it spurred quite different problem solving behaviours among the five students in this group. One student, very practically, converted the given quantities to ounces to easily observe that they are not in the given relationship; his solution was only a few lines. Another student meticulously checked all the conditions of the definition – even the non-essential ones – to produce a rigorous proof that the statement is false; her solution covered a couple of pages. All 5 students in this group displayed features of theoretical thinking, particularly reflective and systemic-definitional thinking and did not confuse quantities with abstract numbers.

Another 10 (26%) students interpreted S as a statement about equivalence of fractions of quantities; the problem they appeared to be solving was to show that a quantity – in general, not the given particular quantity – can be both $\frac{3}{7}$ and $\frac{24}{56}$ of another quantity. The main characteristic of these students' solutions was that they ignored the concrete values of the quantities in their reasoning and thus failed to notice that statements $S1$ and $S2$ were false. In logical terms, the solutions of this group contained the proof of the conjunction of $S1$ and $S2$, *assuming* that $S1$ and $S2$ are true – a condition they failed to check. Some students indeed referred to the quantities as variables, as in the following response:

> A quantity Q1 is 24/56 of a quantity Q2 when measured with a common unit u. When both quantities are measured with a unit u' such that u'=8u then Q1 which was 24u becomes 3u' and Q2 which was 56u becomes 7u'. Q1 is 3/7 of Q2.

Other students in this group featured the particular values of the given quantities in their solutions, but did not use these values, treating them as if they were letter variables. Responses in this group contained sensible arguments, capturing the essence of the problem: the equivalence of fractions of quantities. Students proved the equivalence by demonstrating a quantitative relation between two units used to measure the two given quantities. In doing this they resorted to the theoretical constructs provided in the course and made them operational for producing justifications within the given system (displaying *systemic-definitional thinking*). Their productions read mostly well, as coherent written discourse with few inconsistencies, and mathematical notation used adequately (good *analytical thinking*). All of the students in this group failed to establish the statement as false because they *didn't engage in hypothetical thinking* so as to ask if the relationship between the given quantities is indeed $\frac{3}{7}$ (or $\frac{24}{56}$) before proceeding to show that it is both one and the other of these fractions. This flaw in their arguments could be attributed to a logical miss: proving only that $S1$ implies $S2$ (or that $S2$ implies $S1$) without validating any of these statements.

In the remaining 23 (61%) solutions, the focus on numerical values and omission of the quantitative context of the problem were very present and there were serious deficiencies of theoretical thinking. There was a great variety of interpretations and strategies used. A detailed analysis of these solutions is available in the first author's doctoral thesis (Bobos-Kristof, 2015).

Overall, of the 38 students, 15 or about 40%, engaged in quantitative reasoning, and in justifications using the theoretical tools provided in the course. Yet more than a half of the students remained confined within the numerical domain of their previous knowledge of fractions, often in a procedural fashion, appropriating only superficial aspects of the taught theory of fractions of quantities.

To Add or Not to Add the Barter Definition in the Course

Such distribution of highly positive and highly negative results posed a dilemma for us after the Year II experiment: should we renounce teaching something that is too difficult for some students or should we teach it nonetheless for those who stand to greatly benefit from it? We decided to try again next year, after a revision of the approach. We decided to add the barter definition early in the course, since it certainly simplified reasoning in the verification problems and in many other problems as well. It turned out – as could be expected – that while this change solved some issues, it also created another: it provoked some students to replace theoretical reasoning by performing formal algebraic operations on equations.

There were not many such students; most students had bad experience with algebra in secondary school and did not want to return to it. A few, however, interpreted the barter definition not as a definition, but as a result of multiplying both sides of the equation $A = \frac{a}{b} \times B$ by b. They felt they did not need a definition of the meaning of this equation; after all, they knew from their previous education what it means for one quantity to be a fraction of another quantity. All they needed was a procedure to calculate the A, the $\frac{a}{b}$ or the B, whichever happened to be unknown in the problem. Their ease with algebraic operations worked, in fact, as an obstacle against their engagement with *definitional thinking* about fractions of quantities.

This *algebraic obstacle*, as we called it, did not appear in Year II where only the common unit definition was used. The definitional condition cannot be derived algebraically from the phrase A is $\frac{n}{m}$ of B. This definition is close to the idea of fraction of something and as students had no choice but to use it in many problems about quantities, it gradually superseded the material conception in some students – as many as almost half of them by the end of the course. A few, as discussed above, even exhibited a very high level of mathematical sophistication.

There was evidence, however, in the students' questions and solutions, that some continued to understand the common unit definition as merely an unnecessarily complicated way of saying that a fraction is just a part of a whole, which is *what I know anyway, no need for all those u's and A's*.

The common unit definition is structurally more complex than the barter definition, so verifying if a given quantity is the given fraction of another given quantity using this definition poses a greater challenge. Such problems were relatively easy for Year III students to answer correctly (although their justifications were not always as desired), most using the barter definition. Year II students struggled to even assess the validity of such statements.

Conclusions

This paper is a contribution, on the one hand, to research experimenting with Davydovian approaches to teaching mathematics, and, on the other – to "an enduring discussion of what the community intends that students learn" which Thompson & Saldanha believe is one of the conditions for improved instruction:

> When teachers and teacher educators lack clarity and conceptual coherence in what they intend students learn, they introduce a systematic disconnection between instruction and learning. This becomes a major obstacle to creating instruction that empowers students to think mathematically. (Thompson & Saldanha, 2003, p. 110)

In this paper, we tried to be as clear as possible within the limits of a journal article about what we intended prospective elementary teachers in our design experiment to learn about fractions. We are not claiming that this is what they should learn. We only make clear what we intended them to learn and what challenges we had to face.

Our research shows that connection between the material and the formal conceptions of fractions remains difficult to achieve. At the same time, it also shows that the measurement approach exposes the disconnection and thereby gives the instructor and the students the opportunity to become aware of it and to try to overcome it. The measurement approach rests on two conceptual pillars – quantitative reasoning and theoretical thinking in mathematics – neither of which is well developed in students at the start of their undergraduate studies whether in education or in other domains, such as mathematics. The instructor must be prepared to cope with this state of affairs and teach students to look at material objects from a quantitative angle and at mathematical statements from a theoretical perspective. He or she must also be prepared for not achieving success with all students; they are adults, with well-entrenched habits of mind that may be radically different from quantitative and theoretical ways of reasoning.

One might conclude from our report that the measurement approach presented here is too complicated and that the visual approach such as represented, for example, in (Parker & Baldridge, 2003), is much easier. We have heard this argument when presenting our research to colleagues[16].

It is easy to understand why the visual approach is more popular than the measurement approach. The language is simpler – pictures of pizzas, cakes and partly shaded figures are sometimes used instead of explicit reasoning – and knowledge about standard units of measure is not necessary. There is the impression of comparing objects directly, not indirectly, by means of their measures in concrete units as in the measurement approach. But this is only an illusion:

[16] We take solace in knowing that Thompson and Saldanha have heard such remarks as well, when presenting their theory of understanding fractions but had not for that reason stopped believing in the soundness of their theory (Thompson & Saldanha, 2003).

> … the illusion of an immediate transition from integers to fractional inversion by means of examples such as the division of four apples among three persons. In this case, they have ignored the fact that the operation (4 apples) ÷ 3 = $1\frac{1}{3}$ apples would make sense only if all four apples were identical in size. Here 'one apple' has necessarily acted as the unit of measurement of volume or of weight. (Kolmogorov, 1960), quoted in (Davydov & Tsvetkovich, 1991, p. 104)

Thus, the visual approach overtly speaks of dividing objects into parts, but implicitly treats these objects (one apple, one pizza, one circle, etc.) as some natural or almost standard units of measure and expects children to understand them this way. Some children catch on, some don't.

We are not looking for an easier approach. We are looking for an approach which is grounded in the concrete reality, but is also, ultimately, amenable to engaging in mathematical thinking. We concur with Davydov and Tsvetkovich, that

> [t]he concept of fractional number is best introduced when it is based on their real source of origin – the measurement of quantities. This method is completely justified from the standpoint of mathematics itself…. It is also highly productive from the psychological standpoint because it provides a connection between the concept, the process of its origination, and original material content and creates favorable conditions for the students subsequently to form an abstraction of the relationship between quantities which is contained in the very form of a fractional number. This abstraction is very important both for the entire school mathematics course and for the general development of the students' theoretical thinking…. (Davydov & Tsvetkovich, 1991, p. 107)

Indeed, whatever the approach taken, fractions will remain a difficult subject to teach and to learn, be it anew – for children, or on second take – for prospective teachers. Understanding of fractions cannot be made easy by hands-on activities such as cutting objects into pieces, playing with colourful fraction apps, or displaying fraction signs in ingenious diagrams as in the visual approach[17], nor by making the quantities, units and quantitative operations explicit in careful logical reasoning as in the measurement approach. It cannot be made easy because understanding fractions requires *abstraction of a second order.*

For understanding whole numbers, a person has only to abstract from qualitative characteristics of objects: e.g., abstract from the colour, the size and the mutual position of a collection of marbles and think only of their number. But if the question is what fraction one quantity is of another one, it is necessary to make not only the previous qualitative abstraction to find the measures of these quantities, but also to establish a *quantitative relationship* between these quantities. One must be "thinking in [terms of] abstract relationships" (Davydov & Tsvetkovich, 1991, p. 87). It is necessary to construct a complex system of conceptual schemas related to proportionality, measurement, multiplication and division (Thompson & Saldanha, 2003).

There are, however, choices to be made regarding the mathematical and didactic organizations (Chevallard, 1999; Barbé, Bosch, Espinoza, & Gascón, 2005) of content when using the measurement approach to fractions in courses for prospective elementary teachers. There are choices to be made regarding the definitions of fraction of a quantity, the order in which operations on fractions of quantities are to be studied, what tasks and problems students will be asked to work on, etc. The mathematical and didactic organizations of content in Years II and III were not exactly the same. In particular, in Year III, the barter definition of fraction of quantity was added and we have often reflected

[17] Teachers prepared using the visual approach do not seem to understand fractions well enough for teaching them to children. (Ma, 1999)

on the consequences of this decision on the quality of students' mathematical thinking. Addition of the barter definition simplified the reasoning, but at the cost of rendering less visible the operation of conversion of unit of measurement which is essential in understanding the multiplicative nature of fractional relations (Davydov V. V., 1991). It often worked better to prove *that* something is the case than it explained *why* it is the case. Some students complained about having no feel for the barter definition, even if they had a preference for it in deciding about the validity of statements of the form $A\ is\ \frac{n}{m}\ of\ B$. Their arguments, however, were often symptomatic of the algebraic obstacle. The barter definition appeared to make it more difficult for them to engage in systemic-definitional thinking.

Given these pros and cons, we think that, in a future realization of the approach, we would introduce the common unit definition at the beginning of the course, let students get used to it and work with it on a variety of problems and only in a third or fourth week ask them to think about the validity of the barter characterization and check if it is equivalent to the common unit definition.

But we could not have wished for the algebraic obstacle or other obstacles, such as the tendency to ignore units and focus on numerical values of quantities only, not to appear. Future teachers have them anyway; they have been constructing them all through their elementary and secondary education. If these obstacles are not revealed, the teacher cannot help the students to overcome them. Since the barter definition is a good trigger for especially the algebraic obstacle to reveal itself, it is a useful instructional tool.

The conclusion from our research is not that the approach *doesn't work* and should be abandoned. We could say that *it works if you work it*: the difficulties and obstacles revealed in the research are a warning for both students and teachers that they will have to work hard for understanding to develop. It points to aspects of thinking on which they will have to work particularly hard.

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