

Elementary School Students' and Prospective Teachers' Proportional Reasoning Skills

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The purpose of this study was to investigate how prospective elementary school mathematics teachers and elementary school students' ability to distinguish proportional situations from nonproportional situations differ or are alike. Participants of the study were 319 prospective primary and middle school teachers and 320 elementary school students from fourth to seventh grades. Four problems consisting of two proportional and two nonproportional situations were used as a data collection tool. Participants' strategies were coded as multiplicative, additive, and other. Findings indicated that while fourth, fifth, and sixth graders were dominantly additive reasoners, prospective teachers and seventh graders were multiplicative reasoners. In addition, it was found that the overgeneralization of proportionality seems to begin with the instruction on the cross-multiplication algorithm. Implications for instruction on proportion, mathematics teacher education programs and future research are discussed.

Introduction

Teachers' mathematical knowledge has an important effect on instruction and student achievement (Coleman, 1966; Fennema & Franke, 1992; Hill, Rowan & Ball, 2005; Sowder, Phillip, Armstrong, & Shappelle, 1998). There is no doubt that, for students' substantive learning, mathematics teachers must know the content they will teach in a deep and connected way (Ball, Thames & Phelps, 2008; Ma, 1999). However, prior research findings indicate that, pre-service and in-service elementary teachers don't have a deep understanding of the mathematics they teach (Ball, 1990; Fennema & Franke, 1992; Van Es & Conroy, 2009; Toluk-Ucar, 2009). Many pre-service teachers and mathematics teachers' content and pedagogical knowledge are weak to teach mathematics effectively (Ball, 1990; Ma, 1999; Tirosch, 2000). They generally focus on procedures and rules and base their reasoning on procedures (Ball, 1990; Son, 2006). Thereby, they lacked connected conceptual ideas and procedures (Feiman-Nemser, 2001). Similarly, elementary school students tend to focus on procedural knowledge and learn mathematical concepts in algorithmic manner without a connection to other mathematical concepts (Kwon, Park & Park, 2006). In accordance with this information, we think that it is worth to examine prospective elementary school teachers' and elementary school students' understanding about mathematical concepts. As a result, we may have an opportunity to understand similarities and differences between prospective teachers and elementary school students' thinking on the same mathematical concept and draw some conclusions for teaching and teacher education. In this study, we specifically focused on proportional reasoning as a content.

Literature review shows that studies on proportional reasoning mostly focus on elementary school students and early age children. A few studies exist with prospective teachers and teachers. Also, existing studies focus on teachers' problem solving, problem posing, proportional reasoning skills and strategies (Lo, 2004; Lobato, Orill, Druken & Jacobson, 2011; Masters, 2012; Monteiro, 2003). Based on this idea, we investigated prospective primary and middle school mathematics teachers' and elementary school

students' ability to distinguish additive and proportional situations. By comparing prospective teachers and students' performances on both problem situations we may have a clear understanding of deficiencies, if exist, in participants' understanding of such situations.

Proportional Reasoning

Proportional reasoning concepts are used in everyday life as well as in mathematics and science. If proportional reasoning is not understood conceptually, but algorithmically, it becomes more difficult to transfer and use in daily life (Dooley, 2006). For example, calculating how many hours is taken for travelling 200 km by a bus or by a car or comparing prices of the same product involve proportional reasoning. So, proportional reasoning plays a critical role in a student's mathematical development that it has been called as a pivotal concept (Lesh, Post & Behr, 1988). With Lesh, Post and Behr's terms, proportional reasoning can be considered as the capstone of elementary arithmetic, number and measurement concepts and as the cornerstone of algebra and higher mathematics.

Proportional reasoning has been defined in various ways. Karplus, Pulos and Stage (1983) defined proportional reasoning as involving a linear relationship ($y=ax$, $a \neq 0$) between two variables. On the other hand, Lesh, Post and Behr (1988), stated that "proportional reasoning is a form of mathematical reasoning that involves a sense of co-variation and of multiple comparisons, and the ability to mentally store and process several pieces of information" (p. 93). Despite its importance, proportional reasoning has merely been considered as the ability to solve proportional missing value problems (Cramer, Post & Currier, 1993) and comparison problems (Noelting, 1980). But this is not an adequate view because proportional reasoning needs to use multiplicative relationships to compare the numbers and determine the amount of a quantity based on other values. In other words, proportional reasoning includes a process of understanding relationships involved in a situation (Lamon, 2007).

Teachers should recognize that the development of proportional reasoning is not an easy task. Teaching proportion is more than merely teaching of the cross-multiplication algorithm for solving missing value problems. Rather, they need to know that there are semantically different types of proportion problems and that each problem type requires different cognitive skills. As Lamon (1993) identified, part-part-whole problems (e.g. groups of people with a ratio of 15 women to 20 men), associated sets problems (e.g., pizza and person), well-known measure problems (e.g., speed in kilometers per hour) and growth problems (enlarging or shrinking) elicit different solution strategies regardless of a student's level of understanding of proportional reasoning. After a study with sixth graders, Lamon found that while associated sets problems demand a higher level of proportional reasoning strategies and elicit the language of ratio more naturally, students are inclined to use informal methods when solving part-part-whole problems. Even when students could think proportionally in other situations, they tend to use counting and building-up strategies for part-part-whole problems. Among the problem types, growth problems are the most difficult problem type because unlike the part-part-whole and associated-sets problems, these problems involve continuous quantities that make them difficult for students to model with objects or pictures. Especially in well-known measure problems, because of the familiarity of the context, students tend to apply erroneous strategies such as approaching problems additively (Karplus et al., 1983; Lamon, 1993). Similarly, it is well documented that students tend to overuse proportionality outside its applicability range (van Dooren et al., 2005; van Dooren et al., 2009; van Dooren et al., 2010).

To sum, proportional reasoning is about number reasoning rather than using formal procedures for solving missing value problems. Due to these features, proportional reasoning is much more complex (Tourniare & Pulos, 1985) and comprises difficult skills than of the thought. Development of proportional reasoning requires to distinguish between additive relationship and multiplicative relationship in a given situation (Fernandez, Llinares, Modestou & Gagatsis, 2010). For its development, teachers should provide rich experiences for their students both in different problem situations and in different contexts of additive and multiplicative situations (Dole & Wright, 2013; Vergnaud, 1988). Below, we will discuss additive and multiplicative situations and the transition from additive to multiplicative reasoning.

Additive and Multiplicative Thinking

One of the important stages in the development of proportional reasoning at the end of primary education is that children progress from additive to multiplicative reasoning (Fernandez & Llinares, 2009). As van Dooren, de Bock and Verschaffel (2010) stressed, multiplication and division are more than different operations that are taught after addition and subtraction, and multiplicative thinking goes beyond merely a faster way of doing repeated addition. However, previous research shows that the repeated addition meaning is not sufficient for transition from additive to multiplicative thinking (Greer, 1994; Nesher, 1988; Nunes & Bryant, 1996). Rather, a significant qualitative change is required. These changes are the ability to think relatively and to perceive a ratio as a new entity. Lamon (1993) argues that these skills are two critical factors that separate an additive reasoner from a multiplicative reasoner. Lamon also states that the emergence of relative thinking seems to be a signal that the student is beginning to fill the gap between additive and multiplicative structures. In addition, after an extensive review of the literature on proportional reasoning, Langrall and Swaford (2000) identified four essential components of formal proportional reasoning that may help to explain the deficiencies in students' proportional reasoning. Besides relative thinking and unitizing, Langrall and Swaford claim that the recognition of situations in which using a ratio is reasonable and understanding co-variation are important for proportional reasoning.

It is clear that relative thinking and understanding co-variation are two crucial factors for understanding proportionality. Relative thinking involves that students should recognize the difference between absolute (additive) and relative (multiplicative) change. For example, the question of what change in 4 will result in 8 can be solved additively or multiplicatively. The additive thinking gives that changing 4 to 8 is to "add 4", while multiplicative thinking is to "multiply by 2". According to the additive-change structure, changing 8 from 4 is the same as to get 17 from 13, but this is not true for the multiplicative structures (Behr, Harel, Post, & Lesh, 1992). Regarding absolute change, the original amount is altered by a fixed amount. In contrast, relative or multiplicative change alters the original amount by a quantity relative to the original amount. Relative change is multiplicative because the amount of the change is found by multiplying the original quantity by the rate whereas absolute change is additive because the amount of change is found by calculating the difference (Langrall and Swaford, 2000). From the mathematical perspective, the additive-change structure situation can be formulated by a function of the form " $f(x) = x+a$ " and the second by a function of the form " $f(x) = bx$ ". Thereby, both situations involve different kinds of reasoning (Van Dooren, De Bock & Verschaffel, 2010).

Students who developed sound proportional reasoning are able to differentiate additive and multiplicative situations, and apply appropriate strategies according to the context

(Lamon, 2007). However, studies conducted with different age groups in recent years showed that students have difficulty in identifying whether problem situations are multiplicative or additive (De Bock, Van Dooren, Janssens & Verschaffel, 2002; Van Dooren, De Bock, Hessels, Janssens & Verschaffel, 2005; De Bock, 2008; Lim, 2009; Van Dooren, De Bock, Gillard & Verschaffel, 2009). Students tend to use multiplicative strategies on problems including additive structures (De Bock, 2008; Lin, 1991; Modestou & Gagatsis, 2009) or they tend to use additive strategies on proportional word problems (Ben-Chaim, Fey, Fitzgerald, Benedetto & Miller, 1998; Çelik & Özdemir, 2011; Misailidou & Williams, 2003; Modestou & Gagatsis, 2009; Tourniaire & Pulos, 1985). Similarly, this kind of difficulties were observed among prospective teachers. Studies with prospective teachers indicated that they applied inappropriate strategies on proportional and non-proportional problems: additive approach on proportional problems and multiplicative approach on non-proportional problems (Chang, 2002; Fernandez, Llinares & Valls, 2013; Valverde & Castro, 2012). It is clear that both students and prospective teachers tend to solve word problems that are multiplicative erroneously in an additive way, and, inversely, solve additive word problems multiplicatively. Therefore, we think that it would be helpful to investigate the development with age of the use of additive and multiplicative strategies simultaneously. In addition, a comparison of students' use of both strategies with the prospective teachers' strategies may give some insight about teaching of proportionality.

Method

Participants

This descriptive study was conducted with 319 prospective teachers and 320 elementary school students in 2014-2015 academic year. The number of participants by grade level and programme year were provided in Table 1. All of the prospective teachers were selected from the same university. In Turkey, both prospective primary school teachers and middle school mathematics teachers are trained in 4-year programmes. Because primary school teachers are expected to teach all subject matters in grades 1-4, prospective primary teachers are trained in all curricular subjects, including mathematics. Throughout the programme, prospective primary teachers are required to take only a one-year mathematics content course, which covers an overview of the high school mathematics curriculum and a one-year mathematics methods course in the third year of the programme. In contrast, prospective middle school mathematics teachers who are expected to teach mathematics in grades 5-8 are trained to have a strong mathematics background. They take a lot of mathematics content courses such as calculus, advanced calculus, differential equations, linear algebra, analytic geometry, geometry, and different mathematics methods courses during their study.

Fourth graders were selected from a primary school, students from fifth to seventh grade were selected from a middle school. Average ages of elementary school students varied between 9 and 12 years. In Turkish school mathematics program, multiplication is taught at second grade with an emphasis on repeated addition meaning. In second grade, the focus is on solving simple multiplication word problems (e.g., "1 plate has 3 cookies. How many cookies do 4 plates have?"). In third, fourth and fifth grade, the focus gradually shifts toward the development of the standard multiplication algorithm and the application of this algorithm for solving multiplication problems. Although students encounter solving some missing-value proportional problems, a typical instruction on multiplication focuses usually on the repeated addition meaning of multiplication and thereby fosters additive reasoning. The concept of ratio is introduced at sixth grade and by seventh grade, students are expected

to compare the equality of two ratios and solve missing value proportion problems. Although it is not recommended by the middle school mathematics curriculum, right after the concept of proportion in seventh grade, students are taught the cross-multiplication algorithm for solving missing value problems. In addition, mathematics textbooks for primary school and middle school do not pay attention to contrasting proportional and nonproportional missing-value problems. Sixth and seventh graders were taught the units on ratio and proportion, respectively, before this study was conducted.

Table 1
Numbers of Participants

	Prospective mathematics teachers (PMT)	Prospective primary teachers (PPT)	Elementary school students	
1st year	45	43	Grade 4	85
2st year	42	34	Grade 5	85
3st year	42	44	Grade 6	80
4st year	40	29	Grade 7	80
Total	169	150	Total	320

Data Collection and Analysis

Data was collected using a test including four problems (two additive and two proportional). Problems were adapted from the study by Fernandez, Llinares and Valls (2011) into Turkish. Problems are presented in Table 2.

While both proportional and additive problems on the test deal with co-variation, there is a significant difference between the two situations. In additive problems, as one value increases, other one also increases but the difference between the two values remains same. However, in proportional problems, the multiplicative relationship between the values, not the difference, remains same. Two of the word problems (second and fourth problems) were proportional because students need to find x in the expression $b/a=x/c$. The first and third problems were additive or nonproportional because students need to find x in the expression $b-a=x-c$. Put it another way, while in proportional problems the multiplicative relationship between the quantities was invariant, in nonproportional problems the difference between the quantities was constant. Both problem types were formulated similarly. But the second sentence of each word problem determines whether the problem is proportional or additive. For example, in the first problem, “They load equally fast but Hasan started later” implies that the difference between the numbers of chairs loaded remains constant. In the rest of the article, we will use the terms of nonproportional and additive interchangeably.

Table 2
Problems Used in Test

Types of problems	Problems
Additive	Hasan and Rahmi are loading chairs in a truck. They load equally fast but Hasan started later. When, Hasan has loaded 40 chairs, Rahmi has loaded 100 chairs. If Hasan has loaded 60 chairs, how many chair has Rahmi loaded?
Proportional	Ayşe and Murat are making cookies. They started together but Murat makes faster. When Ayşe has made 4 cookies, Murat has made 12

	cookies. If Ayşe has made 20 cookies, how many cookie has Murat made?
Additive	Arzu and Hakan are packing chickens. They work equally fast but Hakan started earlier. When Arzu has packed 12 chickens, Hakan has packed 24 chickens. If Arzu has packed 48 chickens, how many chicken has Hakan packed?
Proportional	Leyla and Poyraz are pasting stamps on letter. They started together but Leyla pastes slower. When Leyla has pasted 60 stamps, Poyraz has pasted 280 stamps. If Leyla has pasted 120 stamps, how many stamps has Poyraz pasted?

The test was administered in a regular classroom hour. Participants were asked to solve problems and explain their reasoning. Participants' responses were analysed by two researchers by using the framework developed by Van Dooren, De Bock, and Verschaffel (2010). Analysis of the data consisted of two phases. In the first phase, coding was performed on a problem by problem basis. Participants' strategies for each problem were classified as either multiplicative, additive, or other. These strategies are explained below:

Additive Strategies: This strategy includes additive comparison that the difference between given values is computed by subtracting one value from another, and adding the difference to the third one (Van Dooren, De Bock & Verschaffel, 2010). For example, problems like "When Ayse was 5 years old, Ali was 9 years old. When Ayse is 13 years old, how old is Ali?" are additive situations. In this problem, to find Ali's age, one may find the difference between Ayse's age in the first situation and the second situation ($13-5$) and then add the difference to Ali's age ($9+8$). The difference between given values is critical in this strategy. Additive strategies are appropriate for non-proportional situations.

Multiplicative Strategies: This strategy involves multiplicative comparison and is a correct approach for proportional situations. This approach focuses on the ratio between two values and uses multiplication with a third value (Van Dooren, De Bock & Verschaffel 2010). For example, "A car has gone 240 km in 4 hours, how many km has it gone in 2 hours?". This problem has a multiplicative relationship and division is the necessary operation ($240/4$) to find the distance that the car has gone in one hour. Then, the ratio of 60 is multiplied with 2 (60×2) and the answer is found.

Other: The solution displays no multiplicative or additive character, or the given numbers are combined with arithmetic operations other than specified above, or the problem was left blank.

Table 3
Categories and Explanations

Categories	Explanations
Correct	Two proportional problems were solved multiplicatively and two nonproportional problems were solved additively.
Additive	All four problems were solved additively.
Multiplicative	All four problems were solved multiplicatively.
Uncategorized	All cases that did not fit into these profiles.

Second, we also looked participants' individual solution profiles to reveal their reasoning in general. For analysis, we used Van Dooren, De Bock, Gillard and Verschaffel's (2009) classification. Categories and explanations are provided in Table 3. The classification of the responses was conducted by following the rules in Table 3. At this stage of analysis, each participant's four solutions as a whole was coded. When a participant correctly solved all four problems, the participant's answers as a whole were coded as "correct". For example, when a participant used additive strategies for additive problems, and multiplicative strategies for proportional ones, then he/she was labelled as a correct reasoner. When he/she applied additive strategies for all problems, then he/she was classified as an additive reasoner. In other words, additive reasoners used no multiplicative strategy on any of the four problems and multiplicative reasoners used no additive strategy for the problems on the test. The last category "other" was coded, when a participant's responses cannot be included into any of the three categories or didn't attempt to solve problems.

Results

The results are presented in two parts. In the first part, strategies used by the participants for proportional and additive problems are presented with sample solutions. In the second part, we present the participants' individual solution profiles to all of problems.

Strategies Used by Prospective Teachers and Elementary School Students

In this section, participants' strategies for solving additive and proportional problems are presented in Table 4. Taken as a whole, when we look at the elementary school students' solutions in Table 4, the proportional problems elicited more proportional solutions (37%) than the additive problems (27%); and the additive problems elicited more additive solutions (60%) than the proportional problems (50%). Although the differences were not very large, we observe an overgeneralization in both directions. Students frequently used proportional strategies on additive problems and additive strategies on proportional problems. Second, solutions on the test seems to be affected by students' age. While fourth, fifth and sixth graders, regardless of the problem type, tended to give many additive responses (74%, 59% and 70%, respectively), this trend sharply decreased at seventh grade (18%). This trend can better be seen from the Figure 1. On the other hand, an opposite trend was observed for the number of proportional responses. In fourth grade, only 19% of all responses were proportional, but this increased to 76% in seventh grade. As Table 4 shows, it can be said that there is a decrease in additive responses and an increase in proportional responses both for the proportional and for the additive problems. Therefore, the number of correct answers to proportional problems increased with age, whereas on additive problems it decreased. Interestingly, fourth, fifth and sixth graders showed almost similar performances on both of the proportional and the additive problems. This result implies that the sharp decrease in additive responses and the sudden increase in proportional responses at seventh grade is not the result of the growth of students' ages. Rather, this is the result of the instruction because at seventh grade students are taught the cross-multiplication algorithm to solve missing value proportion problems.

Table 4

Percentages of Proportional (P), Additive (A), and Other (O) Answers in Each Group to Problem Types (Correct responses are shown in bold.)

	Proportional Problems			Additive Problems			Total		
	P	A	O	P	A	O	P	A	O
Students (grades)									
4 (n=85)	19	66	15	9	81	10	14	74	12
5 (n=85)	19	54	27	11	61	18	15	59	16
6 (n=80)	25	66	9	18	74	8	22	70	8
7 (n=80)	83	13	4	69	22	9	76	18	6
Total (n=320)	37	50	14	27	60	11	32	55	11
PPT (years)									
1 (n=43)	86	9	5	59	39	2	73	24	3
2 (n=34)	84	11	5	59	31	10	72	21	7
3 (n=44)	95	4	1	44	56	0	70	30	0
4 (n=29)	85	14	1	23	74	3	54	44	2
Total (n=150)	88	10	3	46	50	4	67	30	2
PMMT (years)									
1 (n=45)	98	2	0	36	60	4	67	31	2
2 (n=42)	100	0	0	60	32	8	80	16	4
3 (n=42)	94	0	6	18	73	9	56	37	7
4 (n=40)	100	0	0	40	60	0	70	30	0
Total (n=169)	98	0	2	39	56	5	68	29	3

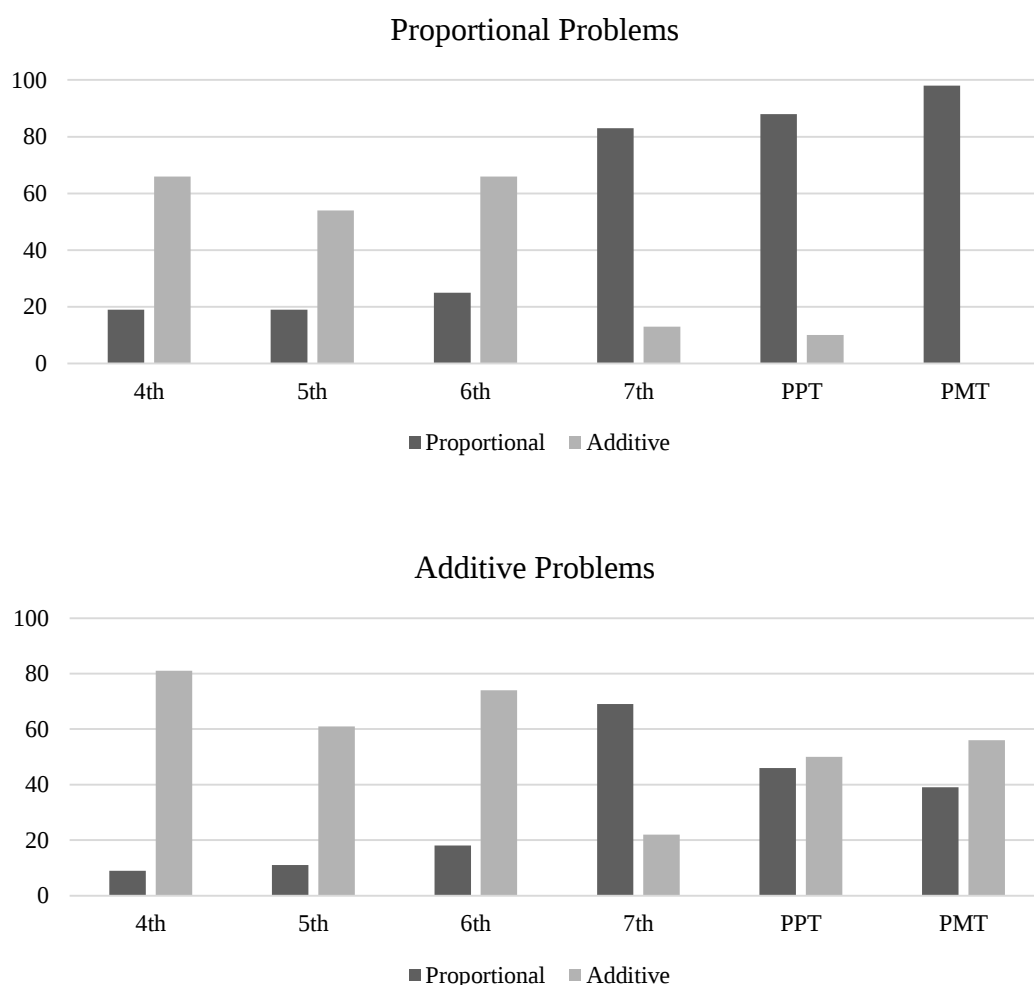


Figure 1. Percentages of proportional (P), additive (A), and other (O) answers in each group to each problem

Regarding the results for prospective teachers in Table 4, both prospective primary and prospective mathematics teachers, regardless of the problem type, provided more proportional answers (67% and 68%, respectively). In contrast to elementary school students, prospective teachers gave more correct answers to proportional problems than to additive problems. Interestingly, a considerable number of prospective primary and prospective mathematics teachers erred in applying multiplicative strategies for additive problems (46% and 39%, respectively). In other words, both groups of prospective teachers are inclined to apply multiplicative strategies on additive problems whereas only 29% of elementary school students showed this tendency. Prospective teachers tended to think that there is a multiplicative relationship between the values given in additive problems. This tendency was reflected by the overuse of the cross-multiplication strategy for all problems by the prospective teachers. For example, in Figure 2, two prospective teachers' erroneous solutions of Problem 1 is given. Although the problem includes a non-proportional situation, these prospective teachers mistakenly used the cross-multiplication strategy and claimed that the problem was a proportion problem. It is also clear that prospective teachers- almost all of PMT and about 90% of PPT, were more successful on the proportional problems than on the additive problems (see Figure 1). This may be due to the prospective teachers' tendency

to overuse multiplicative strategies regardless of the problem type. Put it another way, like seventh graders, prospective teachers committed more proportional errors than fourth, fifth and sixth graders. Therefore, it can be said that in contrast to fourth, fifth, and sixth graders, prospective teachers and seventh graders tended to overgeneralize proportionality. Prospective teachers frequently justified their choice of strategy with the explanation of “because this is a proportion problem.” However, they were not able to explain why these problems were proportion problems.

Hasan ve Rahmi bir kamyonu sandalye yüklemektedirler. İkisi de aynı hızda sandalye yüklemektedir fakat Hasan yüklemeye daha sonra başlamıştır. Hasan 40 sandalye yüklediği zaman Rahmi 100 sandalye yüklemiştir. Hasan 60 sandalye yüklediği zaman, Rahmi kaç sandalye yükler?

$$\begin{array}{r} 40 \text{ H} \quad \times \quad 100 \text{ R} \\ 60 \text{ H} \quad \times \quad x \text{ R} \\ \hline 100 \cancel{R} \cdot 40 \text{ H} = 40 \cancel{H} \cdot x \text{ R} \\ \frac{600}{4} = x \\ 150 = x \end{array}$$

Ozan orantı sorusu olduğundan formüle göre gördüm.

Since it is a proportion problem, I used cross multiplication.

$$\begin{array}{r} \text{H} \quad \text{R} \\ 40 \quad 100 \\ 60 \quad x \\ \hline 40 \cdot x = 100 \cdot 60 \\ x = 150 \end{array}$$

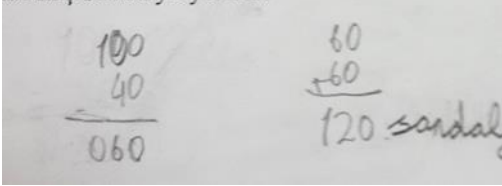
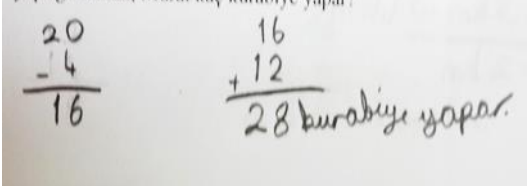
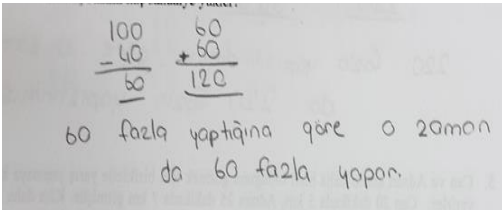
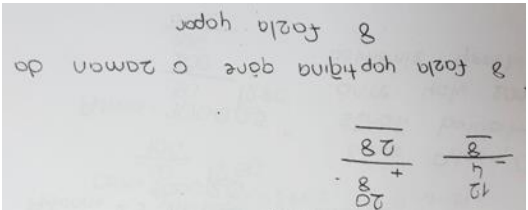
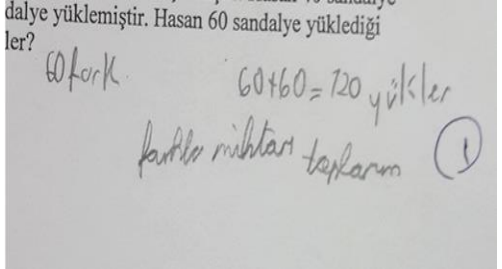
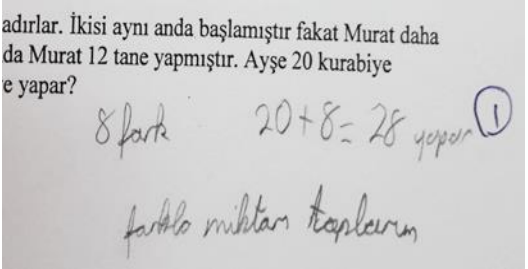
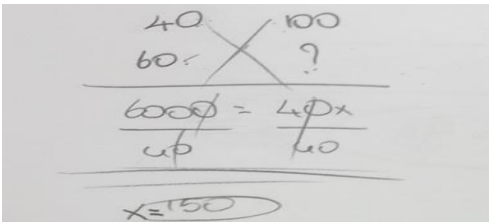
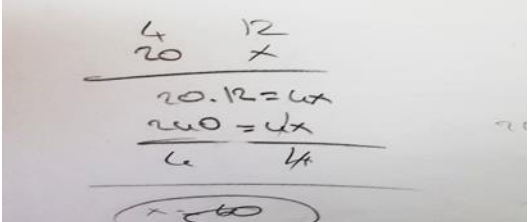
→ Doğru orantı mantığıyla çözüm. Çünkü böyle anlatıldı.

I solved it as it was a direct proportion problem, since it was taught like this.

Figure 2. Solutions of two prospective teachers to an additive problem

In contrast to prospective teachers and seventh graders, fourth, fifth and sixth graders commonly used additive strategies for all problems. In Table 5, examples of elementary school students' solutions of the first and second problems were provided. As it is seen from Table 5, while fourth, fifth, and sixth graders usually found the difference between the given quantities and add it to the third quantity to find the answer, regardless of the problem type, seventh graders dominantly used the cross-multiplication strategy. To summarize, it can be said that fourth, fifth and sixth graders are more likely additive reasoners even in proportional situations whereas prospective teachers and seventh graders are proportional reasoners regardless of the problem type. In addition, it is also important to note that strategies such as building up or replication that are commonly reported in the literature are not observed among the strategies used by the participants. It was striking that while third to sixth graders usually found the difference between the given two initial quantities and add it to the third quantity for both problem situations, seventh and prospective teachers often applied the cross-multiplication method, regardless of the problem type. It seems that the participants ignored the most important part of the problems' statements. Understanding the second sentence in each problem (*They load equally fast but Hasan started later or they started together but Murat makes faster*) was critical for identifying whether the situation is additive or multiplicative.

Table 5
Examples of Solutions of Elementary School Students

Grade	Problem 1- Additive problem	Problem 2-Proportional problem
4		
5		
6		
7		

Individual Strategy Profiles of Prospective Teachers and Elementary School Students

In order to determine the participants' individual strategy profiles, participants' individual strategies for all of the four problems on the test were classified as correct (additive strategies for additive problems, multiplicative strategies for multiplicative problems), additive (additive strategies for all problems), multiplicative (multiplicative strategies for all problems), and other. Table 6 shows the results of this classification.

Table 6
Overview of Solution Profiles of Individual Students (in %)

	Correct	Additive	Proportional	Other
Students (grades)				
4 (n=85)	4	56	0	40

5 (n=85)	1	33	7	47
6 (n=80)	3	54	6	37
7 (n=80)	3	5	55	37
Total (n=320)	3	38	17	42
PPT (years)				
1 (n=43)	21	7	49	23
2 (n=34)	18	6	53	23
3 (n=44)	43	0	36	21
4 (n=29)	52	10	14	24
Total (n=150)	33	5	39	23
PMMT (years)				
1 (n=45)	51	2	31	16
2 (n=42)	24	0	55	21
3 (n=42)	64	0	14	22
4 (n=40)	55	0	33	12
Total (n=169)	49	1	33	17

As Table 6 reveals, almost half of the PMT (49%) were able to correctly solve all of the problems. Similarly, only one-third of PPT and 3% of elementary school students were able to correctly solve all of the problems. One striking observation from the Table 6 is that while fourth, fifth, and sixth graders tended to be additive reasoners (56%, 33%, and 54%, respectively), seventh graders and prospective teachers more inclined to be a multiplicative reasoner. Especially, it was interesting that while 55% of seventh graders tended to apply more proportional strategies for all of the problems, only 3% of them were able to correctly solve them. In comparison to elementary school students, prospective teachers were more successful on the test. However, like seventh graders, they tended to overgeneralize proportionality. In other words, fourth, fifth, and sixth graders tended to commit additive errors more, whereas prospective teachers and seventh graders were more inclined to commit proportional errors.

Discussion

In this study, we investigated prospective elementary (primary and mathematics) teachers' and elementary school students' ability to distinguish proportional situations from non-proportional situations. Below we summarize the findings of the study and simultaneously highlight the potential implications for practice and teacher preparation regarding proportionality. We also outline limitations of the study.

Initially, we investigated participant's strategies on additive and proportional problems. We observed that prospective teachers and students exhibited weak proportional reasoning

skills. Participants used incorrect additive strategies to solve proportional problems and multiplicative strategies to solve additive problems. This result is similar to the findings of other studies (Chang, 2002; Fernandez, Llinares, & Valls, 2013; Masters, 2012; Monteiro, 2003; Valverde & Castro 2012) and students (Kaput & West, 1994; Lin, 1991; Modestou & Gagatsis, 2009; Van Dooren, De Bock, Gillard & Verschaffel, 2009). It can be said that regardless of the age, participants have difficulty in interpreting the difference between additive and multiplicative structures. Especially, it was observed that while fourth, fifth, and sixth graders tended to think additively, seventh graders and prospective teachers tended to think proportionally on both situations. Related to this tendency, Dooren, De Bock, Gillard & Verschaffel (2009) stated that additive reasoning decreases with age and proportional reasoning increases. Previous research (Fernandez, Llinares, Van Dooren, De Bock & Verschaffel, 2012; Degrande, Verschaffel & Dooren, 2016) and findings in this study, supported this claim. However, as proportional reasoning increases, the ability to distinguish proportional situations from non-proportional situations doesn't develop as it is expected. In other words, the dominant additive reasoning among fourth, fifth, and sixth graders is replaced by a dominant proportional reasoning among seventh graders and prospective teachers without any consideration to the structures of situations. Especially, right after the instruction on the cross-multiplication algorithm in seventh grade where the actual teaching of proportionality starts, this tendency becomes apparent. However, the emergence of the overuse of proportionality in seventh grade implies that instruction on proportion reinforce this overgeneralization. In other words, we may say that students are probably provided with an intensive practice of proportional reasoning skills with missing-value problems in this grade. Existence of overgeneralization of proportionality among prospective teachers also lead us to conclude that so many years of schooling including university education do not add too much prospective teachers' proportional reasoning skills. Instead they are equipped with the knowledge of algorithms and are unable to decide whenever to use them. In order to develop sound proportional thinking skills, students need to understand the difference between proportional and non-proportional situations. Lamon (2007), specifically directed the students' attention to learning to "recognize the difference" between multiplicative relationships and additive relationships. Ability to distinguish these two kinds of situations in a given problem or context lies at the heart of proportional reasoning (Sowder, Armstrong, Lamon, Simon, Sowder & Thompson, 1998). Therefore, teachers should draw students' attention to differences in mathematical structure of problems in order to prevent that students base their solution strategies on superficial associations. (Fernandez, Llinares, Modestou & Gagatsis, 2010). At the same time, children not only need to consider differences in the structures of the situations, they also need to compare quantitatively and decide which quantitative relationships are relevant. According to Piaget, proportional reasoning involves understanding the "relation between relations" (as cited in Boyer, Levine, & Huttenlocher, 2008). That is to say, students should be able to compare relative changes of quantities, interpret and decide the nature of comparison.

Another result of this study showed that even if participants used correct strategies or procedures in solving the problems, most of them could not justify their choice of strategy. Their explanations were limited to procedural knowledge and they did not provide any mathematical arguments to whatever they had done or why they had done. Participants mostly claimed that the problem situations were either proportional or nonproportional but they did not provide any justification why they are so. It seems that they base their reasoning on the surface attributes of the problems such as wording. Studies related to proportional reasoning suggest that many elementary students, elementary teachers and prospective

teachers lack a deep understanding of proportional reasoning and their understanding build on procedures such as the cross-multiplication algorithm (Riley, 2010; Canada, Gilbert & Adolphson, 2008; Valverde & Castro, 2012). This was the case in the present study. Because the wordings of both problem types were very similar, a critical attention to the mathematical structure of the problems was needed for a correct strategy choice.

It was also striking that the most of the participants that appropriated multiplicative approach, tend to use cross-multiplication regardless of the problem structure. Different strategies such as building up were not encountered in answers. This finding indicates that both students and prospective teachers have limited ideas in producing different strategies. Same findings have identified by Modestou & Gagatsis (2009) and they stated as a result of teaching, especially elementary students commonly follow same procedures to solve problems regardless of the problem situation. In school, this may be due to the fact that students usually face standard or routine problems that can be solved by using cross-multiplication algorithm. It is also well documented that students choose their solution procedures by relying on surface-level characteristics of problems instead of the deep-level characteristics. Especially, students very strongly tend to pay attention to superficial cues such as key words in the word problem to decide on the arithmetical operation(s) to be performed (Verschaffel, Greer, & De Corte, 2000; van Dooren, de Bock, & Verschaffel, 2010). To prevent this tendency, teachers have to encourage students to solve a variety of non-routine problems as well as different types of proportion problems. More specifically, teachers might pose problems that may be answered using either absolute or relative thinking. Through such experiences, students may have a chance to identify multiplicative and additive situations and distinguish them from each other and from other situations. In addition, before proceeding to the formal symbolism and the cross-multiply algorithm for solving proportion problems, students can be given the time to explore multiplicative situations and to coordinate both the additive and relative perspectives (Lamon, 1993).

For an effective mathematics teaching, not only prospective teachers should know algorithms procedurally but also have to understand how and why these procedures work (Hatano, 1988; Kahan, Cooper & Bethea, 2003). Level of understanding of teachers in relation to effective teaching is critical (Burgess, 2001). As is seen in this study and in the literature, prospective teachers made errors in identifying the relationships between quantities, it is meaning that prospective teacher's proportional reasoning was not sufficient to teach. In addition, Masters (2010) argued that teachers did not recognize the errors made by students because they also made the same errors as the elementary students did. That is to say, because many of teachers and prospective teachers had same misunderstandings and slight conceptualizations as their students, it is unrealistic to expect them to develop their students' proportional reasoning. As a consequence, teachers can feel helpless of not knowing what to do when encountered with student's difficulties, misconceptions and errors in classrooms (Chang, 2002). For this reasons, prospective teachers need to be supported in developing a sound understanding of proportionality. To support conceptual understanding, prospective teachers need to be provided with opportunities in some rich problem contexts so that they will be better prepared to promote conceptual development of their own students.

In literature, it is often argued that teaching of proportional reasoning is limited to solving missing value problems where three numbers are given and a fourth is unknown. For solving these problems, one simply needs to apply learnt algorithms. It is obvious, that when prospective teachers are provided with proportional and non-proportional problems which were similar in wording, they tend to solve them as missing value problems without any consideration to the mathematical structures of the problems. It wouldn't be too strong to

argue that these prospective teachers will teach their students as they were taught. They will carry this limited understanding of proportionality to their students. Based on these findings, it is possible to say that teacher educators shouldn't assume that prospective teachers know the concepts they are supposed to teach in the future. Hence, teacher educators should uncover prospective teachers' prior understandings and provide opportunities to develop conceptually rich understandings so that they can teach their students for understanding. Especially, in teacher education courses, prospective teachers should have an opportunity to learn how students acquire proportional reasoning skills, which difficulties they experience, and how it can be enhanced by instruction. In these courses, prospective teachers should also have a chance to reflect on and revise their existing understanding of proportionality.

We examined proportional reasoning in the context of additive and multiplicative problems formulated in a missing value format and we used integer numbers in problems. Further study can investigate that when we use non-integer numbers in problems, does proportional reasoning change? If reasoning changes, how numbers influence participants' reasoning. Especially, prospective teachers and students' judgement processes and their explanations can be examined. Other hand, this study can be replicated with in-service mathematics teachers. In literature, studies on proportional reasoning usually focused on students, thence more studies needed to analyse in-service mathematics teachers' proportional reasoning. One limitation of this study is that four problems on the test were fixed in order when the test was administered. The first problem on the test was an additive problem, the participants' impression of the first problem may have an influence on the participants' approach to the rest of the problems. To test this, a similar test can be administered by changing the order of the presentations of the problems. Another important limitation of this study is that our analysis of the participants' proportional reasoning skills was totally based on the written responses given by the participants. We strongly recommend that new studies could be done using in-depth interviews with both students and prospective teachers.

Consequently, studies indicated that the teaching of mathematics is not simple transfer relation from the teacher to the student (Modestou & Gagatsis, 2007). Teachers need to come to understand a mathematical idea, so they can teach it conceptually.

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