INFINITE SERIES FROM HISTORY TO MATHEMATICS EDUCATION

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Abstract: In this paper an example from the history of mathematics is presented and its educational utility is investigated, with reference to pupils aged 16-18 years. Students' behaviour is examined: we conclude that historical examples are useful in order to improve teaching of infinite series; however their effectiveness must be verified by the teacher using experimental methods, and the primary importance of the cultural context must be taken into account.

Introduction

Several authors have shown that the history of mathematics can be widely employed by teachers in the presentation of many mathematical topics (Fauvel & van Maanen, 2000); of course this requires some epistemological assumptions: teaching is influenced by teachers' conceptions about the nature and the evolution of scientific knowledge (Moreno & Waldegg, 1993; Heiede, 1996).

A. Sfard states that, in order to speak of mathematical objects, it is necessary to consider the whole process of concept formation and she supposes that an operational conception can be considered before a structural one (Sfard, 1991). As regards the *savoir savant* (Chevallard, 1985) the historical development of many mathematical notions can be considered as a sequence of stages: an early intuitive stage and so on, until the mature stage is reached. This savoir cannot be considered absolute and it must be understood in terms of cultural institutions (Lizcano, 1993; Grugnetti & Rogers, 2000; Furinghetti & Radford, 2002).

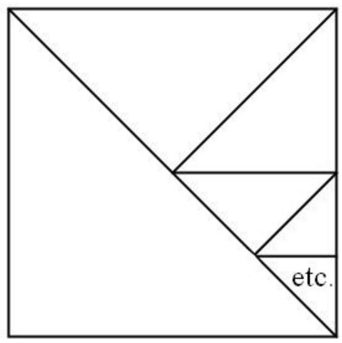
The sociological perspective is clearly relevant to mathematics education. Let us consider the approach introduced by R. Cantoral and R.M. Farfán (Cantoral, 2001; Cantoral & Farfán, 2003): while epistemological approaches sometimes do not take into account the influence of social contexts, in the theoretical approach denominated "Socioepistemology" an extension of theory of didactic situations is proposed in order to show the social construction of the knowledge and the negotiation of meanings.

With reference to the use of the history into didactics on mathematics, several theoretical frameworks can be mentioned in order to link learning processes with historical issues (Cantoral & Farfán, 2004, see in particular the Chapter 8). According to the "epistemological obstacles" perspective

(Brousseau, 1983), a goal of historical study is finding systems of constraints (*situations fondamentales*) that must be studied in order to understand existing knowledge, whose discovery is connected to their solution (Radford, Boero & Vasco 2000, p. 163). It seems that this perspective is characterised by an epistemological assumption: the reappearance in teaching-learning processes of the obstacles encountered by mathematicians in the past. Nevertheless, historical data must be considered nowadays and several issues are connected with their interpretation, again based upon our cultural institutions and beliefs (Gadamer, 1975); according to Radford's socio-cultural perspective, knowledge is linked to activities of individuals and this is strictly related to cultural institutions; knowledge is not built individually, but in a wider social context (Radford, Boero & Vasco, 2000).

In this work we shall discuss the introduction of infinite series by using some well known historical examples (concerning the history of the Calculus see: Edwards, 1994; Hairer & Wanner, 1996). When we introduce infinite series, we must keep in mind that a sum of infinitely many addends is frequently considered by pupils as "infinitely great" (Bagni, 2000a) so first of all we must overcome the misconception "infinitely many addends, infinitely great sum".

Of course it is possible to employ several visual representations (see the picture: the big square, whose side's length is 1, is divided into a sequence of triangles so that it is possible to state: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\ldots=1$. But the educational use of visual registers can cause problems, particularly if pupils' ability to coordinate representation registers is lacking: Duval, 1995; D'Amore, 2001).



The history of mathematics can help us to direct our pupils correctly; for instance, we can mention Zeno of Elea (490-430 BC) and his famous paradox of Achilles and of the Turtle: it is well known that it lead us to consider a convergent geometric series. This example can be very useful and through this, implicitly, we present a sum of infinitely many addends that cannot exceed a finite number. Unfortunately some misunderstandings can arise: in particular, pupils could notice that addends, in that case, are indefinitely small and this condition can be considered wrongly as a sufficient one for the convergence of an infinite series (the harmonic series, studied in the 14th century by Nicole Oresme is useful for overcoming this mistake: Anglin, 1994, p. 134).

We shall present some historical examples that can be employed in classroom practice; then we shall examine students' reactions in a brief experimental survey.

Infinite series: historical remarks

Ut non-finitam Seriem finita cöercet,
Summula, & in ullo limite limes adest:
Sic modico immensi vestigia Numinis haerent
Corpore, & angusto limite limes abest.
Cernere in immenso parvum, dic, quanta voluptas!
In parvo immensum cernere, quanta, Deum!

Jakob Bernoulli (*Ars Conjectandi*, 1713)

A first notion of infinite series may well have a very ancient source: Aristotle himself implicitly underlined that the sum of a series of infinitely many addends (potentially considered) can be a finite quantity (*Physics*, III, VI, 206 b, 1-33). In his *Quadratura parabolæ*, Archimedes considered (implicitly, once again) a geometric series. Several centuries later, Andreas Tacquet (1612-1660) noticed that the passage from a "finite progression" to an infinite series would be "immediate" (Loria, 1929-1933, p. 517); but such a passage is crucial, from the epistemological point of view. In fact, Tacquet made reference to ancient mathematics without any historical contextualization: Greek conceptions strictly distinguished actual and potential infinity (mathematical infinity, following Aristotle, was accepted only in a potential sense so it is meaningless to suppose any explicit consideration of infinite series. Tacquet's position, too, must be contextualised: we cannot suppose the presence of our epistemological awareness in the 17th century; it is necessary to take into account either the period in which the original work was written or the period of its edition or comment: Barbin, 1994; Dauben & Scriba, 2002).

In particular, we are going to examine a well-known indeterminate series. In 1703, Guido Grandi (1671-1742) noticed that from the infinite series 1–1+1–1+... it is possible to obtain 0 or 1:

$$(1-1)+(1-1)+(1-1)+(1-1)+\ldots = 0+0+0+0+\ldots = 0$$

$$1+(-1+1)+(-1+1)+(-1+1)+\ldots = 1+0+0+0+\ldots = 1$$

The sum of the alternating series = 1–1+1–1+... was considered ½ by Grandi (and by several mathematicians in the 18th century). According to him, the proof can be based upon the following expansion expressed using modern notation (nowadays accepted if and only if $|x|<1$):

$$\frac{1}{1+x} = \sum_{i=0}^{+\infty}(-x)^i = 1 - x + x^2 - x^3 + \ldots$$

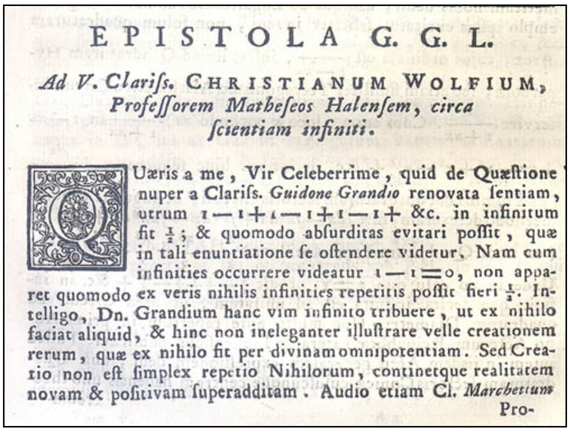
From $x = 1$ (and this is *not* correct!) we should have: 1–1+1–1+... = ½.

From an educational point of view, it can be noticed that this (wrong) result can be achieved by the (wrong) procedure: from s = 1–1+1–1+... we should have: s = 1–(1–1+1–...) and s = 1–s, so s = ½. Of course, nowadays, this procedure cannot be accepted: it is clearly based upon an incorrect use of arithmetical properties and upon the implicit statement that 1–1+1–1+... is a number s, and we know that this is false.

Euler and Fourier also thought that 1–1+1–1+... = ½; Gottfried Wilhelm Leibniz (1646-1716) studied Grandi's series and wrote to Jacopo Riccati (1676-1754):

> "I do not know if Mr. Count Riccati and Mr. Zendrini have looked at the question if 1–1+1–1 etc. is ½, as Grandi stated, in someway correctly. In fact 1/(1+x) is 1–x+xx–x^3+x^4–x^5 etc. so if x is 1, we have 1/(1+1) = 1–1+1–1+1–1 etc. = ½. It seems that it is clearly absurd. In *Acta Eruditorum Lipsiae* I think I have solved this problem" (letter probably written in 1715: Michieli, 1943, p. 579; in this paper translations are ours).

Leibniz studied Grandi's series in some letters (1713-1716) to Christian Wolf (1678-1754); he introduced the "probabilistic argument", that influenced Johann and Daniel Bernoulli.


EPISTOLA G. G. L.

Ad V. Clarifs. CHRISTIANUM WOLFIUM, *Profeſſorem Matheſeos Halenſem, circa ſcientiam infiniti.*

Quæris a me, Vir Celeberrime, quid de Quæſtione nuper a Clarifs. *Guidone Grandio* renovata ſentiam, utrum 1—1+1—1+1—1+ &c. in infinitum fit ½; & quomodo abſurditas evitari poſſit, quæ in tali enuntiatione ſe oſtendere videtur. Nam cum infinities occurrere videatur 1—1=0, non apparet quomodo ex veris nihilis infinities repetitis poſſit fieri ½. Intelligo, Dn. Grandium hanc vim infinito tribuere, ut ex nihilo faciat aliquid, & hinc non ineleganter illuſtrare velle creationem rerum, quæ ex nihilo fit per divinam omnipotentiam. Sed Creatio non eſt ſimplex repetio Nihilorum, continetque realitatem novam & poſitivam ſuperadditam. Audio etiam Cl. *Marchettum* Pro-

Leibniz noticed that if we "stop" the infinite series 1–1+1–1+..., it is possible to obtain 0 or 1 with the same "probability". So the most probable value is the average between 0 and 1, that is ½. Leibniz conceded that "his argument was more metaphysical than mathematical, but went on to say that there was more metaphysical truth in mathematics than was generally recognized" (Kline, 1972, p. 446; Leibnitian letters to Wolf were published in *Acta Eruditorum Lipsiae*, Tom. V. ab an. 1711 ad an. 1719 Epist. G.G.L. ad V. claris. Ch. Wolfium).

Lagrange and Poisson also accepted that argument; Pierre de Varignon (1654-1722) noticed that in order to state:

$$\frac{1}{a+b} = \frac{1}{a} - \frac{b}{a^2} + \frac{b^2}{a^3} - \ldots + \ldots$$

the condition $b<a$ is needed (Loria 1929-1933, p. 673), while the convergence of Grandi's series to ½ can be obtained by $a = b = 1$. Varignon's note can be interpreted as a first consideration of the role of convergence.

Jacopo Riccati criticised the convergence of Grandi's series to ½; in *Saggio intorno al sistema dell'universo* (1754), he wrote:

> "Grandi's argument is interesting, but wrong because it causes contradictions. Let us consider $n/(1+1)$ and, by the common procedure, build $n–n+n–n$ etc. $= n/(1+1)$. If it is remembered that $1–1 = n–n$, or $1+n = n+1$, we have, in this series and in Grandi's, that there is the same quantity of zeroes" (Riccati, 1761, I, p. 87).

Riccati's argument deserves a remark; he writes $½ = 1–1+1–1+...$, "by the common procedure", then he introduces the infinite series: $n/2 = n–n+n–n+...$ Let us compare the considered series; we can write:

$$s = 1–1+1–1+1–1+... = (1–1)+(1–1)+(1–1)+... = 0+0+0+...$$

$$s' = n–n+n–n+n–n+... = (n–n)+(n–n)+(n–n)+... = 0+0+0+...$$

Through this, Riccati concludes that Grandi's procedure is incorrect. Nowadays, this argument cannot be accepted (it is based upon the "common procedure" referred to indeterminate series). However, Riccati's conclusion is correct:

> "The mistake is caused by the use of a series from which it is impossible to get any conclusion. In fact, it doesn't happen that if we stop this series, the following terms can be neglected in comparison with preceding terms; this property is verified only for convergent series" (Riccati, 1761, I, p. 86).

So Riccati's statement can be related to ideas that mathematicians were going to point out in the 18^{th} century; finally, in *Disquisitiones generales circa seriem infinitam* Carl Friedrich Gauss (1777-1855) considered the notion of convergence correctly.

Infinite series: a brief experimental survey

In the first paragraph, we outlined the theoretical frameworks by Brousseau and by Radford: we underlined that they are based upon different epistemological assumptions and we pointed out the necessity of an adequate consideration of original cultural contexts in introducing history into the classroom, in order to present a sense of historical development to the pupils correctly. Besides these theoretical issues, and whatever perspective we are going to choose, the introduction of historical remarks in mathematics education must be carefully controlled.

The consideration of infinite series can cause inconsistencies in students' minds (for instance, if a pupil considers an infinite series as an arithmetical operation, the absence of the sum of Grandi's series can cause many doubts). It is important to point out that it is not enough to consider at the same time two conflicting statements in order to develop in pupils' minds the awareness of an inconsistency and the necessity of second thoughts (Schoenfeld, 1985): the perception of some mutually conflicting elements does not always imply the perception of the situation as a problematic one (Tirosh, 1990).

Our aim is the study of some pupils' reactions to stimulations provided by historical elements in order to point out their influence upon learning. For instance, are our pupils' reactions and arguments similar to reactions noticed in mathematicians in history (Tall & Vinner, 1981)? In particular, we shall briefly consider pupils' opinions regarding Grandi's series.

A test was proposed to students of two third-year *Liceo Scientifico* classes (pupils aged 16-17 years), total 45 pupils, and of two fourth-year *Liceo scientifico* class (pupils aged 17-18 years), in Treviso (Italy), 43 pupils (total: 88 pupils). Their mathematical curricula were traditional: in all classes, at the moment of the test, pupils did not know infinite series; they knew the concept of infinite set (the researcher was not the mathematics teacher of the pupils, however, he was present in the classroom with the teacher and the pupils; the experience took place during a lesson in the classroom).

We gave the following card to every student (time allowed: 10 minutes; no books or calculators allowed):

> In 1703, the mathematician Guido Grandi studied the addition: 1–1+1–1+... (addends, infinitely many, are always +1 and –1). What is your opinion about it?

Answers	the result is 0	26	29%
	the result is 1	3	4%
	the result can be either 0 or 1	18	20%
	the result is ½	4	5%
	the result is infinite	2	2%
	the result does not exist	5	6%
	no answer	30	34%

First of all, let us underline that the greater part of the pupils interpreted this question as an implicit request to calculate the "sum" of the considered infinite series. Only 5 students (6%) explicitly stated that it is impossible to calculate the sum of Grandi's series (and their answers were not provided with clear justifications); it should be remembered that 18 pupils, 20%, gave two "results"; 35 pupils (40%) gave a "result" (a finite or an infinite one). Many pupils gave no answer (34%).

Several students justified their answers in some interviews. Concerning pupils that stated that the sum of the infinite series 1–1+1–1+... is 0, some of them made reference to an argument by Grandi himself, quoted by Riccati, too:

“If I always want to add 1 and –1, I can write (1–1)+(1–1) and so I can couple 1 and –1: so I am going to add infinitely many 0: I obtain 0” (Marco, third year, and 15 other pupils).

Students that stated that the sum of the considered series is ½ made reference to justifications similar to the argument by Leibniz-Wolf; for instance:

“If I add the numbers I have 1, 0, 1, 0 and always 1 and 0. The average is ½” (Mirko, fourth year).

Audio-recorded material and transcriptions allowed us to point out a salient short passage (1 minute and 35 seconds, 9 utterances; following Linell, an utterance is a “stretch of continuous talk by one person, regardless of length and structure”: Linell, 1998, p. 160; original utterances are in Italian):

[1] Researcher: “Why did you write that the result is ½?”

[2] Mirko: “Oh, well, I start with 1, so I have 0, then 1, 0 and so on. There are infinitely many +1 and –1.”

[3] Researcher: “That’s true, but how can you say ½?”

[4] Mirko: “If I add the numbers, I obtain 1, 0, 1, 0 and always 1 and 0. The average is ½.”

[5] Researcher: “And so?”

[6] Mirko: “The numbers that I find are 1, 0, and 1, 0, and 1, 0 and so on: clearly, every two numbers, one of them is 0 and one of them is 1. So these possibilities are equivalent and their average is ½.”

[7] Mirko: [after 12 sec.] “Perhaps my answer is strange, or wrong, but I don’t see a different correct result: surely both the results 0 and 1 are wrong. If I say that the result is one of that numbers, for instance 1, I forget all the other numbers, an infinite sequence of 0.”

[8] Researcher: “So in your opinion both 0 and 1 cannot be considered the correct answer.”

[9] Mirko: “Alright, and in this case what is the result? I wrote that ½ is the results of the operation because ½ is the average, so it is a number that, in a certain sense, contains either 0 or 1.”

Let us point out a consideration: Mirko stated that “every two numbers, one of them is 0 and one of them is 1” (utterance [4]) and “the average (…) is a number that, in a certain sense, contains either 0 or 1”(utterance [9]). So he did not make explicit reference to the probability: he mainly tried to find a result for the considered “operation”. In the 18th century, the probabilistic argument was based upon a different remark, according to which if we “stop” the infinite series 1−1+1−1+..., it is possible to obtain 0 or 1 with the same “probability”. Let us remember the importance of the probability in the mathematical researches in the 18th century (for instance, in *Acta Eruditorum 1682-1716*, we find either the quoted Leibnitian letter to Wolf or Bernoulli's *Specimina Artis Conjectandi*).

So really students' justifications are sometimes similar to some examples from the history of mathematics (this seems to bear out a well-known idea in Piaget & Garcia, 1983, according to which historical development and individual development are linked; of course this does not mean that we support this idea in all educational situations); but the cultural context is different from the context that characterised the historical reference: so can we state that Mirko's and Leibniz-Wolf's arguments are referred to the same epistemological obstacle?

Final reflections

> "The history of mathematics is not just a box of paints with which one can make the picture of mathematics more colourful, to catch interest of students at their different levels of education; it is a part of the picture itself. If it is such an important part that it will give a better understanding of what mathematics is all about, if it will widen horizons of learners, maybe not only their mathematical horizons (…) then it must be included in teaching."
>
> Torkil Heiede (1996, p. 241)

In this research the considered sample is rather small: it would be necessary to identify clearly sampling criteria and, for example, pre-course intuitions: so we cannot give general results. However we can state that some examples from the history do help with the introduction of an important topic of the mathematical curriculum of High School. This utility cannot be stated uncritically: historical examples clearly stimulated many pupils, but an explicit institutionalisation by the teacher is necessary.

This social aspect is meaningful: let us underline once again that recent works (Cantoral, 2001) suggest a deep study of the interactions of the system professor - student - knowledge, taking into account the social construction of the knowledge: this brings to a revision of the notion of interaction, with reference to the Brousseau's didactical contract. So in order to conclude our reflections, we came back to the theoretical framework mentioned at the beginning of this paper.

According to several researchers (Sfard, 1991), the historical development of a concept can be regarded as the sequence of stages: an early intuitive stage, and a mature stage; as we noticed above, with reference to infinite series, several centuries can pass between these stages. In the early stage the focus is mainly operational; the structural point of view is not a primary one: for example, concerning infinite series, in this early stage main questions of convergence were not considered. A similar situation can be pointed out from the cognitive point of view: of course, in the early stage pupils approach concepts by intuition, without a full comprehension of the matter. Then the learning becomes better and better, until it is full.

There is a clear analogy between these situations, and some experimental results seem to suggest that in the educational passage from the early stage to the mature one we can see, in pupils' minds, doubts and reactions that we can find in the passage from the early stage to the mature one as regards the *savoir savant*. Naturally, processes of teaching and learning take place nowadays, after the full historical development of the *savoir savant*, regarding both the early and the mature stage. So the *transposition didactique*, whose goal is initially a correct development of intuitive aspects, can also be based upon the results achieved in the mature stage of the development of the *savoir savant* (with regard to teachers, of course, a deep epistemological skill is needed).

Nevertheless, a correct sociological approach points out some difficulties: since an *operational* conception can be considered before a *structural* one (Sfard, 1991, p. 10), as far as infinite series is concerned the passage from an operational conception to a structural one has been really arduous, because of the necessity of some basic notions, like the limit concept. This has relevant consequences for education. For instance, some remarks in the historical part can suggest the following issue: is it reasonable to introduce convergence in schools without prior introduction of the limit notion, since this is what happened historically? In our opinion a crude paralleling of history with learning processes would connect two cultures referring to quite different contexts, so it cannot be used without an adequate consideration of the social and cultural backgrounds (moreover we cannot forget that frequently a better use of the developed systematic structure of mathematics for teaching runs counter to the direct paralleling of history with learning processes).

The introduction of infinite series in the classroom is not simple and several aspects can be considered. Certainly, for instance, embodiment (Lakoff & Núñez, 2000) is one of the most important issues of research into mathematics education and it is relevant to investigate further connections between perceptions and symbols. However it is important to underline that the fundamental work by Lakoff and Núñez is devoted to cognitive aspects: from a strictly epistemological point of view, the crucial point is the passage from finite to infinite; and metaphorical reasoning, very important from the educational point of view, must be controlled by the teacher in order to avoid dangerous misguided generalisations (Bagni, 2000b).

In our opinion, the main problem of the passage from finite to infinite is a cultural one, and historical issues are important in order to approach it (Radford, 1997). Undoubtedly, the historical approach can be considered together with other approaches (e.g. the educational use of visual representations: Duval, 1995). Several questions are still open: for instance, the reading of primary sources can be an important and effective tool, but it needs a clear consideration of the historical evolution of representation registers (Bagni, forthcoming). Concerning the teacher's role, which is also fundamental as far as teacher training is concerned, perspective teachers should be aware of the possibilities connected to the use of the history into education. Further research can be devoted to clarifying the questions mentioned.

Acknowledgments

The author would like to thank Ricardo Cantoral (Mexico) and Torkil Heiede (Denmark) for the valuable help and for important bibliographical references.

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