Situating Student Errors: Linguistic-to-Algebra Translation Errors

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ABSTRACT

While it is well recognized that students are prone to difficulties when performing linguistic-to-algebra translations, the nature of students' difficulties remain an issue of contention. Moreover, the literature indicates that these difficulties are not easily remediated by domain-specific instruction. Some have opined that this is the case because few frameworks exist which are sufficiently robust and revelatory to provide the insights needed to address these difficulties. This study uses a model situated in both a cognitive-oriented frame and the translation process itself to analyze student activity in the linguistic-to-algebra translation process, defines error types made, and recognizes the frequencies of such.

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The ability to translate the cover story in a linguistic depiction of a mathematical situation (or word problem) into algebraic language (symbolism or an equation) is viewed as a prerequisite skill to understanding mathematics (NCTM, 2000). Subsequently, most educators recognize that students must learn to translate a mathematical situation – usually expressed in a linguistic format – into an algebraic format and retranslate a solution back into the linguistic format. It is often argued that linguistic-to-algebra translations not only provide actualizations of the abstract mathematical ideas students are expected to acquire but also serve as platforms for students to model their previously acquired knowledge in mathematics.

Unfortunately, linguistic-to-algebra translation is precisely what most students have not learned to do (Clement, 1982). Research documents several instances (e.g., distance-rate-time problems, work-rate problems, the students-professor problem, Mindys' restaurant problem, and the country wheat problem) where students struggle or are prone to error when asked to render the linguistic cover story into the symbolic language of algebra (Clement, 1982; Koedinger & Nathan, 2004; Yerushalmy, 1997). So the question naturally arises, what are the sources and nature of student's difficulties in linguistic-to-algebra situations?

As will be seen in following discussions, it has been historically determined that students are prone to make errors during linguistic-to-algebra translations (and the student-professor problem) through research centered primarily on whether or not students could correctly perform the task, and not specifically defining error types within the work. Many of the individual studies have delimited themselves by defining one particular error type at a time and most studies have considered student work singularly in respect to the initial verbal expression and the subsequent algebraic solution. Summarily, based primarily on students' final symbolic representation, many of the previous studies have been able to determine if an error (or a particular type of error) occurred during linguistic-to-algebra translations. However, that leaves the research question at hand still unanswered.

Altogether, as will be later explicated in greater detail, this immediate study builds upon the historic research and extends such in a number of ways. First, this study assesses student understanding and error typology through student work and dialog in respect to three interconnected representations: verbal, diagrammatic, and symbolic. Second, in order to ascertain student understanding and define error types made, this study employs a modern framework to both define types of errors and pinpoint where (or when) they occur in the translation process. Third, this study seeks to determine if student errors occurred at different times in the translation process contingent upon the type of errors created. Fourth, this study situates itself in the perspective of signs and sign use (Ernest, 2006) and specifically investigates linguistic-to-algebra difficulties from a representation-based perspective (Duval, 2006; Kaput, 1987).

Since this investigation builds and extends upon previous research by others and many dimensions are addressed in these studies, the following background necessarily addresses numerous important dimensions associated with this immediate investigation. Some of these dimensions include: perspectives on the source of difficulties in linguistic-to-algebra translations; the representation-based perspective on performing translations between mathematical representations; attributes of linguistic-to-algebra translations; and the Translation-Verification framework used in this study.

BACKGROUND AND LITERATURE REVIEW

This study is built upon a foundational history of research in the realms of mathematical representations, translations between different representations, and, in particular, linguistic-to-algebra translations. In order to develop the modern rationale in which this study is situated, a historic perspective must be considered. The following review of the literature incorporates some of this historic perspective.

Perspectives on Linguistic-to-Algebra Difficulties

Student difficulties in linguistic-to-algebra situations have been studied extensively in the mathematics education community. Findings from these studies have led to the articulation of different prevalent perspectives regarding cognitive events contributing to student difficulties in linguistic-to-algebra translations. For instance, some researchers analyze student difficulties from a mathematical or algebraic perspective. They argue that mathematics as a domain uses symbols (e.g., letters, base ten numeration, =, +, ÷) as parts of its discourse and creates narratives based on these symbols and their operators (Kaput, 1987). Subsequently, in order to comprehend mathematical or algebraic ideas, students need: an adequate grasp of mathematical symbols, semantics, and lexicons; an attention to individual symbols, their referents, and the ways in which the symbols are combined; and an understanding of the relations represented by these combinations (Hall, Kibler, Wenger, & Truxaw, 1989).

From this mathematically-oriented perspective, it can be argued that student difficulties or errors arise due to misunderstandings students have concerning the meaning of mathematical symbols when they either interpret or use these symbols (Küchemann, 1981; Philipp, 1992; Wagner & Parker, 1993). In order to be successful in linguistic-to-algebra situations, students must have implicit processing knowledge of equation syntax and semantics, ability to view the equal sign as an agent of comparison, and knowledge for comprehending symbolic notations and for uncovering implicit and explicit relations (Falkner, Levi, & Carpenter, 1999; Kaput & Sims-Knight, 1983; Kirshner, 1989; MacGregor & Price, 1999). Errors arise, in part, because students misinterpret algebraic letters as abbreviated words, labels for objects, or the equal sign as a separator symbol. This position regarding variable comprehension as the principal reason for linguistic-to algebra difficulties is, however, questioned by some researchers (e.g., Wollman, 1983).

Some researchers analyze student difficulties in linguistic-to-algebra situations from a text comprehension perspective (e.g., Barbu & Beal, 2010; Shaftel, Belton-Kocher, Glasnapp, & Poggio, 2006; Solano-Flores, 2008). They opine that linguistic situations which can be mathematized potentially serve as platforms where students can both model their previously acquired knowledge in mathematics and obtain everyday illustrations of the abstract mathematical ideas they are expected to acquire (Janvier, 1987). Given that the texts in a linguistic-to-algebra situation are not merely concatenations of words and sentences, there is a need for students to adequately grasp the objects, data, syntactic rules (i.e., microstructure of the text), as well as sentences, phrases, and etc. (i.e., macrostructure of the text) used to constrain interpretation of these objects in order to comprehend the text (Cummins, Kintsch, Reusser, & Weimer, 1988).

From a text-oriented perspective, it can be argued that student difficulties or errors arise because students fail to comprehend the text (Mestre, 1989). In order to be successful in such situations, one must use linguistic knowledge to interpret relational as well as assignment statements, distinguish relevant from irrelevant information, translate what is stated, and restate the givens and goals in one's own terms (Nathan, Kintsch, & Young, 1992; Roth, 1996). Errors arise, in part, because students fail to understand the syntax and semantics of the text: the givens, the unknowns, and the ideas articulated in the text (Lewis & Mayer, 1987). This hypothesis regarding text comprehension as the primary generator of student difficulties, however, is discounted by other researchers (e.g., Clement, 1982; Wollman, 1983).

Some researchers analyze student difficulties in linguistic-to-algebra situations from a perspective regarding the processes employed by students in these situations. This process-oriented approach has led to the emergence of two opposing perspectives regarding the source of student difficulties in linguistic-to-algebra situations: the syntactic perspective and the semantic perspective. In order to better explicate these perspectives, we will discuss them in the context of the linguistic situation "there are six times more students than there are professors" and recognize that the semantically aligned algebraic relation would be $6P = S$.

Researchers who adopt a syntactic perspective to linguistic-to-algebra difficulties argue that errors arise, in part, because students perceive that the order of words in the verbal statement must necessarily map to the order of symbols in the algebraic expression (e.g., $6S = P$). These errors are a result of students ascribing logical necessity to the sequence of key words in the verbal statement and doing a direct, sequential, left-to-right algorithmic mapping of words to symbols when rendering the verbal situation into a symbolic algebraic relation. However, the veracity of the syntactic misconceptions hypothesis as a progenitor of linguistic-to-algebra difficulties is brought into question by some studies (e.g., MacGregor & Stacey, 1993).

Researchers who adopt a semantic perspective to linguistic-to-algebra difficulties argue that errors arise due to static comparison processes students adopt when working in the symbolic schemes available in algebra (MacGregor & Stacey, 1993). Consistent with this view is the belief that students correctly discern magnitude distinctions among quantities in the given textual situation, but fail to adequately account for these distinctions when using the conventions and rules available in algebra. As a result, they realize that one quantity is more than another and try to account for this by placing a multiplier next to the symbol associated with the larger quantity. In other words, they fail to discern the difference between semantic (translation for meaning) and syntactic (literal translations) characterization of relations when using the conventions and rules available in algebra.

Altogether, each perspective provides a different lens describing the origin of errors associated with linguistic-to-algebra situations. Although numerous studies have applied these domain and process specific perspectives to investigate linguistic-to-algebra translation errors, the nature of student difficulties, and how best to remediate student errors or conceptual deficits, many have failed in their attempts to mitigate these difficulties (Rosnick & Clement, 1980). The Balkanization of these perspectives potentially creates a false dichotomy that excludes possibly more unified understanding of these difficulties and errors associated with such. Given both the polarization of the perspectives and the questionable success it provides in making significant advancements in student success in linguistic-to-algebra situations, some researchers have argued for a different research focus on student difficulties in linguistic-to-algebra translation. For example, some researchers argue the need for a focus on prior knowledge students bring with them to algebra learning and how best to utilize this knowledge to mitigate linguistic-to-algebra difficulties. Philipp (1992) recommends research that focuses on algebraic variables, student interaction with algebraic variables, and the mitigating impact of problem structure on this interaction. Others argue the need for research that focuses on the nature of information presented in problems and how different linguistic nuances in problems affect student success

(Martiniello, 2008; Wolf & Leon, 2009). However, Duval (2006) argues that analyzing difficulties through concepts and their epistemological complexity may not be sufficient in a domain, such as mathematics, where mathematical objects are only accessible through representations. To avoid replicating historic fragmentation, this paper considers linguistic-to-algebra difficulties from a perspective situated in signs and sign use (Ernest, 2006). The paper specifically investigates linguistic-to-algebra difficulties from a representation-based perspective (Duval, 2006; Kaput, 1987). This is explicated in the following discussions.

Representation-Based Perspective

Central to a representation-based perspective is the conceptualization that mathematical concepts or relationships are never accessible by perception or through physical instruments; the only way of having access to them, and to working with them, is through a *representation system* (Duval, 2006), a system of information or objects whose convention of denotation and transformation of mathematical concepts are established based on rules agreed upon by the mathematics community (Cobb, Yackel, & Wood, 1992). Duval (2002) uses the term *representation register* to denote such representation systems that allow for the possibility of articulating and transforming mathematical ideas. In this paper the terms representation system and representation register will be used interchangeably. Researchers identify necessary components of any representation system. Kaput (1989) describes a notation system or symbol scheme with rules for identifying or creating characters, operating on them, and determining relations among them. Ernest (2006) identified sets of characters and operators that can be written, drawn or encoded; sets of rules for production and transformation; and sets of meaning relationship or convention of denotation.

A perusal of the different types of linguistic-to-algebraic situations found in the literature reveal two distinct representation registers: a *verbal* or *natural language register* and an *algebraic* or *symbolic register*. These two registers are unique in the sense that they each possess their own core set of characters for representing ideas and for constraining interpretation on those ideas. They also each have their own structural conventions and unique rules for transforming represented ideas (Kaput, 1989), for moving from one permitted configuration of the register to another (treatment) and moving from sets of configuration to new sets (conversion). Duval (2006) uses the terms *treatment* and *conversion* to distinguish between the two types of transformations that can be performed in any representation register. In the case of the verbal register, these core set of characters may include alphabets, words, punctuation symbols, parts or speech, grammar, and syntax which can be combined in complex, rule-governed structures to represent ideas and relations inside and outside the domains of math. Furthermore, users can perform treatments in the verbal register to transform the textual description in a given cover story while maintaining the meaning. In the case of the algebraic register, some of the characters may include, letters, constants, variables, and operators (+, -, =, ÷ etc.). Using these characters and the rules associated with the system, users can transform $x+4x=15$ into $5x=15$ and $x=3$ to make the solution to an equation explicit.

Users can transform a given linguistic cover story in a verbal register into an algebraic relation in an algebraic register using cognitions and structures associated with both registers. However, due to the phenomenon of representational determinism (Zhang, 1997), the structure or information captured in a verbal register may not necessarily correspond on a one-to-one basis with the structure captured in an algebraic register. For example, let us consider the work rate situation: “Three workmen can do a piece of work in a certain amount of time. A can perform it once in *x* weeks; B can perform it three times in *y* weeks; and C can perform it five times in *z* weeks. How much time will they spend if they worked together to perform the work?" Herein, the verbal register articulates the contexts and objects of the situation (semantic structure) and hides or distorts the mathematical relation among these objects. The algebraic register on the other hand articulates the relations (mathematical structure) among objects in the situation (i.e., Total time = $\frac{xyz}{yz+3xz+5xy}$) and hides or distorts the semantic structure of the situation.

Subsequently, two structures undergird any linguistic-to-algebra situation: a semantic structure (captured explicitly in a verbal register) and a mathematical structure (captured explicitly in an algebraic register). *Semantic* structure describes the objects, data, and physical elements (quantities) together with the words and phrases used in the cover story to refer to, and constrain interpretation on, them (Hall et al., 1989). *Mathematical* or *quantitative* structure refers to the constants, unknowns/variables, constraints, operators, and their associated symbolic relationship that approximates and describes the relations among the quantities in the problem (Shalin & Bee, 1985).

Linguistic-to-algebra situations are thus designed as analogies between their semantic and mathematical structures (Bassok, Chase, & Martin, 1998). Success in a linguistic-to-algebra situation depends on the student's ability to infer and correctly align the semantic structure of the cover story depicted in the verbal source register with an isomorphic mathematical structure via an algebraic target register. In this paper, we describe *translation* as the process involved in aligning the semantic structure of the linguistic cover story with an analogous symbolic mathematical structure.

Notably, employing the representation-based perspective for this investigation serves to modernize findings in comparison to much of the previously cited research through which perspectives and error types were identified. Additionally, the representation-perspective understandably builds upon earlier research from previous perspectives.

Linguistic-to-Algebra Translations

Most researchers agree that any translation between source and target representations involve a complex integration of cognitions associated with each representation as well as knowledge structures concerning the relationships between these representations (Adu-Gyamfi & Bossé, 2014; Bossé, Adu-Gyamfi, & Chandler, 2014; Kaput, 1989). In the case of a linguistic-to-algebra translation, this involves a complex integration of cognitions associated with source verbal or natural language register and target algebraic register and the interrelations among these registers. Researchers (e.g., van Dijk and Kintsch, 1983; Duval, 2006; Janvier, 1987; Lesh, Post and Behr, 1987; Sternberg, 1984; Wolman, 1983) identify different processes involved in a translation. For example, Wolman (1983) outlines: understanding the sentence; understanding the algebraic equation; having a method for generating the equation from the sentence; and having a method for checking that the equation generated is correct. Lesh et al. (1987), similarly maps: analysis of the verbal situation; synthesis of meaning of the verbal situation; algebraic representation formulation; transfer of content of the verbal situation into the algebraic representation form; and restructuring of the algebraic representation. van Dijk and Kintsch (1983) articulate: text comprehension; situation model formulation; and algebraic model formulation. Sternberg (1984) describes: *selective encoding* (sifting out relevant from irrelevant information); *selective combination* (combining selected information in such a way as to render it interpretable); and *selective comparison* (rendering newly encoded or combined information meaningful by comparing its relations to old information previously).

By analyzing and synthesizing these differently defined translation processes, and the frameworks proposed by others (e.g., Clement, 1982; Duval, 2006; Kaput, 1987; MacGregor & Stacey, 1993; Kaput & Sims-Knight, 1983), a common core of activities emerge. According to the literature, during translation from a source to a target representation, students are either: interpreting information presented in representation registers; performing (executing) a translation action (heuristic) that converts elements and ideas depicted in one representation register into another; or assessing (or ensuring) the equivalence in information articulated between two representation registers. Adu-Gyamfi, Stiff, and Bossé (2012) distilled these activities into the *Translation-Verification Model* to account for student activities in the translation process. Without pedantically repeating the frameworks from which constructs of the model were synthesized (e.g., Clement, 1982; van Dijk & Kintsch, 1983; Duval, 2006; Kaput, 1987; Lesh et al., 1987; MacGregor & Stacey, 1993; Kaput & Sims-Knight, 1983; Sternberg, 1984; Wolman, 1983), the model is herein, described.

The Translation-Verification Model

As stated above, within the translation process from a source to a target representation, students are either interpreting, performing, or assessing. Although it may seem that this is the logical order through which students perform translations, past research demonstrates that this is frequently not the case (Adu-Gyamfi & Bossé, 2014; Bossé, Adu-Gyamfi, & Chandler, 2014) and that the actual order of these components is idiosyncratic in respect to the translator. Indeed, student work demonstrates that, when performing translations, steps are often not linear, steps are often skipped, and the order of the steps (source-to-target versus target-to-source) is inconsistent. Furthermore, by recognizing that these steps can be codified by attribute interpretation, implementing transitional processes, and evaluating conceptual veracity and consistency, these three activities can encapsulate all actions that are performed in the translation process. Thus, the constructs of *Implementation*, *Attribute*, and *Evaluation* can be employed to situate all activities associated with the translation process. Moreover, since errors arise within translation process, all errors are also captured within these constructs.

Much research reveals that students are prone to make errors during linguistic-to-algebra translations, but few studies attempt to pinpoint where (or when) in the translation process errors are made. This study attempts to address this open question. The Translation-Verification Model (Adu-Gyamfi, Stiff, & Bossé, 2012) is employed in this study as a framework to analyze student errors in linguistic-to-algebra situations and determine where (or when) these errors occur in the translation process. Notably, the three *constructs* of the model (Attribute, Implementation, and Evaluation) are situated locations for translation activities; interpretation activities occur in the Attribute construct, execution activities occur in the Implementation construct, and assessing activities occur in the Equivalence construct. Figure 1 illustrates the different paths through the Translation-Verification Model. The figure indicates that in a linguistic-to-algebra translation, students can begin the translation process at any construct (albeit beginning at the Equivalence construct would be unusual), pass from the initial construct to any other (or series of others), iteratively traverse the different constructs and exit the model at any point (even if the translation process is not completed).

The power of the model is that it captures the situated locations for translation activities, and, thus maps out locations in the translation process where possible errors may arise. If one or more of the constructs of the model are not properly traversed, then a step is broken and errors may subsequently arise. The location of the break(s) could thus serve as a means to account for student error(s). Using the model, errors can be situated in the: Attribute construct (if the source and or target registers are not properly interpreted); Implementation construct (if a translation algorithm or heuristic is incorrectly utilized); and Equivalence construct (if an omission or commission occurs due to failure to verify source and target representational for consistency).

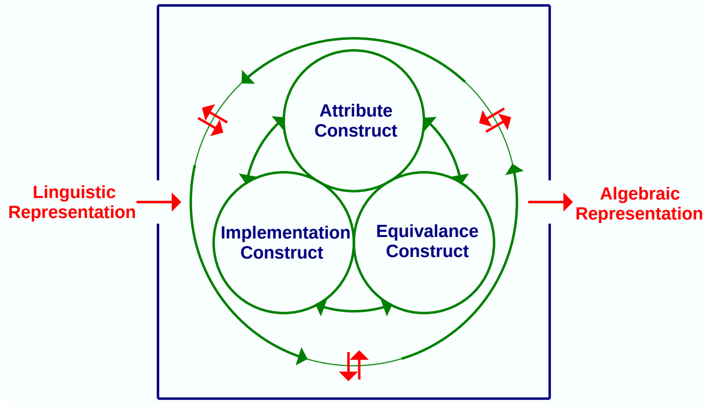


Figure 1. Translation-Verification Model.

The capacity of the Translation-Verification Model to account for student errors has the potential to greatly enhance research regarding linguistic-to-algebra translations. Later in this paper, as findings are reported, a fuller understanding of the Translation-Verification Model and its application for the locating and defining of student errors will be developed.

RESEARCH METHODOLOGY

The nature of this research carries a degree of novelty. While previous research regarding linguistic-to-algebra translations (and the student-professor problem) centered on whether students could correctly perform the task, far less of this research has focused on the types of errors made by students in performing the translation tasks, and virtually none have attempted to assess precisely where (or when) the errors occur in the translation process. Furthermore, this study seeks to determine if student errors occurred at different times in the translation process contingent upon the type of errors created. These questions were addressed through the following research structure.

Participants

Participants for the study were selected from a pool of undergraduate elementary education majors at a large southeastern university in the United States. These students were either in their second or third year and almost all had taken a mathematics methods for teachers course that included a more traditional examination of elementary algebraic concepts and functions with some exposure to the conceptual development of these ideas. Few participants had exposure to algebra significantly beyond investigation of quadratic functions.

The three-item research task (defined below) was formulated based on ideas synthesized from the literature on mathematical representations in general and linguistic-to-algebra situations in particular (e.g., Clement, 1982; Kintsch, 1988; MacGregor & Stacey, 1993). The first of the three-item task was used to determine potential research participants. A large pool of elementary education majors were asked to rewrite the verbal statement, "there are six times more students than there are professors," into algebraic form (symbolic task item). Students whose solutions were found to be in error (n=150) were invited to further participate in the study; 88 volunteered to participate. These participants were then asked to meet, individually, to complete the two remaining items on the task and to articulate their reasoning and processes. In order to fully understand the meaning and notions behind their responses on these items, participants were each interviewed as they completed the tasks, and their verbal articulations were recorded.

Tasks

Among the previously mentioned linguistic-to-algebra scenarios commonly recognized in the literature (e.g., distance-rate-time problems, work-rate problems, Mindy's restaurant problem, and the country wheat problem), the student-professor problem was selected for this study because it is an adequate exemplar of linguistic-to-algebra problems and it has a long and rich research history surrounding it. However, within the history of research, distinct findings are made which often seem more complementary than consistent. While some could argue that it would be of greater benefit to investigate other, or novel, scenarios with a lesser historical research base, it was believed that the nature of the Translation-Verification Model could provide both additional findings to previous research and help all previous research findings cohere.

Although participants had already completed the symbolic task item, they were asked to complete all three research tasks (including the symbolic task item again). For the verbal statement "there are six times more students than there are professors," students were asked to: (A) rewrite the problem in their own words (sentential task item); (B) draw a picture representing the verbal statement (diagrammatic task item); and (C) rewrite the verbal statement in algebraic form (symbolic task item). These tasks were to be completed in no particular order. These tasks were selected because: (1) altogether, the three tasks cover most commonly recognized mathematical representations; (2) students working with multiple representations allows them to interact with their preferred representational modality and with others; and (3) observing student work through multiple representations is more revelatory of student understanding, processes, misunderstandings, and errors, any or all of which may not be observable through work in any one representation. Further detail regarding the nature and rationale for each of these tasks follows.

The sentential task item required that students paraphrase the student-professor problem in their own words and produce a sentential representation that served as accurately as possible the main ideas they discerned from the given cover story. Research reveals that when reading a word problem, a student first constructs a propositional text base model representing the semantic structure or meaning of the problem statement (van Dijk & Kintsch, 1983). Such a model represents (to the best of the student's knowledge) essential details, ignores nonessential details, presents things qualitatively, and provides situational constraints. Kintsch (1988) suggests that it is through this model that students can recall the text, answer questions about the text, and summarize and reproduce the text itself. Therefore, it was assumed that what students recognize as important in the original text would be highlighted in the sentential representation they produce and that this ultimately furnishes clues into the nature of their interpretation regarding the propositional model for the problem. This sentential item served as a means to gain insight into student reasoning regarding the semantic structure for the problem at hand.

The diagrammatic task item required that students generate a diagram to illustrate their thinking on the quantitative relationships discerned from the student-professor cover story. Research reveals an interrelationship between student generated drawings/diagrams and student thinking about relationships on quantities in a problem situation (Kersch & McDonald, 1991).

Clement (1982) asserts that such drawings help in making cognitive explanations more explicit, visualizable to the student, and amenable to elaboration. Larkin and Simon (1987) further argue that such diagrammatic representations are meant to preserve explicitly the information about the topological and geometric relationships among the components of the problem. Therefore, it was assumed, herein, that correspondences and connections students made in the diagrammatic representation they generated would be revelatory of their thinking regarding the inherent mathematical relationships they discerned in the cover story. Because these diagrams were constructed using rules and conventions of representational systems created by the students themselves, it was assumed that relations articulated through them would ultimately furnish insight into the nature of student knowledge regarding the problem's inherent mathematical structure (Shigematsu & Sowder, 1994).

The symbolic task item required that students formulate a relationship for the student-professor problem using symbolisms and schemes available in algebra. Skemp (1982) argues that the symbolism and conventions associated with algebra form a symbol system that can be used to refer to a set of relations between concepts. Facility with the algebraic register requires attention to the individual symbols, their referents, the ways in which the symbols are combined, and the relational constructs represented by these combinations. von Glasersfeld (1987) opines that, since such written symbols are non-iconic and are based on agreed upon conventions, each learner constructs his or her own meaning or ideas to make sense of their experiences. It is often argued that learners who attend to the relations between symbols apart from the concepts and their interactions are attending to the syntactic surface structure of algebraic expressions. On the other hand, those who attend to the concepts represented by the symbols and to the relational structure reflected in their combinations are attending to the semantic structure of algebraic expressions. Thus, in this study, this item serves as a means to discern the way students interpret, manipulate, and configure symbols in algebra and to categorize, and further analyze, errors that recur in student work.

Regarding student difficulties in linguistic-to-algebra situations, most previous studies have contributed by focusing on domain knowledge and processes students employ in these situations. Simultaneously considering student developed sentential, symbolic, and diagrammatic representations in response to the student-professor problem distinguishes this study from many of its predecessors. It was believed, herein, that the totality of students' work within each of the different registers of representations and the connections or interrelations they make among the different representations would better illuminate the nature of student understanding and difficulties.

Data Codification and Analysis

The analysis of data proceeded through two stages. In the initial stage, individual student data were examined and compared to get a general sense of the data and to reflect on their overall meaning. Common themes within and across each student work were elicited and then compared. In the second stage, the data was examined via the lens of the Translation-Verification Model. Student formulated equations, diagrams, and paraphrased sentences were analyzed and then coded according to the constructs of the model. The analysis was conducted in a qualitative manner with the purpose of finding answers to the following questions in respect to linguistic-to-algebra translations in the context of the student-professor problem: What types of errors do students commit? In what particular construct in the Translation-Verification Model can student error be situated? Does the type of error committed associate to specific processes students employ?

Student formulated equations, diagrams, and paraphrased sentences were coded using the Translation-Verification Model, and common themes were identified. Using the model, student errors were coded as due to: either a misinterpretation of the verbal source register or a misrepresentation in the algebraic target register (situated in the Attribute construct); an algorithmic misstep (situated in the Implementation construct); or an introduction of an incorrect constraints or the overlooking of a critical constraint in both representations (situated in the Equivalence construct). In the following discussions, examples of the analysis are provided in respect to the constructs of the model in which they are situated.

STUDY FINDINGS

The findings of this study are presented in two parts: individual work and generalized results. The former assists to validate the revelatory power of the Translation-Verification Model while simultaneously illuminating the work and errors identified in the work of individual students. The latter provides a more global understanding of the location and timing of student errors in the linguistic-to-algebra translation process.

Notably, in many cases, student responses on the three-item task painted an incomplete picture of their understanding (and misunderstanding) of the student-professor problem. As demonstrated below, student interview responses were complementary and provided additional insight into the verbal, diagrammatic, and symbolic articulations students provided on the tasks. Students' interview responses were particularly invaluable when their written responses were difficult to interpret or when there seemed to be inconsistencies between written responses provided by any one student. In the following discussions, written and verbal responses are intertwined to best interpret students' understandings.

Findings Regarding Individual Student Work

Attribute construct. Given that a translation can sometimes be performed syntactically with little or no attention given to the object of manipulation, analyzing and vetting the appropriateness and correctness of these actions, while necessary, may not be sufficient in linguistic situations (Clement, 1982; MacGregor & Stacey, 1993). For example, if asked to write an algebraic relation representing a particular verbal situation, it is clear that knowing that analytic-modeling is the appropriate action to use is probably not enough to avoid dissonance between linguistic and algebraic syntaxes (Clement, Lockhead, & Monk, 1981), since the organizational order for the meaningful concepts and attributes in the verbal source representation is not necessarily in a direct left-to-right order in the algebraic target representation (Duval, 2006). If errors are to be avoided in such situations, a translator must necessarily have word and syntax awareness in a verbal context as well as symbol and syntax awareness in an algebraic context (MacGregor & Price, 1999). They must distill meanings from the words used in the linguistic situation and algebraically articulate the distilled relationship. To do so, it is necessary for a translator to understand: (1) the attributes of both source and target representations (their characters, how they encode information and also what processes are possible in each of them) and (2) the relation between source and target representations (which attributes of the source representation are confounding and need to be ignored and which ones are valuable and need to be considered during the translation) (Kaput, 1987). Errors may arise if a student incorrectly ascribes, mischaracterizes, or misrepresents attributes or properties of either the source or target representation. The Attribute construct, thus, provides a lens through which the critical attributes or meaningful concepts identified in the source representation are correctly interpreted and appropriately encoded in the target representation.

In general, errors were coded as situated in the Attribute construct if they were a result of either a misinterpretation of the semantic structure of the cover story or a misrepresentation of the mathematical structure in the problem situation. Figures 2 and 3 show examples of Attribute-situated errors that resulted from a misinterpretation of the semantic structure of the cover story. Figures 4 and 5 show examples of Attribute-situated errors that resulted from a misrepresentation of the mathematical structure of the problem.

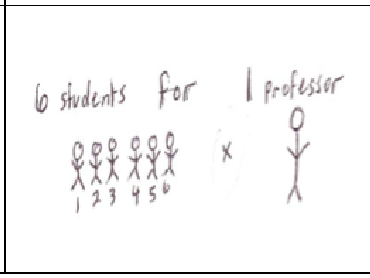
Paraphrased Sentence	Generated Diagram	Linguistic-to-algebra Error
"The student population at the university is six times the professors"		6s • p

Figure 2. Misinterpretation Error #1

As denoted in Figure 2, while the student correctly identified the essential quantities in the cover story of the problem (i.e., populations of students and professors), interview data suggested that the student misinterpreted the words and phrases used to refer and constrain interpretations on these quantities. For example the student noted, "I know that it is 6 times something, but I keep flipping back and forth between 6 times students and 6 times professors". Additionally, the student stated, "I think there is supposed to be an equal sign somewhere but I'm not too sure where". While the student's generated diagram and algebraic representations were correctly aligned with the sentential representation she produced, when probed for why this is the case, the student expressed surprise and responded, "Why…mmm…because it is the same question; so your answers have to be the same." Since the three representations generated by the student correctly aligned with each other, it was inferred that the error resulted from a misinterpretation of the semantic structure of the cover story; so this was coded as an interpretation error situated in the Attribute construct.

Paraphrased Sentence	Generated Diagram	Linguistic-to-algebra Error
"If you multiply six by the number of students, your answer will be greater than the # of professors b/c there are more"		S(6) > P

Figure 3. Misinterpretation error #2

Ancillary to the work provided in Figure 3, the student asserted, “six times, so the number of students will always be greater than the number of professors.” While the student’s assertion seemed to initially indicate an understanding of the semantic structure of the cover story, interview data showed that this was not necessarily the case. Upon being asked when the numbers of students will be equal to that of professors she added, “never, because even 5 times more students than professors will still make more students.” The student interpreted the cover story as singularly representing an inequality – that there are a greater number of students than professors. Interestingly, the student correctly developed both diagrammatic and algebraic responses consistent with her incomplete interpretation of the semantic structure of the cover story. Subsequently, the student’s error was coded as a misinterpretation error situated in the Attribute construct.

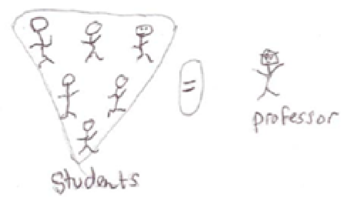
Paraphrased Sentence	**Generated Diagram**	**Linguistic-to-algebra Error**
“The Universities population of students is growing such that there are 6 times the number of students as professor”		$6S = P$

Figure 4. Misrepresentation error #1

Figures 4 and 5 show examples of Attribute-situated errors resulting from a misrepresentation of the mathematical structure of the problem situation. Both figures show paraphrased sentential representations that seem to accurately mirror the semantic structure of the problem’s cover story. Both figures also show diagrammatic depictions that correctly align with generated symbolic representation but misalign with the paraphrased sentence. In Figure 4, the student articulates a relationship between students and professors through the diagrammatic depiction of six characters equal to one unique character. However, interview data suggested that the diagram generated by the student was created based on characters and conventions that focus on the literal relationship in the problem. For example, it was revealed during the interviews that the phrases “students and professor”, as used in the diagrammatic depiction and the letters *S* and *P* in the algebraic relation, were used to label objects in the problem. She interpreted the cover story to represent a literal 6 students and a literal 1 student and not 6 times any number of professors. The excerpt below provides an insight into the student’s reasoning regarding her diagram. (In all following transcripts, *I* represents the *Interviewer* and *S* represents the *Student*.)

S: Okay, so I have six students and one professor (pointing to the problem statement). So I draw six here, one, two, three, four, five, six (pointing to the left-hand-side [LHS]) and one there (pointing to the right-hand-side [RHS]).

I: I see your character for student is quite different from that of the professor. Why is that the case?

S: Yes, because they are different. I have to show that the professor and the student are different.

I: Okay, so can there be more than six here (pointing to the LHS)?

S: No, because the problem states that there has to be six students.
I: How about the professor? Can there be more than one of them?
S: Hmmm. No because it says one so there can only be one.

At no point during the interview did the student acknowledge that there could be a possibility of multiple configurations for the number of professors and students in the population of the university. Furthermore, it was revealed that the relationship represented in both the diagram and the algebraic equation was syntactically produced through a left-to-right identification of key words and phrases. That is, the student utilized syntactic cues to generate the relations between characters and symbols in these registers. Though the student accurately captured the semantic structure of the cover story, this error was coded as a misrepresentation error because the student misrepresented the mathematical structure in the problem situation and attended to the characters in the problem separately from the concepts and their interactions.

In Figure 5, the student represented the relationship between students and professors through a two-column depiction of images of groups of characters. At first glance, the student's spatial depiction of a large group for *S* and a much smaller group for *P* seems to indicate that she comprehended the problems' inherent mathematical structure. Interview data, however, suggested that the student's diagrammatic depiction was based on a convention that misrepresents the quantitative relation between the number of students and the number of professors. For example, the student utilized the same character to depict the different entities in the problem; six characters for the letter *S* and one character for the letter *P*. When asked why she used the same character for both, she responded, "They are all people, so I put 6 in *S* and 1 in *P*." Moreover the letters *S* and *P*, as used in the student's response, symbolized graphic or iconic representation of the entities in the problem and not algebraic symbols as required by the problem's inherent mathematical structure. Thus, her diagrammatic depiction was based on a convention of relative size between the two groups – size disparity between students and professor – rather than equal numbers as required by the problem's inherent mathematical structure, thus leading to the symbolic representation $6S = P$. This misrepresentation error is consistent with the static comparison error or semantic translation misconception identified in the extant literature (Kaput, 1987; MacGregor & Stacey, 1993).

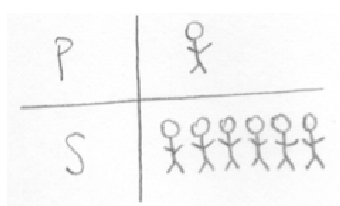
Paraphrased Sentence	Generated Diagram	Linguistic-to-algebra Error
"There is one professor for every 6 students"		6S=P

Figure 5. Misrepresentation error #2

In general, misrepresentation errors resulted because students failed to correctly discern the quantitative relations among problem elements or objects. The question arises as to why misinterpretation errors arose, given that these same students correctly discerned and articulated text-based models that accurately mirrored the semantic structure of the cover story (Kintsch, 1988). A possible answer may be found in the phenomenon of representational determinism (Zhang, 1997). Since each representation register captures and hides or distorts different aspects of a concept or mathematical idea, an ability to successfully interact with an idea in one representation register (e.g., verbal register) may not necessarily imply facility with another (e.g., algebraic register). From a mathematical comprehension perspective, it could be argued that students who committed misrepresentation errors failed to discern: the implicit mathematical structure in the situation; the mathematical symbols, semantics, lexicons, and their referents; and the ways in which the symbols are combined and the relations represented by these combinations (Hall et al., 1989).

Implementation construct. In any given linguistic-to-algebra situation, a translator must map information expressed in words (source representation) into an algebraic relation (target representation). Such an activity may be implemented syntactically in a literal, step by step, procedural manner guided by the structures of the source and target representations with little or no attention paid to the object of manipulation or the meaning of the information (Kaput, 1989). However, since these syntactic actions follow algorithmic-like conventions and depend on the nature of representations involved, they could be a contributing source for errors if there is an algorithmic misstep or if an inappropriate action is used. The Implementation construct thus furnishes a lens for tracking and vetting syntactic actions.

In general, errors were coded as situated in the Implementation construct if they resulted from an algorithmic misstep or as a result of incorrect treatments in the algebraic register. These were mostly students who correctly interpreted the cover story and successfully discerned the mathematical relation between students and professors but made errors as they interacted with the syntaxes and conventions of the algebraic register. Figures 6 and 7 exemplify these errors. In both cases, the semantic structure of the cover story is correctly interpreted and accurately represented with an isomorphic mathematical structure. Students' sentential responses and diagrammatic depictions attest to their correct interpretation of the semantic structure of the cover story as well as accurate representation of the problem's inherent mathematical structure. However, both students articulate incorrect equations.

Figure 6 shows a generated equation that is inconsistent with both the paraphrased sentence and the diagrammatic representation. The student was then asked to explain her generated equation.

I: What do *S* and *P* stand for?
S: The *S* is number of students and *P* is the professors.
I: So how did you come up with "$6S=1P$"?
S: I knew that however many professors there are, there are 6 times that amount of students. So for every 5 professors there are 30 students. If I had 2 professors then there would be 12 students. So, at this particular university, there is a ratio of 6:1 students to professors.
I: You mentioned that the ratio is 6:1. How is that related to your equation?
S: I don't understand, what you mean?
I: So the equation you got is $6S=1P$, and you mentioned that the ratio of students to professor is 6:1. How are the two related?
S: Hmm, I'm not too sure how to answer, I don't really know.
I: Can you explain to me how your equation works?
S: Sure. It just means that, if I have 6 students, I should have 1 professor and, if I have 12 students, I should have 2 professors.
I: But in your equation you have $6S=1P$.
S: Yes.

I: So are you saying that 6 times the number of students is equal to the number of professors?
S: No, because that will mean that if I have one student there will be 6 professors.

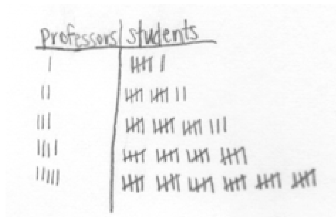
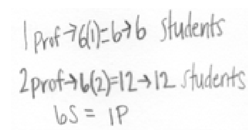
<table>
<tr><th>Paraphrased Sentence</th><th>Generated Diagram</th><th>Linguistic-to-algebra Error</th></tr>
<tr><td>"If there are 10 professors at the university there are 60 students. The ratio is 6:1"</td><td>
</td><td>
</td></tr>
</table>

Figure 6. Implementation error #1

In the course of the interview, the student was adamant that the written juxtaposition of 6 with *S* and 1 with *P*, did not mean $6 \times S = 1 \times P$, since that would lead to an incorrect interpretation of the problem. When asked to further elaborate on the meaning of $6S = 1P$, she wrote "$\frac{6}{S} = \frac{1}{P}$." This relation, while addressing the mathematical structure inherent in the problem situation, was highly unexpected; it was unclear how the student transformed the relation $6S = 1P$ into $\frac{6}{S} = \frac{1}{P}$. Through further communication, the researchers inferred that the student incorrectly depicted $6S = 1P$ to refer to the relation $6{:}S = 1{:}P$, thereby leading to $\frac{6}{S} = \frac{1}{P}$. The error in the student's generated equation was seen to be a product of an algorithmic misstep and was thus coded as an Implementation error.

Through the sentential and diagrammatic representation in Figure 7, the student reveals that she correctly interprets both the sematic and mathematical structure of the problem. However the student's generated algebraic relation is inconsistent with both the paraphrased sentence and diagrammatic representations. When asked to explain her algebraic relation, the student stated, "The ratio of the number of students to professors is 6 to 1." However, she symbolically represented $6S{:}P$, which incorrectly captured her intended meaning. When she was asked to explain her reasoning regarding the algebraic relation, the following conversation ensued:

I: You said the equation will be 6S:P.
S: Yes.
I: You also mentioned that if there are 10 professors there will be 60 students.
S: Yes.
I: Can you explain to me how you got your 60 students from 10 professors?
S: I have ten circles and 6 in each circle that makes it 60.
I: Can you show me with the equation?
S: I have 6 students to 1 professor. So, if I have 5 professors, it will be to 30 students. And if I have 10 professors, then 60 students. So it checks.

While it was initially unclear how the student was deriving her answers in her algebraic representation, through further communication it was inferred that her notation of $6S{:}P$ was erroneously being used to denote a direct proportional relation between students and professors (i.e., $6S \rightarrow P$), where S and P are units of the quantities under consideration; so $6S \rightarrow P$ will imply $30S \rightarrow 5P$ and $60S \rightarrow 10P$. The error in the student's generated equation was seen to be a product of incorrect syntax awareness (Falkner et al., 1999; Kaput & Sims-Knight, 1983; Kirshner, 1989; MacGregor & Price, 1999) and was coded as having an Implementation source.

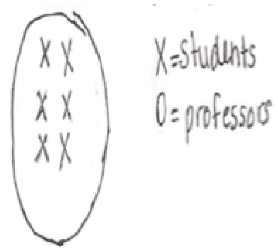
Paraphrased Sentence	Generated Diagram	Linguistic-to-algebra Error
"The ratio of students to professors is 6 to 1"		6S : P

Figure 7. Implementation error #2

In general, Implementation-situated errors resulted from either an algorithmic misstep or incorrect use of syntax in the algebraic register. Some may argue that students who committed Implementation errors did so because of a lack of mathematical knowledge. However such a view does not adequately account for why these same students were able to articulate correct quantitative relationships between problem elements in both their generated diagrams and their verbal reasoning accompanying their incorrect symbolic relations. Goldenberg (1995) distinguishes between surface confusions about any one representation and true misconstructions of the concept. Students who committed Implementation errors in this study demonstrated surface confusions about the algebraic register rather than true misconstruction of the mathematics. From a representation-based perspective, it could be argued that the Implementation errors arose, herein, due in part to difficulties with: characters and conventions of algebraic registers; processing knowledge of equation syntax and semantics; and the ability to view the equal sign as an agent of comparison (Falkner et al., 1999; Kirshner, 1989).

Equivalence construct. A one-to-one mapping between concepts and attributes of the source and the target representation may not be possible (Duval, 2006). Subsequently, in a linguistic-to-algebra situation, there should be a commensuration of attributes of both source and target representations. When there is no commensuration of attributes, dissonances may result between natural language syntax and algebraic syntax due to either errors of commission (when an incorrect concept or attribute is introduced in the target representation) or errors of omission (when a critical concept or attribute in the target representation is overlooked). The act of commensuration in the translation process is tantamount to an iterative source-to-target and target-to-source verification that necessary concepts and attributes from both source and target representations map correctly to corresponding concepts and attributes in either representations without being perturbed by idiosyncrasies of either representations. The Equivalence construct of the Translation-Verification Model furnishes a lens through which structures or attributes not explicitly demonstrated in the source, but nevertheless encoded in the target and vice versa, could be vetted for its consistency across the two representations.

Notably, since equivalence verification between two mathematical representations necessitates repeated interpretations of attributes, features, and characteristics of the two representations and the mathematical notions that they encode, equivalence verification may be incorrectly recognized as little more than repeated occurrences of the Attribute construct. However, herein, the Equivalence construct is recognized as significantly more than determining what representational attributes mean and what concepts they encode; the Equivalence construct also considers whether the representational attributes of two representations encode similar mathematical concepts. Therefore, while the Attribute construct incorporates interpretation, the Equivalence construct incorporates interpretation in conjunction with comparison (i.e., coordination). Altogether, therefore, since the process of verifying equivalence between representations is (A) an iterative process of interpretation and implementation between the representations, at least in part, and (B) interpretation and implementation errors are more readily observed and discerned, then it is most likely that errors associated with the determination of equivalence are captured as either interpretation or implementation errors prior to being properly coded as errors associated with the Equivalence construct. Furthermore, data sets from this and previous studies seemingly lack the ability to discern many errors determined specifically within the realm of assessing representational equivalence. Therefore, the lack of finding errors attributable to the Equivalence construct is readily understood. This rationale is commensurate with the findings in this study, where no discovered errors were singularly discernable as equivalence errors distinct from errors associated with the Attribute and Implementation constructs.

Non-situated errors. Student errors were coded as *non-situated* when connections could not be established among any of the three representations generated by the student and the work and accompanying errors could not be associated to either the Attribute, Implementation, or Equivalence constructs. Figure 8 exemplifies a student's work coded as non-situated. The example shows that all three of the student's generated representations are inconsistent with each other; the diagrammatic depiction is inconsistent with the paraphrased sentence and algebraic relation and vice versa.

Interview data suggested that the student selected different aspects of the cover story and used for each representation. For example, explaining what aspects of the word problem she considered in order to generate a diagram, she stated, "I look at 'six times professors.'" Explaining the development of her final symbolic representation, she responded, "I know that the problem wants me to find a total. So, that means addition." Moreover, when asked how her different solutions to the items were related, the student's conversation and actions connoted her complete surprise. "I don't know if they are related, other than they all come from parts of the same word problem. I just did what I thought I needed to for each part." This student was not certain that a link existed, or could be established, between the three distinct representations she generated.

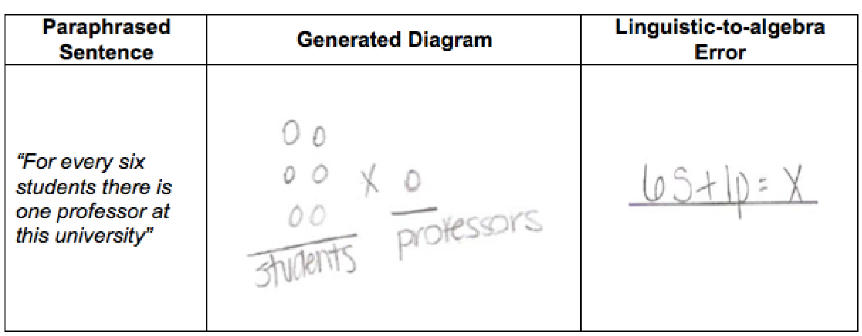


Figure 8. Non-Situated error.

Since her work clearly demonstrated a lack of connections among representations, it may be immediately queried why these errors are not situated in the Equivalence construct. As previously mentioned, Equivalence construct errors result when there is a dissonance between representation registers due to either errors of commission (when an incorrect concept is introduced in the target representation) or errors of omission (when a critical concept in the target representation is overlooked). Notably, this student's work was fraught with errors well before she considered equivalence between the representations, and she never independently considered equivalence between representations. Since it was, therefore, determined that this student's errors were not consistent with Equivalence construct errors, it was difficult to situate student error in any of the constructs of the Translation-Verification Model.

Generalized Results

This study analyzed linguistic-to-algebra translation difficulties from a representation-based perspective through the Translation-Verification Model. Students' oral explanations and reasoning, equations, diagrams, and paraphrased sentences were analyzed and then coded according to the constructs of the model. Table 1 reports the results of the analysis and details the constructs of situation of the errors, the source of the errors and their frequencies.

	Source of Error				
Situated locations	Misinterpretation	Misrepresentation	Algorithmic misstep	Unknown	Total N
Implementation Construct	-------	--------	20	----	**20**
Attribute Construct	29	29	------		**58**
Equivalence Construct	------	------	-----	------	**0**
Un-situated	-----	------	------	10	**10**
Total	**29**	**29**	**20**	**10**	**88**

Table 1. Error Source, Frequency and Location of Situation

Of the errors recognized in this study, regarding linguistic-to-algebra translations, two constructs of the Translation-Verification Model were found to be prevalent: 66% of the errors were situated in the Attribute construct and 23% were situated in the Implementation construct; 11% could not be situated in any of the constructs of the model. Of the 58 Attribute-situated errors identified, 28% were traced to have resulted from a misinterpretation of the semantic structure of the cover story and 35% were traced to have originated from a misrepresentation of the inherent mathematical structure of the given problem situation.

To perform any translation between mathematical representations, students must distinguish among features, characteristics, and attributes in the source representation which are mathematically relevant and those which are irrelevant (Duval, 2006). In parallel discussions: Lesh et al. (1987) discuss the analysis of the source representation and the synthesis of meanings of the source representation; Kaput (1987b) articulates reading the representation; Sternberg (1984) considers selective encoding (sifting out relevant information from irrelevant information in order to select information for further processing); and Bossé, Adu-Gyamfi, and Chandler (2014) define this process within unpacking the source, the first stage in the translation process. In a linguistic-to-algebra situation, students must interpret the cover story, extract and reduce key information, and discriminate between relevant and irrelevant information. This requires that they interpret combinations or configurations of characters in the verbal register in order to discern encoded relationships.

Later in the process, students must reformulate the encoded relationship via algebraic symbolism. This step necessitates the interpretation and articulation of another set of characters according to syntactic rules available in an algebraic register. This requires thinking about mathematical operations, the meaning of symbols, and how letters are used in equations to represent variables. To complete the translation, students must again interpret configurations of characters in both verbal and algebraic registers to ensure alignment of the semantic structure of the cover story with an isomorphic mathematical structure. These activities parallel findings from others. For instance, Duval (2006) proposes that, in order for students to be able to identify the same object denoted in two different representations, there is a need to progress beyond networking of facts between the representations to making associations between representations. Lesh et al (1987) captures these notions through the processes of: target representation formulation, transfer of the content of the source representation into the target representation, and restructuring of the target representation. Sternberg (1984) identifies this process as: selective encoding, selective combination, and selective comparison. Bossé, Adu-Gyamfi, and Chandler (2014) encapsulate these ideas through the processes of preliminary coordination, constructing the target, and determining equivalence.

Notably, all Implementation-situated errors resulted even after students clearly demonstrated understanding of the cover story as well as the quantitative structure of the given problem. While students correctly interpreted the cover story and discerned the quantitative relation between students and professors, in the act of algebraically articulating the relation, they performed an implementation error. Thus, errors can be made in the doing of mathematics, despite correctly interpreting the mathematics verbally defined and seemingly understanding how the mathematics is to be done. The data in this study shows that no errors situated in the Equivalence construct were observed. This result is later discussed in greater detail.

Interestingly, 11% of the errors identified in this study couldn't be accounted for via a representation-based perspective or constructs of the Translation-Verification Model. Immediately, it may be wondered if, and how, this study's construct-situated errors compare with perspectives on linguistic-to-algebra translation errors identified in the extant literature (i.e., text comprehension, variable comprehension, etc.). To better understand how a representation-based perspective allows previously incomparable perspectives on linguistic-to-algebra errors to be juxtaposed, compared, and contrasted, student data were examined and compared via perspectives identified in the literature. Table 2 details the juxtaposing of perspectives and constructs and ties results from this study to the previously cited literature.

	Perspectives on Errors					
Representation	Text Comprehension	Mathematical Comprehension	Syntactic Misconception	Semantic Misconception	Unaccounted	Total
Implementation	-----	6	----	-----	14	20
Attribute	16	2	20	15	5	58
Un-Situated	-----	-----	----	----	10	10
Total	**16**	**8**	**20**	**15**	**29**	**88**

Table 2. Juxtaposing Perspectives and Constructs of the Translation-Verification Model

Of the 58 Attribute-situated errors identified: 28% were traced to have text comprehension as their source or origin; 35% were traced to have originated from syntactic processes; and 26% resulted from static comparison processes. Notably, 8% of the Attribute-situated errors could not be accounted for through a perspective identified in the literature. Of the 20 Implementation-situated errors identified, 30% resulted from variable miscomprehension and 70% could not be accounted for through any other perspective identified in the extant literature. Notably, no one particular perspective identified in the literature could adequately account for all the situated errors identified in this study.

Error types connected to error sources. Four distinct error types were recognized within student responses and work on the symbolic item of the task. Therefore, in order to more fully understand the connection between error types and their respective sources, student work was grouped, reorganized, and analyzed again in respect to the type of error students made. Three of the groups committed errors consistent with linguistic-to-algebra errors identified in the extant literature: total equation errors (variable quantities are summed to get a total: $6(S)+P$ = university, $P+6S$ = total, $6S+P$= Pop, etc.,); product errors (variable quantities are multiplied to get a product: $(6S)\times(1P)$ = # of students per professor, $6S\times P$, etc.,); and reversal errors (variables in the equation are reversed: $6S=1P$, $6S=P$, and $P=6S$). Errors not consistent with the three error types were described as *other errors* (e.g., $S(6) > P$ and $6S:1P$). The student groups were denoted: Total Equation Group (n=25); Product Group (n=14); Reversal Group (n=43); and Other Group (n=6). Table 3 reports the results of the analysis of students work in each group. The table details the groups, source of error, frequencies and their constructs of situation.

	Source of Error					
Group	Implementation	Misinterpretation	Misrepresentation	Equivalence Construct	Unknown	Total N
Total Equation	---	12	3	---	10	25
Product	---	13	1	---	---	14
Reversal	15	3	25	---	---	43
Other	5	1	-------	---	---	6
Total	**20**	**29**	**29**	**0**	**10**	**88**

Table 3. Tracing the Source of Errors for the Different Error Groups

Categorization of student errors on the symbolic item into the four broad categories provides insight into how students in each group performed on the task items. Notably, on the sentential item of the task, the Reversal Group and the Product Group had the highest and lowest respective percentages of students that provided valid responses. While the Other Group had very high rates of success regarding their text-based models, the diminutive size of this group makes generalizations problematic. Of all the recorded errors in the Total Equation Group, 48% resulted from a misinterpretation of the cover story (i.e., students failed to accurately represent the semantic structure of the given situation) and 12% were due to a misrepresentation of the quantitative relations between objects in the problem (i.e., students failed to accurately discern the mathematical structure and this ultimately contributed to their errors). Of the observed errors in the Product Group, 92% of the product errors resulted from a misinterpretation of the semantic structure of the given situation. Of the recorded errors in the Reversal Group, 58% were due to a misrepresentation of the mathematical structure in the problem. Of the recorded errors in the Other Group, 83% resulted from an algorithmic misstep (i.e., students incorrectly utilized syntax available in an algebraic register and this ultimately contributed to their errors).

Findings, above, from individual student work and global findings provide much information regarding the timing and placement of errors in linguistic-to-algebra translations in context of the student-professor problem. In the next section, we add discussion, implications and conclusions to these findings.

DISCUSSION, IMPLICATIONS AND CONCLUSION

The data in this study reveals that investigating linguistic-to-algebra translation difficulties through a representation-based perspective not only allows for focus on errors previously recognized in the literature but also transcends some previous studies in a number of ways. First, many of the previously cited studies were designed to define only one or two errors types. Thus, errors which could be investigated in one study could not be investigated in another. Altogether, only through a gestalt of previous studies could a more inclusive categorization be accomplished.

While other studies have investigated and categorized student errors in linguistic-to-algebra situations and many have focused on reversal errors, this study extends the literature by analyzing additional types of errors committed by students in the Reversal Group and considering errors committed by students in other groups. Moreover, previous studies have provided conflicting accounts regarding the cognitive events contributing to student errors. It is, therefore, not surprising to note that, while students' difficulty in linguistic-to-algebra situations have being studied extensively for over 30 years, the findings from these studies have yet to provide favorable insights for instruction.

Second, most previous studies have considered student work when translating singularly from one representation to another. As previously mentioned, a number of studies have focused on verbal representations, others on symbolic representations, and still others on diagrammatic representations, and most of these have made findings regarding student understanding and misunderstanding in respect to one representation at a time. Quite infrequent are studies, such as this one, which consider translations and transformations from one representation to three others. In so doing, the findings of previous studies were allowed to speak to each other and further illuminate student understanding in respect to linguistic-to-algebra translations.

Third, most previous studies were designed to determine whether an error occurred and, if so, what type of error it may be. Few studies have situated their investigation in the translation process to be able to determine where in the process the error occurred.

Further discussions and implications are offered in the following sections under the topics of Symbolic Errors, Situated Errors, Error Types and Error Sources, and Additional Implications through Micro-Concepts.

Symbolic Errors

Since all participants in this study made errors in the symbolic component of the three-task activity, it might be erroneously assumed that the task was insufficiently defined to direct students toward acceptable solutions. In other words, it may seem that the task was flawed. However, student participation in this study was predicated by their initial errors precisely on this task item. Thus, when participants completed the three-task activity – including the symbolic task a second time – it is entirely logical that they replicated the mistakes that gained them entree into the study. Conversely, if a significant number of participants did not produce erroneous symbolic results, this result would have been questionable in and of itself.

Situated Errors

Since it can be seen that interpretive acts permeate the translation process (and no less so in linguistic-to-algebra translations), it is not surprising that 66% of student errors were coded as situated in the Attribute construct. While it makes sense that the second highest percentage of errors (23%) is situated in the Implementation construct, this may be somewhat misleading. By definition, all Implementation construct errors result after students demonstrate interpretive understanding of the situational and quantitative structure of the given problem scenario. Once an error situated in the Attribute construct is made, it often sufficiently hinders the translation process so that students are either stymied in the process or subsequent errors situated in the Implementation construct are more difficult to discern. Therefore, a higher percentage of Implementation construct errors may have been possible if not perturbed by preceding interpretive errors.

Notably, all Implementation-situated errors resulted even after students clearly demonstrated understanding of the situation as well as the quantitative structure of the given problem. While students correctly interpreted the cover story and discerned the quantitative relation between students and professors, in the act of algebraically articulating the relation, they performed an implementation error. Thus, errors can be made in the doing of mathematics, despite correctly interpreting the mathematics verbally defined and seemingly understanding how the mathematics is to be done.

The data in this study shows that no errors situated in the Equivalence construct were observed. Initially this may be due to the fact that students did not attempt to verify equivalence between their verbal and algebraic representations and, if they had, they might have discovered that the linguistic and algebraic representations were incommensurate and may have corrected such (Wollman, 1983). Or, as was the case with fewer Implementation-situated errors than Attribute-situated errors, the number of Equivalence-situated errors may be mostly due to the fact that after a previous error is made, then the analysis process for determining which type of error occurred ceases. Additionally, since the process of evaluating for representational equivalence necessitates repeated interpretation across two representations, it may be possible to incorrectly situate these errors in the Attribute construct.

The lack of findings in respect to the Equivalence construct may connote a weakness in either the Translation-Verification Model used as the framework for this study or in the type of data collected for analysis. Since, as previously stated, verifying translational equivalence between representations requires the iterative process of interpretation and implementation and equivalence verification is more difficult to observe and discern than activities associated with interpretation and implementation in the translation process, new designs are needed for data collection which will either provide definitive evidence of student work regarding equivalence verification or determine the impossibility of discerning errors associated with the Equivalence construct in the model. This may imply the need for the revision of the Translation-Verification Model. Future research may speak to these issues

As seen in Table 2, no one particular perspective identified in the literature could adequately account for all the situated errors identified in this study. This may have been the case because previously cited studies, by their very nature, were designed, either directly or indirectly, to uncover errors of particular types (e.g., Barbu & Beal, 2010; Philipp, 1992). Moreover, quite infrequent are studies, such as this one, which consider translations and transformations in three distinct representations and which situate their investigations in the translation process to uncover the nature and source of student difficulties. This study, thus, transcends some previous studies as it provides a much more inclusive categorization of linguistic-to-algebra errors. In summary, the data reveals that a representation-based perspective not only captures errors previously recognized in the literature but also uncovers new ones.

Error Types and Error Sources

This study reveals that different types of mathematical error (total equation, product, reversal, and others) can originate in interpretive activities situated in the Attribute construct of the Translation-Verification Model. This demonstrates the need for instructional practices that transcend simply asking students to transform ideas between representation registers and focus attention on student interpretation as well as re-presentation of those ideas in the different registers of representations. However, the mathematical product error and the total equation error, when the origin of the latter could be accounted for, only originated from the Attribute construct. This allows teachers to not only recognize what mathematical mistakes students make during linguistic-to-algebra translations, but also that these errors occur during interpretive acts regarding one or more of the mathematical representations. This demonstrates the need for instructional practices to focus more attention on student interpretation of mathematical representations.

While this study reveals a disconnect between students' understanding of the semantic and mathematical structures associated with the investigated problem, it reveals that teachers should make efforts to ensure that students be given learning experiences which address both. As a pedagogical technique, students should often be asked to explain what they know, rather than this being assumed as they attempt either linguistic-to-algebra translations or other multi-representational translations.

Using a representation-based perspective this study reveals that errors committed by the reversal group are either situated in Implementation and Attribute constructs; thus, reversal errors either originate during interpretation of the semantic or the mathematical structure of the problem or in the process of reformulating information into an algebraic register. This knowledge can significantly assist teachers in recognizing the potential causes for student reversal errors. Similarly, knowing that product error and the total equation error, when the origin of the latter could be accounted for, originated from the Attribute construct allows teachers to not only recognize what mathematical mistakes students make during linguistic-to-algebra translations, but also that these errors occur during interpretive acts regarding one or more of the mathematical representation. With this understanding, teachers can focus upon instructional techniques which could focus more attention on this potential error and help students to avoid such.

Additional Implications through Micro-Concepts

The findings and subsequent discussions begin to paint a picture of what happened when students attempted linguistics-to-algebra translations in respect to the student-professor problem in this study. However, missing from this account are elements explaining why students performed as they did and had the difficulties and misunderstandings they demonstrated. Employing differing nomenclature, a number of researchers (e.g., Ainsworth, 1999; Cobb et al., 1992; Ernest, 2006; Goldin, 2002; Kaput, 1987; Zhang, 1997) describe a distinction between the mathematical ideas and relationships encoded within a mathematical representation and the codification of symbols, features, operations, characteristics, and rules and conventions for depicting such in these representations. Herein, these characteristics, features, and attributes constituent within a register (and not the ideas encoded by such) are denoted the *micro-concepts* of a representation (Bossé, Adu-Gyamfi, & Chandler, 2014; Bossé, Adu-Gyamfi, & Cheetham, 2011a); mathematical ideas can be encoded in a representation register through associated micro-concepts. Therefore, linguistic-to-algebra situations necessitate the use and traversing of micro-concepts associated with both verbal and algebraic registers. Therefore, learners who attend to the representation's micro-concepts without adequately considering the ideas they encode are merely considering the syntactic structure of the representation and those who attend to the representation's constructs perceive the semantic structure of the representation.

When reading a word problem, the student's construction of a propositional text base model denoting the semantic structure of the problem statement (van Dijk & Kintsch, 1983) demonstrates his understanding of: essential micro-concepts; the representation's encoded constructs; nonessential micro-concepts, and situational constraints inherent in both encoded constructs and micro-concepts. Therefore, the micro-concepts and constructs that students recognize as important in the original text are highlighted in what they produce; this informs the teacher regarding the nature of student representational interpretation. Similarly, student generated drawings/diagrams interpreting verbal representations reveal student understanding regarding the micro-concepts and encoded constructs of both representations (Clement, 1982; Kersch & McDonald, 1991).

In a linguistic-to-algebra situation, students must work in both a verbal and an algebraic register, learn to interact with micro-concepts available in those registers, and interpret the mathematical idea encoded in the register. The nature of verbal registers lends to greater difficulty in interacting with information articulated through such. Some (Bossé, Adu-Gyamfi, & Chandler, 2014; Adu-Gyamfi, & Bossé, 2014; Adu-Gyamfi, Stiff, & Bossé, 2012; Bossé, Adu-Gyamfi, & Cheetham, 2011a, 2011b) believe that the greater difficulty associated with interacting with verbal registers is due to their low attribute density and the high number of micro-concepts which can be interpreted in different ways – including a number of misinterpretations. A number of researchers note the global nature of interpreting verbal representations (Duval, 2006; Leinhardt, Zaslavsky, & Stein, 1990) and recognize that these global interpretations are more complex and difficult than are local interpretations needed to understand other representation registers.

Conclusion

In this study, the Translation-Verification Model was borrowed from Adu-Gyamfi, Stiff, and Bossé (2012) to extend beyond their investigation of the source of errors among symbolic, tabular, and graphical mathematical representations and to look at linguistic-to-algebra translations. Still absent from this combined body of work are investigation specifically focusing on translations from nonverbal representations into verbal representations and verbal representations into tabular and graphical representations. While these studies may be left to others, it is hoped that in the end a meta-analysis will synthesize all these studies and form an ever more coherent picture of student thought, actions, and error patterns in respect to understanding mathematical representations and performing translations between such. Altogether, it is hoped that this concretization of understanding will lead to instructional processes that will help students learn and avoid errors.

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