Joerg MEYER

Election Paradoxes: Social Choice

1. Introduction

Let us start with an example: last year, the speaker of a committee was elected by a run-off procedure. Charles was the winner. By the way, that it was Charles and not Eliza, depended not only on the voters but on the electoral procedure as well, as will be seen later on. Because Charles was a good speaker, the number of his supporters increased within the last year. Yesterday, the annual election happened once again, and although Charles had more persons in favor of him than one year ago, Eliza was the winner! Paradoxical? Yes, indeed, but paradoxes may happen not only with run-off elections but with every type of electoral procedure.

What is a "good" or "just" procedure for committee elections? We will see that demanding quite weak and self-evident properties leads to inconsistencies. This is the famous Impossibility Theorem by Kenneth Arrow which will be proven here in a new and very simple way.

A person is to be elected by a committee (social choice). If the absolute majority (more than 50 %) of the voters support candidate A that person will be the winner. There is no problem at all in this case. But absolute majorities cannot be guaranteed if there are more than two candidates. So one has to agree upon a certain procedure for the general case (and that agreement has to be made before the election takes place). There are several possibilities:

The winner is the one

- who gets the relative majority of the votes (plurality voting), or
- who is able to beat each other candidate ("Condorcet winner"; see below), or
- who wins by run-off, or
- who gets most "Borda points" (see below).

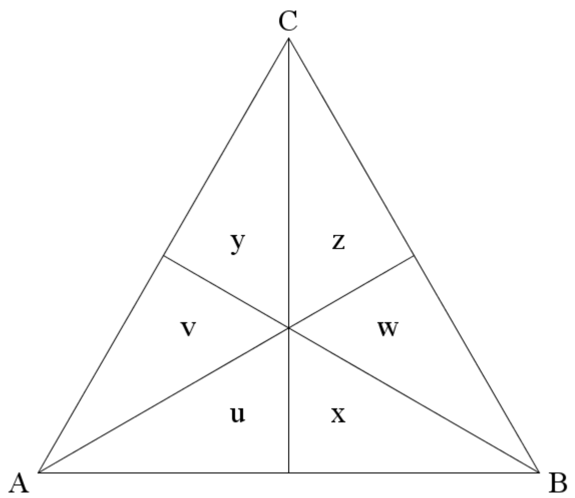
Of course, there are other procedures, but for purposes of remaining brief, they shall not be listed here. Each of the listed procedures (and of the not listed ones, as well) has paradoxical effects, which may not be conjectured by a naive mind.

The existence of paradoxes cannot be seen by studying the outcomes of past elections. Nor does it suffice to ask all voters of their supported candidates. To make paradoxical effects visible, we need more information. We need of each voter his / her individual preference list: whom does he /she think the best, the second-best, the third-best, ..., the worst. We look into the head of each voter.

Let's look at an example with three candidates A, B, and C:

u	v	w	x	y	z
A	A	B	B	C	C
B	C	C	A	A	B
C	B	A	C	B	A

Exactly u voters think A the best and C the worst candidate and so on. As we will almost exclusively deal with three candidates, we use the following geometrical visualization:



The electoral procedure determines not only the winner out of the individual preference lists, but determines the collective preference list, e.g.

B
A
C

. This symbol (which is always framed) means: B is the winner. In case he dies then A will be the winner.

It is reasonable to assume the individual preference lists are strongly ordered, i.e. no voter thinks that two candidates have equal quality. (Even with this assumption the effects are bad enough!)

One last word to the numbers which we will be used in the upcoming examples. They are very small (in many cases even minimal) and won't show up that small in reality. But small numbers have the advantage that their patterns are easier to grasp. Of course, one gets realistically great numbers just by multiplying the given numbers by an appropriate factor.

2. Examples for electoral procedures

2.1 Relative majority (plurality voting)

He who gets the greatest number of supporters is the winner.

2.1.a The winner of the relative majority can have the absolute majority against him.

3	2	2
A	B	C
B	C	B
C	A	A

The relative majority thinks A is the best candidate while the absolute majority thinks he is the worst.

2.1.b First inconsistency of preferences

4	2	3
A	B	C
B	C	A
C	A	B

A
C
B

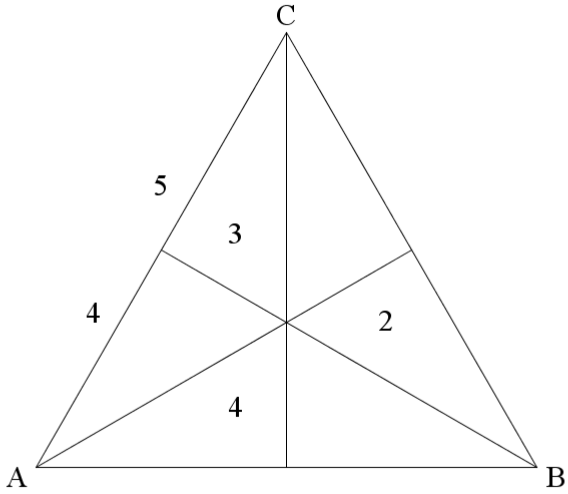
Most (4) of the voters think A to be the best, therefore he is on the top of the collective preference list. Least (2) of the voters think B to be the best. Therefore he is on the bottom of the collective preference list.

At the next election, the individual preference lists have not changed – with one exception: the (collectively) worst candidate, B, is no longer a candidate. This should not bother A:

4	2	3
A	C	C
C	A	A

C
A

Surprisingly, the order of the collective list has turned upside down! This effect can be visualized by passing from the numbers inside the triangle to their respective sums at the edges:



2.1.c Second inconsistency of preferences

For $n > 1$ let the preference pattern be

n	1	
A	C	A
B	A	C
C	B	B

At the following election, the individual preferences have not changed with one exception: the winner A is no longer considered a candidate:

n	1	
B	C	B
C	B	C

But now not the “old second best” C is not going to be the winner but B!

If, with plurality voting, a winner and his substitute are to be obtained there are two possible procedures:

- He who gets second most votes becomes the winner’s substitute, or
- He who gets the greatest number of votes at a new election (when the winner is missing) becomes substitute.

The second inconsistency of preferences shows that both procedures may result in different outcomes.

2.1.d Third inconsistency of preferences

4	3	2	
A	B	C	A
C	C	B	B
B	A	A	C

If there had been several one-on-one contests then C would have beaten A and B, and B would have beaten A. In that case, we would have got the collective preference list (C, B, A).

So pair-wise contests may result in different outcomes than triple-wise contests.

2.1.e The Condorcet paradox

1	1	1
A	B	C
B	C	A
C	A	B

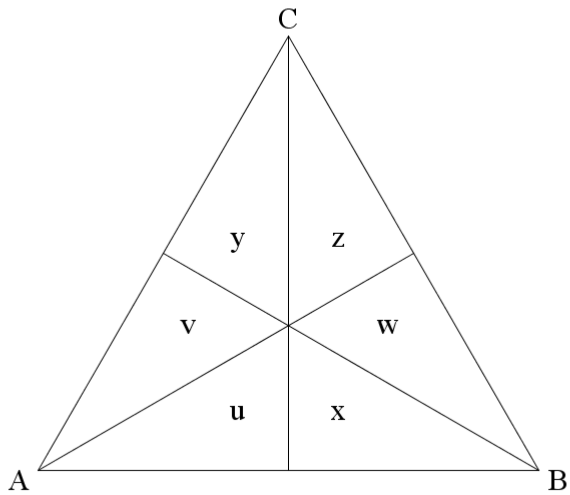
In this case there is no winner based on relative majority.

In one-on-one contests A would have beaten B, B would have beaten C, and surprisingly C would also have beaten A. So the collectice preference list is not transitive!

This paradox is named after Marie Jean Antoine Nicolas Caritat, Marquis de Condorcet (1743 – 1794). After the French revolution, electoral procedures and their paradoxes became interesting, and Condorcet was one of the first election theorists. Later on we will see that the Condorcet paradox is fundamental und unavoidable.

2.2 Condorcet winner

A Condorcet winner can beat each other candidate in an one-on-one contest.



This is the case if and only if and only if the inequalities $u+v+y > x+w+z$ and $u+v+x > w+y+z$ both hold.

2.2.a Condorcet winners may not exist

Here is a more general example:

1	n	n
A	B	C
B	C	A
C	A	B

A beats B, B beats C, and C beats A.

2.2.b A Condorcet winner may be supported by only one voter

For $n > 1$ let the preference pattern be

1	n	n
A	B	C
B	A	A
C	C	B

Only one voter thinks A is the best. In spite of that A is Condorcet winner.

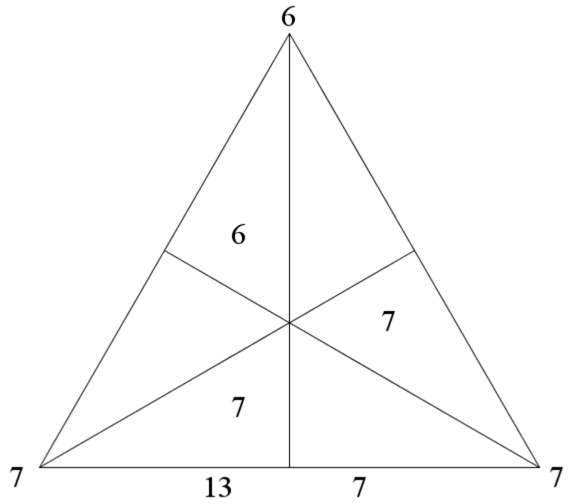
2.3 Run-off elections

If no candidate has got the absolute majority, those two with the greatest number of voters are sent to an one-on-one contest ("duel").
For the sake of simplicity let us assume that the individual preference lists do not change between the two voting acts. Then these two acts can be considered as one act.

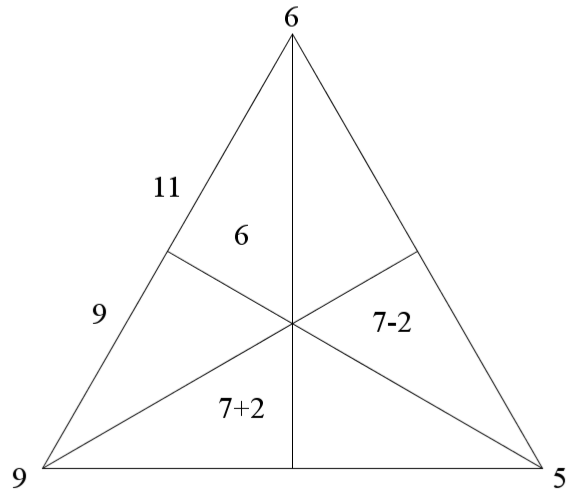
2.3.a Run-off elections are not monotonic, I

7	7	6
A	B	C
B	C	A
C	A	B

A and B are sent off to a duel which will be won by A.



A does his job so well that he gains supporters (from B) within the time of being in office; the rest of the individual preference lists does not change.



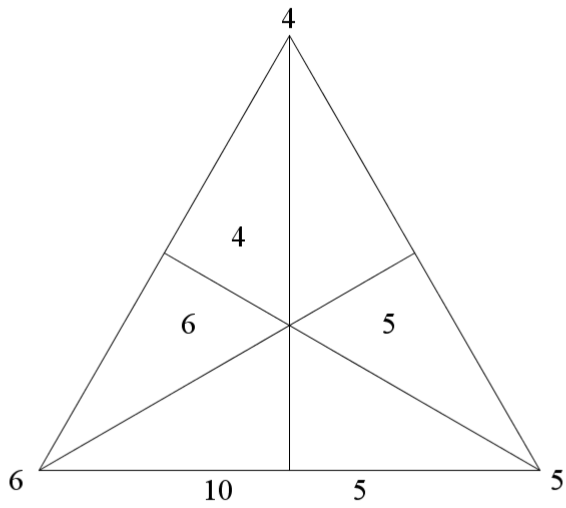
Now, A and C are sent to a duel which will be won by C!

Because of having gained supporters, A now loses the election (cf. introduction).

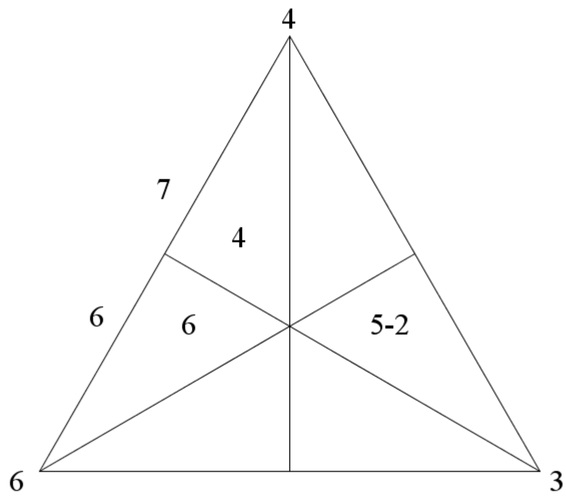
2.3.b Run-off elections are not monotonic, II

6	5	4
A	B	C
C	C	A
B	A	B

A and B gain most of the votes, A wins the entire election.



The voters of the second column are against A. Two of them do not vote at the second run-off election; the rest of the individual preference lists remains unchanged.



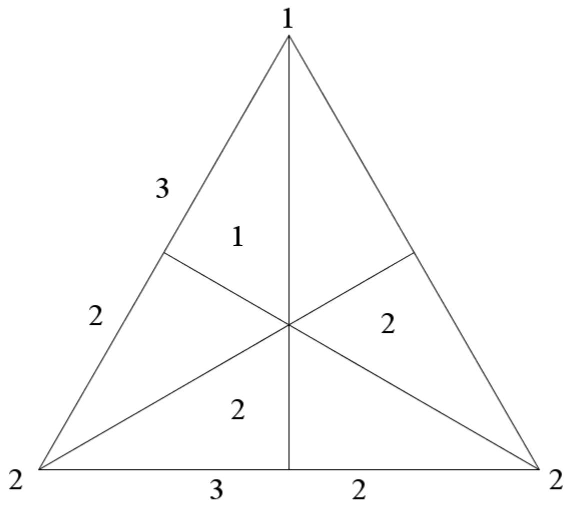
Now, A and C are sent to the duel which is won by C!

That means: if you want to cause a certain election outcome, it may be the best not to vote at all!

<u>2.3.c Inconsistency of preferences</u>

2	2	1
A	B	C
B	C	A
C	A	B

A and B are sent off to the duel, A wins ultimately. The collective preference list is A B C.

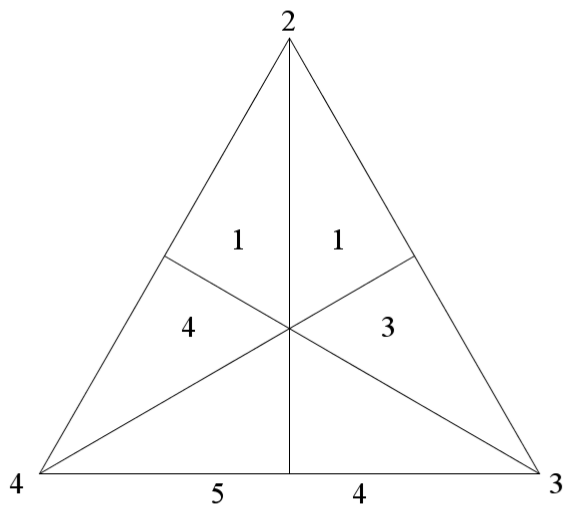


At the next election B is no longer candidate. This causes C to win the election, the collective preference list then being [C / A].

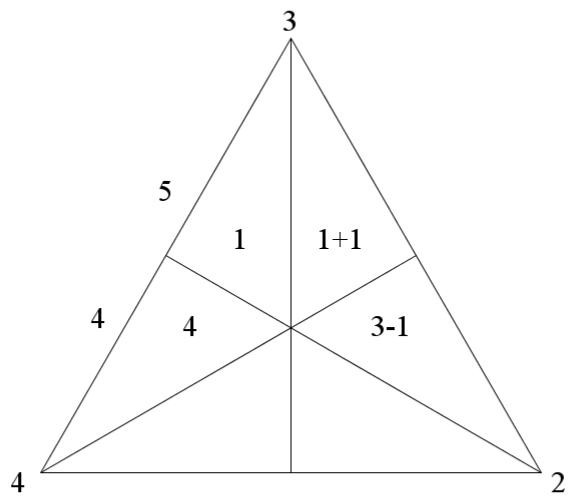
2.3.d The run-off winner is dependent on voters who are against him

4	3	1	1
A	B	C	C
C	C	A	B
B	A	B	A

A and B are sent off to the duel, A wins ultimately.



The voters of column 2 think A to be the worst candidate. If one of those voters had voted for C instead of B, the outcome would be the following:



Now C is the ultimate winner.

2.3.e The run-off winner may be a non-winner in each sub-committee

The first sub-committee may have the following individual preference lists, resulting in winner B:

2	3	4
A	B	C
B	A	A
C	C	B

In the second sub-committee, we have B as the ultimate winner as well:

10	7	5
A	B	C
C	C	B
B	A	A

If we put both sub-committees together, A is the ultimate winner:

2	3	4	10	7	5
A	B	C	A	B	C
B	A	A	C	C	B
C	C	B	B	A	A

2.4 Borda points

The Frenchman Jean–Charles de Borda (1733 – 1799) had the idea of taking the entire individual preference lists into account. It is explained by the following example:

factor	5	4	6	3
2	A	A	B	C
1	B	C	A	B
0	C	B	C	A

5 voters think A the best candidate and C the worst etc.

A gets $2 \cdot 5 + 2 \cdot 4 + 1 \cdot 6 + 0 \cdot 3 = 24$ points.

B gets $1 \cdot 5 + 0 \cdot 4 + 2 \cdot 6 + 1 \cdot 3 = 20$ points.

C gets $0 \cdot 5 + 1 \cdot 4 + 0 \cdot 6 + 2 \cdot 3 = 10$ points.

A is Borda winner because he got the greatest number of Borda points.

2.4.a The candidate with absolute majority does not have to be the Borda winner

factor	2	3
2	A	C
1	B	A
0	C	B

A has 7 points, B has 2 points, C has 6 points. A is Borda winner in spite of C having won the absolute majority.

2.4.b A Borda winner may be supported by only one voter

factor	1	n	n
2	A	B	C
1	B	A	A
0	C	C	B

A has $2+2\cdot n$ points; B has $1+2\cdot n$ points; C has $2\cdot n$ points.

2.4.c Inconsistency of preferences

factor	2	2	1
2	A	B	C
1	B	C	A
0	C	A	B

A has 5 points, B has 6 points, C has 4 points. The collective preference list is B / A / C.

At the next election, B is no longer a candidate. The rest of the individual preference lists has not changed.

factor	2	2	1
1	A	C	C
0	C	A	A

A has 2 points, C has 3 points. The collective preference list has turned upside-down to C / A.

If, instead of B, the "former worst candidate" C is no longer available, one gets:

factor	2	2	1
1	A	B	A
0	B	A	B

A has 3 points, B has 2 points. Also here the collective preference list has turned upside-down, to A / B.

2.4.d The Borda winner is dependent on voters who are against him

factor	3	2
2	A	B
1	C	C
0	B	A

A is Borda winner.

The two voters of colum 2 think A to be the worst candidate. If one of those voters had voted for C instead of B, C would have been Borda winner.

2.4.e The Borda triple is essential

Up to now the "valuation triple" $\begin{pmatrix}2\\1\\0\end{pmatrix}$ was used. Different valuation triples result in different Borda winners:

triple 1	triple 2	triple 3	2	3	4
3	3	6	A	A	B
1	2	5	B	C	C
0	0	0	C	B	A

Triple 1: A has 15 points, B has 14 points, C has 7 points. A is winner.

Triple 2: A has 15 points, B has 16 points, C has 14 points. B is winner.

Triple 3: A has 30 points, B has 34 points, C has 35 points. C is winner.

2.5 Dependence of the winner on the procedure

The different procedures (relative majority, Condorcet, run-off, Borda) may result in different winners, even if the individual preference lists are the same:

factor	1	3	2	1	1	3
3	A	A	B	C	C	D
2	B	C	C	B	D	B
1	C	B	D	D	A	C
0	D	D	A	A	B	A

A has the relative majority, B is Condorcet winner, C is Borda winner, D is run-off winner.

3. The Arrow Impossibility Theorem

We now look for an electoral procedure which is not supposed to have any of the above paradoxes. The following axioms seem to be desirable and quite self-evident:

3.1 Axioms

(1) The procedure shall produce out of individual preference lists (transitively ordered with no ties) a collective preference list (transitively ordered, ties allowed).

(2) The procedure shall respect unanimity: if every voter thinks A to be better than B, the collective list shall show A above B.

(3) No voter shall be privileged, so

n	1		collective
A	B		B
B	A		A

is impossible for $n > 1$.

(4) There shall be consistency of preferences: the positioning of candidates other than A and B in the individual preference lists does not have any effect on the order of A and B in the collective preference list.

(5) The procedure shall be monotonic: if a winner gains support he shall remain winner.

3.2 First consequences

The axioms are not independent ones: (5) is a consequence of (1) to (4) because of

Lemma 1: The preference pattern

n	m		collective
A	B		A
B	A		B

implies

n + 1	m − 1
A	B
B	A

$\boxed{\begin{matrix}A\\B\end{matrix}}$

Proof: We introduce a third candidate C who will not disturb the old preference lists:

n	1	m − 1
A	B	B
B	A	C
C	C	A

The supposition and (4) imply $\boxed{\begin{matrix}A\\B\end{matrix}}$. Because of (2) we have $\boxed{\begin{matrix}B\\C\end{matrix}}$, because of (1) we have $\boxed{\begin{matrix}A\\C\end{matrix}}$. Leaving B aside und using (4) we get

n + 1	m − 1
A	C
C	A

$\boxed{\begin{matrix}A\\C\end{matrix}}$

Because of (3) this preference pattern also applies for candidates A and B. Therefore lemma 1 holds.

Lemma 2: Irrespective of the electoral procedure the following preference pattern always holds for $n > 1$:

n	1
A	B
B	A

$\boxed{\begin{matrix}A\\B\end{matrix}}$

Proof: Because of (3) $\boxed{\begin{matrix}B\\A\end{matrix}}$ is impossible. Which of the remaining possibilities $\boxed{\begin{matrix}A\\B\end{matrix}}$ or $\boxed{A\quad B}$ (tie) holds can again be seen by introducing a third candidate C who will not disturb the given preference pattern:

n	1
A	B
C	A
B	C

Everyone wants $\boxed{\begin{matrix}A\\C\end{matrix}}$. $\boxed{\begin{matrix}B\\C\end{matrix}}$ cannot hold, so we have $\boxed{\begin{matrix}C\\B\end{matrix}}$ or $\boxed{B\quad C}$. Because of the transitivity we arrive at $\boxed{\begin{matrix}A\\B\end{matrix}}$. This proves lemma 2.

Now look again at the Condorcet paradoxon:

1	1	1
A	B	C
B	C	A
C	A	B

According to lemma 2 we always have $\boxed{\begin{matrix}A\\B\end{matrix}}$ and $\boxed{\begin{matrix}B\\C\end{matrix}}$ and $\boxed{\begin{matrix}C\\A\end{matrix}}$ at the same time. This is not compatible with the transitivity of the collective preference list. So with every electoral procedure, the Condorcet paradoxon may occur and is unsurmountable. (The tie solution $\boxed{A\quad B\quad C}$ will not work because of lemma 2.)

This shows for 3 voters and 3 candidates: The axioms (1) to (4) cannot be fulfilled at the same time.

This consequence can be generalized to more voters and more candidates. The famous Impossibility Theorem was first proven by the American economist Kenneth Joseph Arrow (*1921), who won the Nobel prize in 1972 (in honor of his theorem, amongst other things).

Theorem (Arrow): There is no electoral procedure for 3 or more candidates and 3 or more voters satisfying the axioms (1) to (4).

Proof: We need one more lemma.

Lemma 3: The preference pattern

n	m
A	B
B	A

$\boxed{\begin{matrix}A\\B\end{matrix}}$

implies

n – 1	m + 1
A	B
B	A

$\boxed{\begin{matrix}A\\B\end{matrix}}$

Proof: Look at

n – 1	1	m
A	C	B
B	A	C
C	B	A

According to the supposition we have $\boxed{\begin{matrix}A\\B\end{matrix}}$, lemma 2 implies $\boxed{\begin{matrix}B\\C\end{matrix}}$. The transitivity in axiom (1) results in $\boxed{\begin{matrix}A\\C\end{matrix}}$.

Therefore we have

n – 1	m + 1
A	C
C	A

$\boxed{\begin{matrix}A\\C\end{matrix}}$

Lemma 3 follows because of axiom (3).

Proof of the impossibility theorem: Repeated application of lemma 3 results in the following effect: With

n	1
A	B
B	A

$\boxed{\begin{matrix}A\\B\end{matrix}}$

we also have

1	n
A	B
B	A

$\boxed{\begin{matrix}A\\B\end{matrix}}$

The last preference pattern contradicts lemma 2. So the axioms (1) to (4) cannot be satisfied simultaneously.

Dr. Joerg Meyer
Schaefertrift 16
D-31789 Hameln
Germany
J.M.Meyer@t-online.de