

The Use of Alternative Algorithms in Whole Number Computation

Stephen Norton

Griffith University

s.norton@griffith.edu.au

Abstract

Pedagogical reform in Australia over the last few decades has resulted in a reduced emphasis on the teaching of computational algorithms and a diversity of alternative mechanisms to teach students whole number computations. The effect of these changes upon student recording of whole number computations has had little empirical investigation. As reported in this paper, Years 4 to 7 students across three schools were tested for their ability to carry out written computations. A range of recording methods were documented, many of which seemed to be adaptations of mental methods of computation. Students who used alternative methods tended to be less successful than students who used traditional algorithms. The results suggest there is merit in conducting further research into the effects of using alternative written computational methods upon students' learning of mathematics.

Key Words

Errors, whole number computation, alternative, algorithms

Introduction

This paper investigates the changing emphasis on teaching, how they are taught and the effects this is can have upon student learning. In this paper *traditional written algorithms* refer to the four *standard processes* for addition, long and short multiplication, and division laid out vertically emphasising place value and described by Plunkett (1979, p. 2) as “the four rules of number.” Over the past few decades there has been considerable criticism of teaching of formal written algorithms for whole number computation, in part because many children fail to master them (Anghileri, Beishuizen, & Putten, 2002). Ruthven (2001) cited cognitive passivity and suspended understanding as a further disadvantage in the traditional way mathematical algorithms were taught and used. Frequently the learning of traditional written algorithms was associated with an endless sequence of memorizing and forgetting facts and procedures that make little sense to the students (Marshall, 2003) or, as described by Hughes (1986, p. 8), “in many cases their performance consists entirely of meaningless manipulations of symbols”. In effect these criticisms echo that over emphasis on teaching traditional written algorithms resulted in “frustration, unhappiness and deteriorating attitude to mathematics” (Plunkett, 1979, p.3).

There has also been a change in emphasis in regard to mental computation or “a shift towards calculating in the head rather than on paper” (Thompson, 1995, p. 11). Plunkett (1979) argued that there were critical differences between mental algorithms and standard written algorithms. He argued that written algorithms were written, standardised, contracted, efficient, automatic and symbolic, and generalizable; whereas mental algorithms tend to be fleeting, flexible, variable, and less generalizable. Thompson (1995, p. 13) used the term “mental calculation strategies.” This distinction between written and mental is blurred since the English National Numeracy Strategy (*Framework for Teaching Mathematics from Reception to Year 6* Department for Education and Employment, 1999) considers the use of pen jottings during mental calculations as appropriate.

It noted that the way traditional written algorithms were taught did not align well with the way children intuitively learnt mathematics. Dutch reform included recommendations that more attention be paid to mental arithmetic and estimation, with much greater emphasis on realistic context and less emphasis on formally taught traditional algorithms (van de Heuvel-Panhuizen). Across the channel Julia Anghileri and co-researchers have carried out considerable investigation contrasting the traditional emphasis on written algorithms “that have been taught for successive generations” (Anghileri et al., 2002, p. 150) with informal strategies, such as those described by the Realistic Mathematics Education (RME) movement in the Netherlands. The Dutch reform places an emphasis on development of “naïve skills such as counting and doubling, and involves holistic approaches to number within a calculation in contrast with the place value approach developed within English curriculum” (Anghileri et al., 2002, p. 151). In essence the Dutch and English reforms reflected each other’s findings. The work of Anghileri and her colleagues carries on the early work of others (e.g., Plunkett, 1999; Thompson, 1995), who supported the use of informal strategies for the teaching of addition, subtraction, multiplication and division. Thompson (1999) described a progression for calculation, starting with mental approaches, passing through jottings, informal written methods, and then formal algorithms. Thompson’s later work (2007;2008) detailed various methods such as the use of the empty number line, partitioning, and expanded method in columns for addition and subtraction. The alternative methods described by Thompson for multiplication included partitioning with and without various grids. Similarly, division could be achieved by portioning and various expanded methods.

The Dutch and English approaches were based on research that supported the idea termed “guidance principle” which essentially means that children should be given the opportunity to re-invent mathematics, or to construct mathematical insights themselves (van de Heuvel-Panhuizen, 2001). This approach that emphasises student discovery in mathematics is predated by the British Cockcroft Report (Department of Education and Science, 1982) and Hughes (1986). Similarly across the channel, Beishuizen and Anghileri (1998) argued that the teaching of formal written algorithms be postponed and subsequently build upon children’s informal mental strategies. Essentially, reforms recommended that while “mechanization of the standard procedure have their advantages, it is important that the calculation does not become distracted from the meaning of the problem” (Anghileri, 2001, p. 151). These sentiments are reflected by other authors, for example Torbeyns, Verschaffel, and Ghesquiere (2005), who argued that mathematics learning should go “beyond routine expertise, that is the ability to solve mathematical tasks fast and accurately by means of standardised school-taught strategies” (p. 1). Rather, these authors stated that students should develop adaptive expertise, flexibly creating and employing meaningful strategies. Part of the reason for this was the belief that mental strategies including compensation strategies improved essential number facts as well as developing self-confidence in self-initiated strategies.

Anghileri (2001) has noted that differences in classroom practices and different approaches to calculator usage may also play a role in superior test results of Dutch students compared to English students. That is, the effect of particular teaching approaches can be confounded by the other variables including the general quality of teaching. This point becomes relevant in interpreting the results of this present study.

In part as a response to the criticisms associated with traditional written algorithms and the research on how some students work intuitively there has been a general curriculum trend across the western world (e.g., Australia, England, Holland, & USA) to teach mental strategies to children before they are introduced to formal written algorithms, The argument being that mental calculation lies at the heart of numeracy (Straker, 1999). For example, in the USA (e.g., Yackel, 2001) and Australia (e.g., Callingham, 2005; Heirdsfield & Cooper, 2004), research echoed the growing Dutch and English shift of emphasis onto teaching of mental computation and alternative methods. Some empirical evidence that teaching algorithms might be detrimental to student learning of mathematics has been presented. For example, Kamii and Dominick (1997) studied second, third and fourth grade students’ errors on addition tasks, and reported: “It was found that those children who had not been taught any algorithms produced significantly more correct answers. It was concluded that “algorithms ‘unlearn’ place value and hinder children’s development of number sense” (p. 51). Some authors took a firm stance against the teaching of traditional written algorithms: the Victorian Department of Education and Early Childhood Development (2009) cited Gravemeijer in cautioning the teaching of algorithms, rather advocating “instructional sequences in which the students act like mathematicians of the past and reinvent procedures and algorithms” (p. 121). The Victorian Department of Education documents also refer to authors who recommend, “students should not be taught algorithms, but should invent their own methods instead” (p. 4). The use of emerging technologies has been a further factor in the emphasis placed upon written algorithms.

It was reported by the Cockcroft Report (Department of Education and Science, 1982) that calculators were readily available and this necessitated a rethink of the need for students to learn calculation methods that were to a significant degree no longer used in the wider community. Support for the use of calculators is widespread and there is explicit support in the Australian national and state syllabus documents and advisory associations for widespread use of calculators in mathematics learning. For example the Australian

Association of Mathematics Teachers (1996, p. 6) recommended: “It is NOT necessary for students to learn about numbers and number operations before they can effectively use calculators; rather, such understandings and skills follow naturally from appropriate access to technology.”

It is not that the importance of written algorithms has been completely diminished, for example the Australian Draft Consultation Version of the National Syllabus (Australian Curriculum, Assessment and Reporting Authority (ACARA), 2010) has recognised the importance of number knowledge in the establishment of four proficiency strands: understanding, fluency, problem solving, and reasoning. The fluency strand has the descriptor: “Students develop skills in choosing appropriate procedures, carrying out procedures flexibly, accurately, efficiently and appropriately, and recalling factual knowledge and concepts readily” (p. 3). ACARA (2010) does not suggest what the appropriate procedures might be nor do the state-based curriculum documents *Essential Learnings* (Queensland Studies Authority (QSA), 2010) and *Years 1 to 10 Syllabus* (QSA, 2004). Most western nations have curriculum documents that recommend fluency with written algorithms, but there has been a decline in their relative importance and a range of different approaches to teaching written algorithms.

The change in emphasis, in some cases even reluctance to teach specific written algorithms is evidenced in teacher preparation handbooks used nationally and internationally (e.g., Reys, Lindquist, Lambdin, & Smith, 2009; Van de Walle, 2007; Van de Walle, Karp, & Bay-Williams, 2010). These reflect the criticism for teaching traditional written algorithms reported in the literature above and argue that there ought to be reduced emphasis on pencil and paper computation and that too much time has been spent on such skills at the expense of sense making, mental computation, estimation and calculator use.

Review of the literature on error patterns in traditional algorithms

One justification for criticism of student learning of traditional vertical written algorithms comes from the fact that quite often students make mistakes in applying algorithms. Errors in processing and recording basic operations such as addition, subtraction, multiplication and division have been well documented (e.g., Ashlock, 1976; Booker, Bond, Sparrow, & Swan, 2010; Brown & Burton, 1977; Burrows, 1976; Buswell & Judd, 1925; Reisman, 1978). Most student errors associated with standard vertical algorithms tended to be related to errors with place value, number facts, or applying the wrong algorithm, or non-sense procedures such as adding all digits for addition. With subtraction some students took the smallest number from the largest (always) and when multiplying, failed to multiply all the components.

Alternative methods of computation and recording of computation

Consistent with the recommendations to allow students to develop their own methods of computing whole numbers, a range of mental strategies and algorithms are taught in Australian schools and are recommended by teacher education handbooks locally and internationally. In general these methods follow the Dutch findings that some students’ intuitive methods included using the open number line for addition and subtraction, repeated addition and doubling for multiplication, and repeated subtraction for division (Anghileri, 2001) and included partitioning, expanded and grid methods described by Thompson (2007, 2008). The *decomposition* of numbers reported in the Dutch interventions is well represented in teacher preparation handbooks and student text books today. Torbeyns et al. (2005) recommended that first grade (Year 1) students be taught a range of strategies in order to add two single-digit numbers. One strategy was termed decomposition; for example, when adding $8+9$, since 9 is $2+7$, the students could add $8+2+7=17$. In the remainder of this article the idea

of breaking a number into parts is termed “splitting;” other terms used in text books for this process include “decomposition,” “partitioning” and “chunking.”

Texts associated with study schools

The textbooks that are examined below were chosen because they are widely used in teacher education mathematics curriculum courses in Australia and some of the classroom teachers in the study schools acknowledge using them as part of their teacher training or as resources in the primary schools. Aspects of Van de Walle (2007) and Reys et al. (2009) are indicative of the teacher education texts that recommend alternative methods. The two classroom resources (Baker & Baker, 2009; Burnett & Irons, 2009) are widely used as resources in Queensland schools, including the study schools. The study schools did not have strict text requirements and teachers used a variety of sources to guide their mathematics teaching. The various methods of teaching computation contained in these resources are summarised below.

Van de Walle (2007, p. 40) suggested that there are a number of valid ways to add 48 and 25, all with horizontal setting out. For example:

$48+25$	$48+20=68$
$40+20=60$	$68+2$ (from the 5) = 70
$60+8=68$	Then I still have that 3 from the 5
$68+5=73.$	$70+3=73.$

Reys et al. (2009) noted: “The notion of ‘splitting’ a number into parts is a useful strategy for all operations. Both the word split and the use of visual diagrams, as shown, have been found to help students develop strategies” (p. 218). For example, adding 46 and 38 can be modelled as $40+10+10+10 = 70$ then 6 and 8 is 14 which is added to 70 to make 84. The strategy has been termed the “count on strategy to add” (Burnett & Irons, 2009). Sometimes this strategy is taught as a mental method to add numbers.

Some texts use a number line and count on by tens for double-digit addition, for example $\$93+\32 is described as a process of adding on a number line with the suggestion, “I started at $\$93$ and jumped $\$30$ to $\$123$. Then I jumped $\$2$ more to $\$125$. That is a difference of $\$32$ ” (e.g., Burnett, & Irons, 2009, p. 52.5). Van de Walle (2007, p. 225) also used a variety of word descriptions accompanied by number lines with arrows to describe $73-46$. The strategies were described as “invented.” For example, “Take tens from tens, then subtract ones...70 minus 40 is 30. Take away 6 more is 24, now add in the 3 ones (is) 27.” An alternative invented strategy was “73 take away 50 is 23. That is 4 too many. 23 and 4 is 27.” The text illustrated this with a number line with arrows modelling this process. It is not that the standard recording of addition and subtraction computations are completely neglected in the resources listed above; rather, teachers are encouraged to use a variety of algorithms and methods of recording of them.

A similar range of algorithms are recommended for multiplication and division. For example 63×5 was set out as follows: 63 doubled is 126, this was done twice (252), and then 63 was added to 252 to give 315. The process was modelled on a branching tree model with lines connecting the added numbers to the solutions (Van de Walle, 2007, p. 230). Various other methods described as partitioning were illustrated, including 27×4 described as $4 \times 20 = 80$ and $4 \times 7 = 28$ so 80 and 28 = 108. The same problem was illustrated as $10 \times 4 = 40$, repeated, to which was added $7 \times 4 = 28$ and arrows illustrated the joining of 80 and 28 as 108.

Among teacher education and student oriented textbooks, there is less variation in methods for division. The splitting model was illustrated in a teacher source book (Burnett & Irons, 2009). For the division of \$285 by 5, \$285 was first split into \$200, \$80 and \$5. Each of these parts was divided by 5 ($200 \rightarrow 40$; $80 \rightarrow 16$ and $5 \rightarrow 1$) and a bracket was used to indicate the addition of \$40, \$16 and \$1 to give the solution \$57. The alternative presentation to this was to split \$285 into \$250 and \$35; each of these was divided separately by 5 to yield \$50 and \$7 which were added.

Considering the diversity of alternative recommendations for teaching whole number computation, the focus of this study is to document how primary school students are performing basic whole number calculations. This is important since it is well known that a weak starting point, such as an inability to do fundamental computations, is likely to slow down the pace of future learning and impact negatively in problem solving (e.g., Matsuda, Lee, Cohen, & Koedinger, 2009). The following research questions form the basis of the study:

1. How are students recording whole number computation in a sample of primary schools and what errors were evident?
2. Are alternative methods equally effective in assisting students to carry out fundamental computations?

Method

Subjects

The subjects in this study were the cohort of Year 4 (age 9), 5, 6 and 7 (age 12) students ($n=465$) at three state schools in Queensland (Schools A, B, and C). The three schools were randomly chosen. The schools were located in suburbia in Queensland Australia and school mean results on national tests (2010 NAPLAN) indicated that they were a little below average schools (Ministerial Council on Education, Employment, Training and Youth Affairs (MCEETYA), 2010). The data were collected in November 2010 in each school.

Testing instruments

Students were given a short pencil and paper test supervised by the teacher. The students could not access a calculator and were encouraged to record their working; there was no time limit. Teachers collected the scripts when they felt the students had done as much as they could. While some of the questions were set in context, the contexts were considered to be relatively simple and it was not intended to challenge the students in deciding which operation to perform. The contexts ought to have been familiar to all students having completed Year 4, since multiplication and division concepts are typically introduced in Year 3 and the teaching of computational methods associated with multiplication and division are recommended by state and national curriculum bodies (e.g., ACARA, 2010; QSA, 2004; QSA, 2010). The *Queensland Essential Learnings* (QSA) document for Year 3 recommends that multiplication and division of whole numbers to 10 can be calculated using arrays, skip counting, doubles and near doubles, turn around and sharing of concrete materials. QSA stipulated that problems of the form $161 \div 7$ be taught in Year 5. Addition and subtraction to thousands is recommended for Year 5. Thus the multiplication context and the sharing context ought to have been very familiar to students at the end of Year 4. It was not expected that students in Year 4 and 5 would have been taught questions of the form 6 and 8. The syllabus documents indicate that upper primary students were expected to be competent in whole number computation. The structure of the test is illustrated in Table 1.

Table 1

Description of Test Items

Question form	Concepts assessed
1. Addition of two, 3-digit numbers e.g., 578 +745	Addition facts, place value, renaming numbers (Addition algorithm)
2. Addition of three, 3-digit numbers, e.g., 956+176+784	Same as above, but the use of 3-digit numbers makes compensatory methods more difficult to apply (Addition algorithm)
3. What is 45 take away 18.	Subtraction concept, subtraction facts, place value and renaming of numbers (Subtraction algorithm). Checking to see if students took littlest from biggest.
4. A 4-digit number take away a 3-digit number.	Same as above, except the larger numbers involved make mental and compensatory methods more difficult to apply (Subtraction algorithm)
5. A cattle train has 8 carriages, each can carry 26 cows. What is the maximum number of cows that can be carried?	Multiplication concept, multiplication facts, place value, naming and renaming of numbers, addition (Multiplication algorithm)
6. Multiplication of a 2-digit number by a 2-digit number. For example: Julie has a salary of \$27 per hour, if she works for 39 hours how much will she earn?	As above, except the larger digits make additive strategies, grouping and splitting strategies difficult to apply (Multiplication algorithm)
7. 98 divided by 7	Division algorithm with no remainders. Students need to have multiplication facts, place value and ability to rename numbers and to use subtraction processes.
8. A class of 26 shared \$1794 between them. How much did each child in the class receive?	As above, except the larger numbers necessitate knowledge of the formal division algorithm.

Method of analysis

In 2007 and 2008 Thompson described primary strategies approach to written methods, and to a considerable degree these seem an extension of mental strategies. For example the mental addition strategies described by Foxman and Beishuizen (2002), when recorded, could be interpreted as forms of algorithms. This paper analyses students' written recordings; all jottings that do not conform to the four vertically recorded traditional algorithms are termed as *alternative* methods, whether they are based on dramatic or mental methods. Kwaku Duah (2009) reviewed the literature and distinguished between mental calculation methods and mental strategies, but such fine-grain classification is beyond this study. This study just focuses on how students recorded their thinking when calculating addition, subtraction, multiplication and division problems.

Earlier work on the analysis of student recording of their computations (e.g., Ashlock 1976; Booker et al., 2010; Reisman, 1978) confirms examining student scripts is a valid way to determine student understanding of fundamental processes and concepts in arithmetic, such as place value, renaming, number facts for addition and multiplication as well as what methods were taught or were developed in the classroom. Student working that reflected the

splitting of numbers, horizontal setting out and adaptations of mental short cuts as recommended in the teacher education texts (e.g., Reys et al., 2009; Van de Walle, 2007) and classroom resources such as Baker and Baker (2009) and Burnett and Irons (2009) and described above were classified as *alternative recording* and it is reasonable to assume that they reflect alternative methods. Setting out that illustrated vertical setting out traditionally associated with written algorithms was classified as traditional. In Figure 1 for example it is clear that the student has set out the algorithm vertically and this was classified as a *traditional* vertical approach. By contrast in Figure 2, the student has split the number to add the parts; such an approach has been classified as *alternative*. In trying to interpret student scripts the author acknowledges Brown and Burton (1977) observation, “there may be several possible explanations for any particular answer to one problem” (p. 27). Despite this it is argued that reasonable inferences can be made as to how the children were taught, based on their recordings. The percentage of students who gained correct responses by using alternative methods of recording was compared with those who used the traditional vertical algorithm to record their working.

Student use of traditional and alternative recording and their relative success is reported according to year level. It was not intended to compare schools so year level results are aggregated. A further justification for this approach is that no school had a clearly stated work program or mandatory textbook. Teachers in classrooms exercised their professional judgment. Since teachers were regularly transferred between schools in the state system the location of the school in relation to a particular training institution was largely irrelevant. Show of hands survey of teachers indicated they had been trained over several decades in a range of tertiary settings. The use of three schools simply increased the sample size. Since students are expected to improve in basic computation as they progress through the year levels and are exposed to additional teaching, the results are reported according to year.

Use of textbooks

No school possessed a comprehensive school-wide mathematics program that stipulated which resources ought to be used or even which concepts ought to be emphasised. Rather, teachers selected their own resources and tended to teach independently. Thus, it was very difficult to attribute a child’s response to the use of a particular resource. The student response could have been a reflection of what the current teacher had taught or a previous learning experience. The school-based specific data reflected a clustering of the use of particular methods in some classes, but there was no overall school-based pattern that could be reliably linked to a specific approach. It could only be inferred that students would respond in a way that they had been taught at some point in their learning of mathematics.

Results

The results are reported first by describing the different ways by which students recorded their working. This gives the reader a sense of the diversity of recording method and practically distinguishes *traditional* from *alternative* recordings. The relative frequency of success or otherwise across schools is reported according to year level.

Recording methods

The recording methods are presented according to computational type (addition, subtraction, multiplication and division). Where the error type has been previously unreported it is labelled as *new*; in effect this is the reporting of error patterns associated with alternative methods. Errors associated with the traditional methods of recording out have been very well

described and no new patterns were observed. As noted above, the way a student records their thinking is a reasonable indication as to the methods they were taught.

Addition methods

Addition errors associated with vertical algorithm

The error patterns associated with the traditional vertical algorithm conform to earlier descriptions that have been well documented. The majority of errors associated with the vertical algorithm were linked to errors in basic facts; there were also instances of failure to rename or errors in renaming. One such example is documented in Figure 1. The script shows a lack of understanding of naming numbers as well as renaming numbers:

$$\begin{array}{r} 745 \\ + 578 \\ \hline 12113 \end{array}$$

Figure 1. Year 6–Student does not rename.

Addition recording associated with alternative methods (new)

Most of the recording associated with alternative methods of addition was of the forms exemplified in Figures 2 to 4. In Figure 2 the student has attempted to add $578+745$ (Item 1). In the computation the student has split the number into place value components and begun to add the ones, tens and hundreds each independently, without finding any total. The working indicates an inability to name numbers as well as an inability to add any entity. It appears the student has begun by adding the ones ($8+5$) then tens ($7+4$) and finally hundreds ($5+7$) but has not known how to proceed.

$$\begin{array}{l} 8+5 \\ 7+4 \\ 5+7 \end{array}$$

Figure 2. Year 5–Splitting method applied to addition.

An alternative example of splitting is seen in Figure 3, again an attempt to add 578 and 745.

$$\begin{array}{l} 578 + 745 \\ \hline \end{array}$$
$$1020 + 110 + 13 = 1143$$

Figure 3. Year 6–Splitting method of addition.

The student has added 5 hundred to 7 hundred and recorded 1020, then added 70 to 40 and correctly recorded 110, and finally added 8 to 5 and correctly recorded 13. The student then added the parts to 1143. A further example of splitting numbers is seen in Figure 4.

Add 578 + 745.

$$\begin{array}{l} 5+7=12 \\ 7+4=11 \\ 8+5=13 \end{array} \quad 1306$$

Figure 4. Year 6–Working from left to right with splitting of numbers.

In the solution sample in Figure 4, the student has split the number and added from left to right, first adding 5 (hundreds) to 7 (hundreds) and recording 12 (presumably hundreds), then 7 (tens) and 4 (tens) recording 11 tens and finally $8+5=13$ ones. Since the student had not recorded place value, short term memory has presumably been used to add 12 hundred and 11 tens and 13 ones. It is difficult to speculate how the final solution was attained. It looks like the student was unable to put the solution together, and reconstitute the parts via renaming. The student was unable to apply this method to adding larger numbers.

The example in Figure 5 illustrates an alternative horizontal setting out and application of the splitting of numbers. In this example place value is recorded by showing 7 hundreds as 700 to which 500 was added (incorrectly recorded as 1300), the 70 and 40 were added to 110 and this added to the 1300 to give 1410. Then finally the student added the 13 to give the answer of 1423.

Question 1: Add 578 + 745.

$$\begin{array}{l} 700 \\ 500 \\ =1300 \end{array} \quad \begin{array}{l} 70 \\ 40 \\ =110 \end{array} \quad \begin{array}{l} 8 \\ 5 \\ =13 \end{array}$$

1410

1423

Figure 5. Year 6–Horizontal and splitting addition method.

The impact of errors associated with place value became amplified when students were required to add three, 3-digit numbers.

Subtraction methods

Subtraction errors with vertical algorithms

As with addition, most of the subtraction errors associated with the vertical algorithms occurred as a result of errors with basic facts. There were also errors associated with place value, renaming or taking the smallest number from the biggest number, as illustrated in Figure 6.

$$\begin{array}{r} 1,932 \\ - 745 \\ \hline 6522 \end{array} \quad \begin{array}{r} 117132 \\ - 745 \\ \hline 4482 \end{array}$$

Figure 6. Year 7–Errors with place value, renaming and taking smallest from biggest (left) and at right, renaming without care with place value.

Errors of these forms have been documented previously (e.g., Booker et al., 2010; Burrows, 1976).

Subtraction errors associated with alternative methods (new)

About half of all alternative methods of subtraction involved representations of number lines of some form. Even students in Year 7 used pictorial methods, sometimes successfully with smaller numbers, (eg., 45-18) as illustrated in Figure 7.



Figure 7. Year 6–Use of pictorial methods to subtract (45-18).

Several Year 6 and 7 students in each school used a pictorial or tally approach to carry out the subtraction (see Figure 8). These strategies did not work with larger numbers.

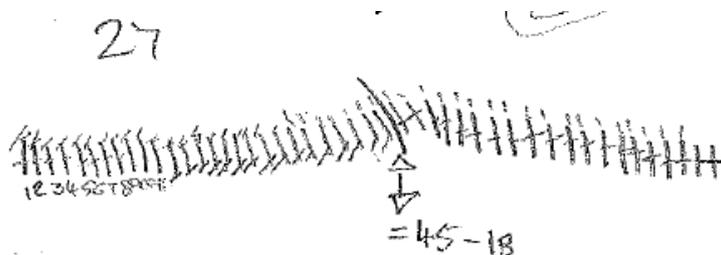


Figure 8. Year 7–Use of crossing out of a tally to subtract.

Figure 9 illustrates a number line method.

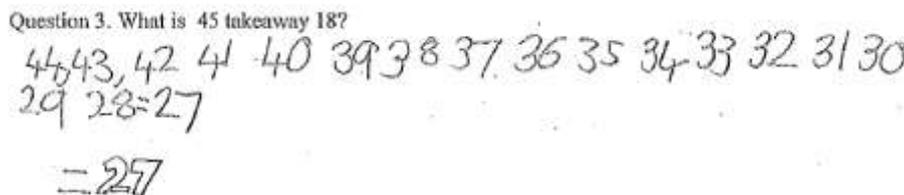


Figure 9. Year 6–Number series method for subtraction, counting down.

Some classrooms used connecting lines to add and subtract numbers and used the term “rainbow” to describe the process. The student whose working is illustrated in Figure 10 shows the rainbow shape and also took “smallest from biggest” in (unsuccessfully) carrying out a subtraction computation.

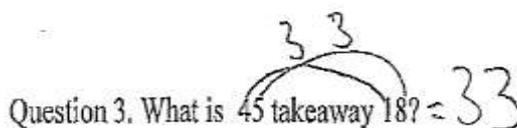


Figure 10. Year 6–Use of a rainbow method for subtraction.

A few students used horizontal setting out and split the place values, presumably able to rename them mentally to record the correct solution of 27. When faced with a 3-digit subtraction these students could not apply the strategy and responded, “I can not do this”.

The remaining half of the students who used alternative methods related to mental strategies including splitting numbers used a variety of recording such as that illustrated in Figures 11 to 14. The work displayed in Figure 11 illustrates that the student has worked from left to

right and recorded bottom up to subtract 18 from 45. It appears that the student took 10 from 40, recording this as 4-1, then 5-8 are recorded. The correct response of 27 is recorded at the top.

$$\begin{array}{r} 27 \\ 5-8 \\ 4-1 \end{array}$$

Figure 11. Year 5–Horizontal methods of subtraction for 45-18, probably based upon mental strategies.

An alternative method that appears to be an adaptation of a number board based strategy is exemplified in Figure 12. The student starts by taking 10 from 45 to record 35 (moving up one row on the number board) then splitting the subtraction of 8 from 35 into two steps, taking 5 from 35, then 3 from 30.

$$\begin{array}{r} 45 - 10 = 35 \\ 35 - 5 = 30 \\ 30 - 3 = 27 \end{array}$$

Figure 12. Year 6–Subtraction working probably based upon mental strategies.

The example in Figure 13 illustrates a similar strategy, except the 10 is taken from the 40 initially, then $5-8 = -3$ and this is added to 30. The use of directed numbers in this way is interesting.

$$\begin{array}{r} 40 - 10 = 30 \\ 5 - 8 = -3 \\ 30 - 3 = 27 \end{array}$$

Figure 13. Year 6–Subtraction working showing the use of directed numbers.

Students rarely succeed in using such methods with larger numbers, as illustrated in Figure 14. The computation was $1932-745$. The student had forgotten the thousand, taken 700 from 900 and correctly recorded 200, then taken 30 from 40 rather than 40 from 30, and similarly taken the smaller from the larger number with the ones place value.

$$\begin{array}{r} 900 - 700 = 200 \\ 40 - 30 = 10 \\ 5 - 2 = 3 \\ \hline 213 \end{array}$$

Figure 14. Year 6–Subtraction working showing right to left working.

Multiplication methods

Multiplication errors associated with the traditional algorithm

A fair proportion of students tried to apply the wrong algorithms, for example addition when multiplication was required, particularly in Years 4 and 5, as illustrated in Figure 15.

Figure 15. Year 4—Computing addition rather than multiplication.

Failures among those who used the standard algorithm were associated with failing to cross multiply (multiply all the parts), errors of multiplication facts or errors in renaming or adding the renamed parts. In the example in Figure 16, the Year 7 student has not multiplied all the parts for 27×39 . That is, 7 is multiplied by 9 to give 63, and two tens are multiplied by three tens to give 6, but two tens are not multiplied by 9 ones and 7 ones are not multiplied by 3 tens. The student then ignored place value in adding the parts, so 6 and 63 are simply added to record 69.

Figure 16. Year 7—Student does not cross multiply and ignores place value.

Multiplication methods and errors associated with alternative methods (new)

Other than the traditional algorithm the methods adopted for multiplication ranged from use of splitting methods, use of material based models, repeated addition, and doubling strategies.

Repeated addition strategies were used in upper primary years by several students as illustrated in Figure 17. Sometimes this was accompanied by estimation attempts. In this instance 27 was grouped into three groups of 10 and one of 9. Unfortunately the student did not have a strategy to accurately add the parts. Young students' inclination to use repeated addition for multiplication has been reported previously (e.g., Lemaire & Siegler, 1995).

Figure 17. Year 6—Use of repeated addition for multiplication of 27 by 39.

Some students in all grades attempted to use a set model to multiply numbers. On one occasion the set model was accompanied with numbers. In the instance in Figure 18, the student allocated different numbers to some sets.

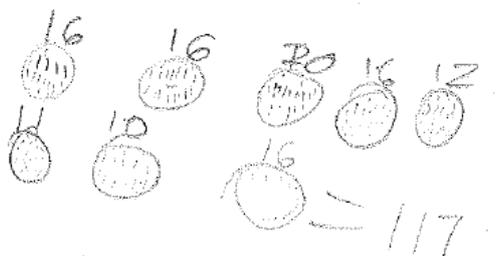


Figure 18. Year 4–Student allocates different numbers to sets before adding.

When students allocated equal numbers to sets and added correctly, they were successful in applying this approach to multiplication, as illustrated in Figure 19.

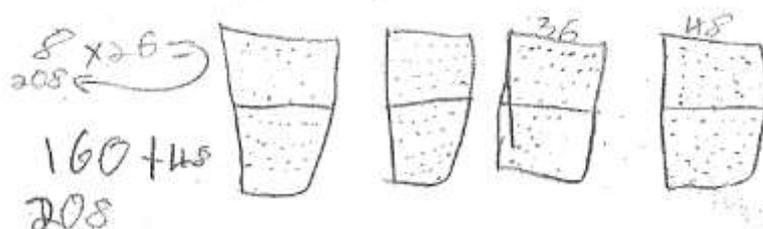


Figure 19. Year 7–Use of splitting and diagrammatic models to multiply 26 by 8.

In Figure 19, 20 is multiplied by 8 to give 160 and 8 is multiplied by 6 to give 48 then 160 is added to 48 to give the correct response.

In the example in Figure 20, the student seems to have split 26 into 20 and 6 and used the lines to link the two tens, counting 16 (tens) and lines link the 6s that are subsequently added to 48. The student then neglects place value and adds 48 and 16, as shown by connecting vertical lines and it looks like the student has used splitting to add 48 and 16. The student then summarises 26 by 8 as 64 at the bottom left of the method.

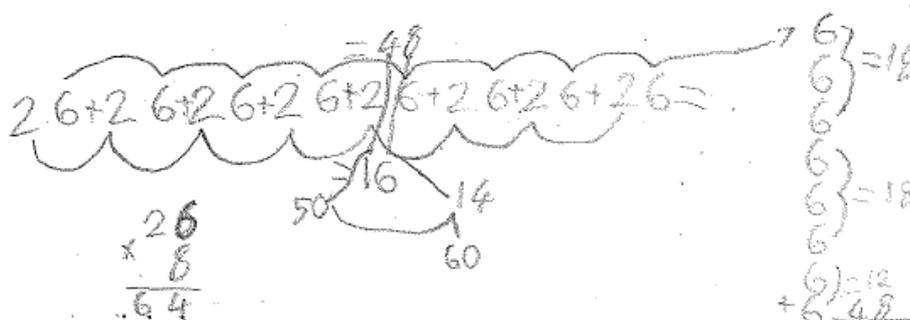


Figure 20. Year 4–Attempt to use splitting to multiply 26 x 8.

The Year 7 student who completed the solution shown in Figure 21, succeeded in using this strategy with 26 by 8, but did not succeed with 27 by 39. The student first multiplied 27 by 30 then 27 by 9 and added the results. However, the student made a computational error in multiplying 27 by 9.

$$27 \times 30 = 810 + 27 \times 9 = 983$$
$$= 983$$

Figure 21. Year 7–Use of splitting approach to multiply double-digit numbers.

Division Methods

When applying the division algorithm students frequently made number fact errors in multiplication or subtraction.

Division methods and errors associated with alternative methods (new)

The methodology associated with division errors was generally related to attempts to use material based models such as allocating marks in boxes in the lower grades, and guess and check multiplication or alternative splitting strategies in the higher grades. A relatively high proportion of students who did not use the standard algorithm for division relied upon diagram based methods (see Figure 22).

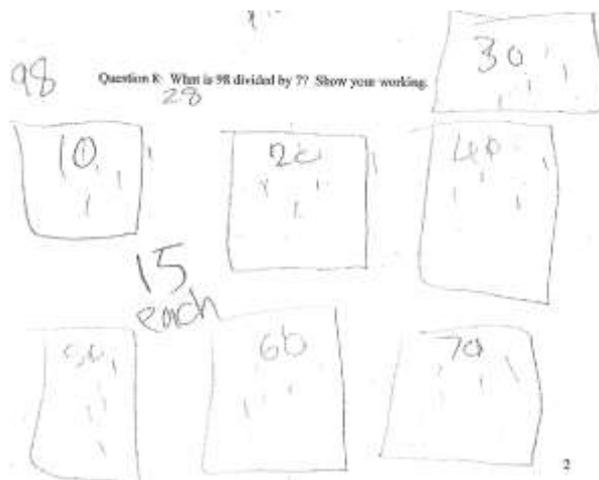


Figure 22. Year 7–Use of a grouping model to divide 98 by 7.

On several occasions grouping and multiplicative approaches were used with division. A method similar to this was described by Ashlock (1976) as “The Doubling Method” and recommended by Van de Walle et al. (2010) for double-digit by single-digit multiplication.

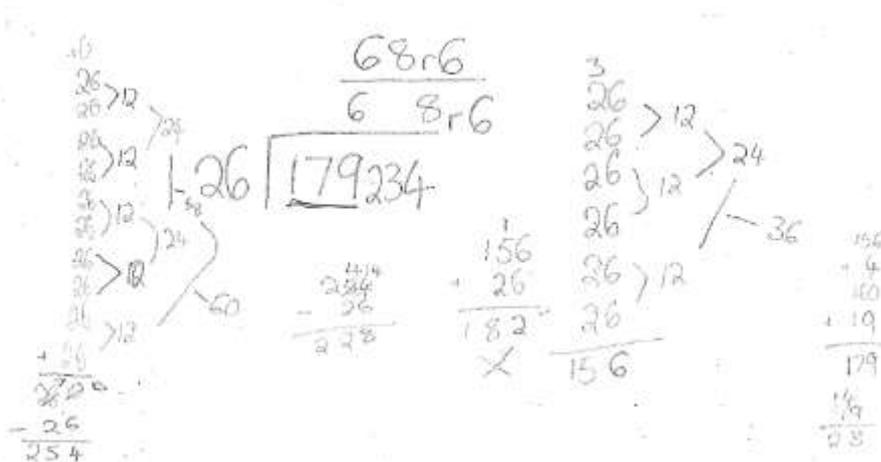


Figure 23: Year 5 student use of multiplication methods to get a good approximation to 1794 divided by 26. This was a unique solution.

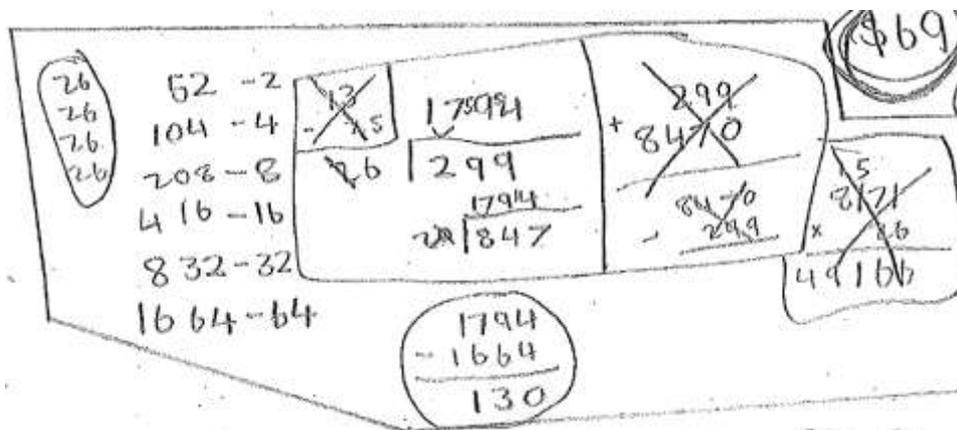


Figure 24: Year 7 student calculates the correct solution to 1794 divided by 26 by doubling and adding method.

A few students using alternative methods succeeded in using high levels of number sense to achieve correct solutions.

Summarising the findings

The first research question asked how students are recording whole number computations. The evidence shows a range of computational methods have been used by students. Table 2 is a summary of the nature of the student errors identified. These errors have been classified into those that have been documented previously by various researchers, and those that have not been documented before. In this regard the work contributes the description of “bugs” detailed by Brown and Burton (1977). In essence those that have been documented previously are associated with the traditional algorithm, while errors associated with alternative methods are newly reported.

Table 2

Summary of Error Patterns

	Computations			
	+	-	×	÷
Previously reported error patterns associated with the traditional algorithm				
Number fact errors	P	P	P	P

Renaming errors: failure to rename, ignoring renamed value	P	P	P	P
Place value errors: placing renamed value in wrong place, operating on wrong value, ignoring place value	P	P	P	P
Wrong numbers operated on		P		
Failure to operate on numbers			P	
New error patterns based on alternative methods				
Prolonged and inappropriate use of material based methods	P	P	P	P
Prolonged use of repeated addition or subtraction for multiplication and division			P	P
Place value errors associated with partitioning or splitting	P	P	P	P
Partitioning and doubling errors where numbers were not operated upon or the wrong digit was operated upon			P	P
Adaptation of mental methods for addition and subtraction resulting in errors or omissions	P	P		
Number fact and omission errors associated with working backwards				P
Errors associated with multiplying, including doubling and then adding to compute				P

Note. P=present in student scripts.

Assessing the success of students who used traditional compared with alternative algorithms was relatively straight forward, but interpreting the results is problematic. In many instances students did no working or did not attempt a solution. This was particularly the case with the questions involving larger numbers, where few Year 4 and 5 students attempted solutions. Since many students did not show any working the response numbers are much smaller than the populations. In each school some students used alternative methods to calculate solutions irrespective of the year level. The clustering of use of alternative methods in particular classes indicates that individual teachers used their discretion as to what algorithms and computational methods they emphasised. As noted in the method section it was not the intention to compare schools, but simply to use more schools to increase the sample representation and numbers. Thus, the results are reported as percentage success rate at each year level irrespective of school. The relative success rates of traditional and alternative methods are tabulated below.

Table 3

Success Rate according to Year Level and algorithm form recorded

Question	Year 4 (n=128)		Year 5 (n=114)		Year 6 (n=113)		Year 7 (n=110)	
	Trad	Alt	Trad	Alt	Trad	Alt	Trad	Alt
<i>Q1 eg.,</i> 578+745	87/119 73%	4/9 44%	93/105 88%	6/9 67%	96/108 89%	4/5 80%	95/102 93%	5/8 63%
<i>Q2 eg.,</i> 956+176+784	70/100 70%	14/28 50%	87/108 81%	0/6 0%	89/107 83%	2/6 66%	73/110 66%	0/4 0%
<i>Q3 eg.,</i> 45-18	35/62 57%	13/30 44%	44/65 67%	4/8 50%	53/72 73%	3/9 33%	58/77 52%	0/5 0%
<i>Q4 eg.,</i> 1932-745	12/24 50%	4/24 17%	23/48 48%	3/18 17%	51/71 71%	2/8 25%	42/59 71%	6/12 50%
<i>Q5. eg.,</i>	2/10	1/21	35/57	14/43	35/52	10/24	51/68	4/26

26×8	59%	5%	61%	32%	67%	41%	75%	15%
<i>Q6 eg.,</i>	0/6	0/21	8/50	0/23	45/79	5/24	60/84	3/13
27×39	0%	0%	16%	0%	56%	20%	71%	23%
<i>Q7 eg.,</i>	0/2	1/6	12/35	8/34	45/79	5/24	65/87	5/18
$98 \div 7$	0%	16%	33%	24%	57%	21%	75%	28%
<i>Q8.eg.,</i>	0/0	0/1	2/26	1/33	9/31	0/11	6/20	4/22
$1794 \div 26$	0%	0%	8%	3%	29%	0%	30%	18%

Key: Trad = traditional methods recorded; Alt = alternative methods recorded.

Students in younger year levels in each school performed relatively poorly on addition and subtraction of large numbers as well as on multiplication, suggesting that they had not yet mastered the various algorithms or methods or learnt basic number facts. Overall there was a gradual improvement in success in whole number computation through to Year 7, but it was clear many students struggled with basic computations at the end of primary school.

An analysis of student recording indicates that they used a range of strategies for solving the various problems. There were many instances of previously documented errors associated with the traditional vertical algorithms; what is new is the range of error patterns associated with alternative methods and these were reflected in alternative recordings. Students from Year 4 to Year 7 attempted to use pictorial methods such as crossing out markers for subtraction, counting back on a number line, using splitting methods for addition and subtraction, or variations of splitting numbers or doubling and halving strategies for multiplication and division. There were parallels in the written methods described above and the mental methods described by Foxman and Beishuizen (2002) for addition including sequencing; compensation, complementary addition, mixed methods and partitioning (splitting). The prevalence of alternative recordings was higher in younger year levels especially for addition and subtraction. Still, the recordings indicate that in Year 6 almost a third of students attempted to use alternative methods for multiplication and about a quarter attempted to use alternatives including pictorial based methods to divide. The data in regard to success rates indicate that students who recorded using traditional algorithms usually had greater success in carrying out these fundamental computations in all year levels, but particularly so when numbers became larger.

Discussion and Conclusions

The review of the literature revealed that several arguments against the teaching of formal traditional vertical algorithms were in circulation. The first was that teaching algorithms could undermine student learning (e.g., Gravemeijer, 2003; Heirdsfield & Cooper, 2004; Kamii & Dominick, 1997). In part this argument is based on a rejection of rote or passive learning of algorithms without understanding the underlying principles. A more moderate approach was taken by British, Dutch and US reformers who recommended that formal written algorithms be built upon students' intuitive understandings and these included mental strategies and methods (e.g., Anghileri et al., 2002; Kwaku Duah, 2009; Thompson, 1999, 2007, 2008; van de Heuvel-Panhuizen 2001; Yackel, 2001).

A further argument supporting the reduction of time spent learning algorithms was that the nature of mathematics has changed. It was reported that the time spent developing proficiency in carrying out written computations could be better used developing reasoning and problem solving (e.g., Reys et al., 2009; Suggate, Davis, & Goulding, 2010; Van de

Walle, 2007; Van de Walle et al., 2010). Irrespective of the motive, few authors would argue that the role of traditional algorithms in learning mathematics has less priority than it did a few decades ago.

Data collection for this paper was done in South East Queensland where alternative methods have been emphasized in teacher education texts used at universities (e.g., Reys et al., 2009; Suggate et al., 2010; Van de Walle, 2007; Van de Walle et al., 2010) and school-based resources (e.g., *Natural Maths* (Baker & Baker, 2009); *Go Maths* (Burnett & Irons, 2009)). The latter textbooks are also used in the US. Earlier editions have been used in Queensland schools for over a decade and these textbooks had been used by some teachers at the study schools for various periods of time.

Overall, the data indicated that irrespective of the methods taught and how they recorded their methods, many students struggled with basic facts, place value, and applying algorithms or methods of any form. As was the case with research by Kwaku Duah (2009) and Borthwick and Harcourt-Heath (2007), division was particularly problematic for many students. This is hardly surprising since division computation relies on place value, multiplication facts and subtraction competency. It was also clear that, as Borthwick and Harcourt-Heath (2007) found, many students struggled to choose efficient and effective methods. For example the few students who used alternative methods to achieve success with division by a 2-digit number did so with long and tedious processes. The level of overall success on fundamental computations is concerning since a number of authors have commented (e.g., Matsuda et al., 2009; Rittle-Johnson, Siegler, & Alibali, 2001) that the development of conceptual and procedural knowledge is an iterative process. That is, weak procedural knowledge is likely to hinder the development of conceptual knowledge and weak conceptual knowledge will limit students' ability to compute. A similar view was presented by Brown and Burton (1977, p. 41): "Many mathematics skills involve interplay between strategic and automatic knowledge."

When the relative success rates of students across the year levels are compared, it was found that those students who had mastered the traditional algorithms and recorded their working using the traditional algorithms were more successful than those who used alternative methods and recorded their working in ways that reflected these methods, especially when numbers become larger. This finding contradicts the findings of earlier authors including Borthwick and Harcourt-Heath (2007) who reported, "this research shows that when children use a strategy, which is based on mental methods, they usually reach the correct solution" (p. 17). Similarly, these results do not support the assertions of Torbeyns et al. (2005) who state that "even relatively low achieving children, who are taught multiple reasoning strategies on sums over ten are able to apply these strategies effectively and adaptively" (p. 18).

The analysis of student scripts illustrate that many students used alternative methods that appear to be variations of methods recommended by teacher preparation textbooks used in Queensland universities and in Queensland school-based resources. This finding parallels what Kwaku Duah (2009) found, that is, there was evidence that students are being taught "specific strategies and that hence the methods were not idiosyncratic and self-developed" (p. 15). It was also evident that many of the students did not develop the intended deeper understanding of number, and progression to effective algorithms was not attained. In summary, there was evidence that alternative methods were not developed intuitively, but were probably taught, and not in a coherent and systematic way that helped develop efficient algorithms; rather, the alternative methods tended to become a grab bag of procedures that lacked a deep structural underpinning. In this way the added variety of methods was

achieving little more than giving students a further range of algorithms to mix up and make mistakes with.

An alternative explanation as to why students who used alternative algorithms were generally less successful could be that more capable students had moved to efficient algorithms and chose to use traditional methods. However, the high levels of number sense displayed by a few students who succeeded with multiplication and division with larger numbers using alternative methods contradicts this assertion (see, e.g., Figures 23 and 24). It is more likely that most students tried to apply the methods they had been taught or, if they had been taught several methods, chose the methods they considered themselves most competent with. There are other possible explanations; one that needs to be considered is that meaningful teaching of traditional algorithms is effective.

One of the main criticisms of teaching the traditional algorithms is that it is dissociated from sense making, but this does not have to be the case. Booker et al. (2010) set out a series of teaching descriptions based on materials where the logic of written algorithms is explicitly linked to vertically written algorithms with specific use of natural language. Booker et al. reported:

While the role of materials and the patterns they develop is fundamental, materials by themselves do not literally carry meaning. ...It is language that communicates ideas, not only in describing concepts but also helping them to take shape in each learner's mind. (p. 21).

The materials used for teaching standard traditional algorithms include ten frames and counters followed by the use of bundle sticks and place value charts for the learning of the four vertical algorithms of addition, subtraction, multiplication and division. The language that is used is designed to promote meaning. For example if 18 is to be taken away from 45, the connecting language would be "Start with the ones, I have 5 ones, can I take away 8 ones? No. Rename one ten into 10 ones, now there are 15 ones..." Such explicit language makes the relationships between what is being modelled with the materials and the symbolic recording of the processes meaningful. Booker et al. (2010) contend that the repetitive use of the focusing questions and explanatory nature of the language associated with teaching and doing the algorithms helps students to remember the processes and meaning underpinning the algorithms.

A further criticism of teaching traditional algorithms is that it wastes time. However, when students carry out computations with algorithms they repeatedly use addition facts, multiplication facts, estimation, and the processes of naming and renaming numbers and consolidating place value. Recent research in brain plasticity (e.g., Brown, Fenske, & Neporent, 2010; Doidge, 2010) suggests that practice on any tasks, particularly when there is a concentration on meaning, strengthens memory. In effect new neural pathways are created that make a particular way of thinking or performing habitual or relatively easy; it takes time for the proteins that connect various neural pathways to be produced (Carr, 2010). This is an explanation of observations previously reported by cognitive load research (e.g., Cooper, 1990; Kirschner, Sweller, & Clark, 2006; Owen & Sweller, 1989) which claimed that with meaningful repetition schema acquisition could be acquired and algorithms and basic facts could be committed to long-term memory.

Kirschner et al. (2006) described the critical role of long-term memory in thinking and problem solving. Good problem solvers have committed more information to long-term memory and this can be accessed automatically. In contrast, short-term memory or working memory is controlled by cognitive processing (Kirschner et al., 2006). When working

memory is processing novel information it has limited duration and capacity, being able to retain between four and seven elements. In the context of this study, if a student has committed an algorithm to long-term memory, when confronted with a large computation such as multiplying 45 by 39, the student can access basic multiplication facts by accessing long-term memory, and by keeping a written record of the algorithm steps working memory is freed up to do the next step. Conversely, when working memory is focused on trying to remember basic facts and to invent or adapt algorithms, it can readily be over loaded, a process called cognitive overload. If the new demands exceed the mind's ability to store and process the information, the individual is unable to retain or process the information or to draw connections to other information stored in long-term memory and new information cannot be translated into new schema (Carr, 2010). According to Carr (2010, p., 124), "the depth of our intelligence hinges on our ability to transfer information from working memory to long-term memory and weave it into conceptual schemas." Cognitive load theorists and brain plasticity research are essentially giving a theoretical explanation for Brown and Burton's (1977) comment about the interplay between strategic and automatic knowledge.

Brain plasticity and cognitive load research offers an explanation as to why students who relied on alternative algorithms fared so poorly. When students are taught to add, subtract, multiply or divide having been taught a range of different alternative strategies, they have increased choices available to them. This has two effects: if we take addition as an example, first students have to decide which particular algorithm is to be used—the pictorial based method, the number line, the splitting of numbers or some other strategy. The data illustrate that the wrong choice of algorithm for the wrong size numbers inevitably leads to error. Further, the act of having to decide which algorithm to use may well have the effect of distracting the student from the problem at hand, adding to cognitive load. The second aspect of teaching a variety of strategies is that there is less time and practice to commit each algorithm to long-term memory. This does not mean there should be repetitive solving of large numbers of conventional problems; rather, rule automation and schematic knowledge can be developed via practice with worked examples on a range of problems (Owen & Sweller, 1989). It would seem reasonable that developing algorithms for each operation, that efficiently serve the students from small numbers to large numbers and that can be used in a variety of contexts, has merit. Clearly this line of research warrants further investigation.

The cognitive load argument could potentially be used to support the meaningful teaching of alternative algorithms. After all, a few students who used alternative methods did succeed on the division of 1794 by 26. That is, if students learnt place value and developed fluency in splitting, doubling and adding of numbers, it might be no greater load on short-term memory than being fluent with the standard division algorithm with its emphasis on place value, multiplication and subtraction. However, there were few students like those whose work is depicted in Figure 24 who were able to adapt alternative methods for solving large number computation. Further, there was no evidence in this paper that the teaching of the alternative methods was a productive stepping stone to efficient computational methods. The data do not contradict the assertion that the "ability to calculate mentally lies at the heart of numeracy" (Straker, 1999, p. 43). However, it is worth noting that to be fluent with standard computational algorithms necessitates relatively high levels of mental computation and in the case of division, estimation as well. In short, we should be careful of selling short the value of learning algorithms to spend the time on developing mental methods.

From a practical perspective the great variation in success between schools, between year levels, and even within a year level at a school suggests there is a need for each school to consider a school-wide approach to teaching mathematics rather than relying on the judgment of individual teachers to choose their own resources and methods in relative isolation. Some

teachers may have chosen strategies wisely, but it is clear that some had not. The data also indicate that a substantial number of the students have been taught in ways that did not help them become competent in computation.

The paper contributes to the discussion on the appropriate role and form of algorithms to be learnt in mathematics and in particular what methods are used to teach the recording of fundamental computations. The data all suggest that teachers need to be aware of the complexity of interplay between various methods of teaching computations.

The main limitations of the study are related to the relatively small sample size and the emphasis placed on inferring student intentions from their recordings. However, data has been presented that supports the assumption that such a process is reasonable. Clearly, in-depth student interviews and interviews with teachers about how they taught computation would strengthen the study.

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