ALGEBRA STUDENTS' ABILITY TO RECOGNIZE MULTIPLE REPRESENTATIONS AND ACHIEVEMENT

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ABSTRACT

Learning about linear relationship with one unknown, middle school students may often demonstrate a certain degree of proficiency manipulating algebraic symbols. When encouraged, they can verbalize and explain the steps they perform, thereby demonstrating awareness of the procedures with symbols according to fixed rules. It is well known that correct and seemingly fluent procedural skills are not always supported by conceptual understanding. Previous research suggests that one of the indicators of conceptual understanding is the ability to recognize structurally the same relationship posed via multiple representations. The purpose of this study was to examine whether there is an association between middle school students' achievement level on standardized test, their ability to recognize structurally the same relationship presented in different modes and their ability to solve problems involving linear relationship with one unknown posed in different modalities.

The study was conducted with a relatively large sample size (N = 443) of 7th and 8th grade students from one underperforming urban district. The student achievement level was measured by mathematics portion of the statewide Comprehensive Assessment System (sCAS).

INTRODUCTION

Learning about linear relationships with one unknown, middle school algebra students may often demonstrate a certain degree of proficiency manipulating algebraic symbols. When encouraged, they can verbalize and explain the steps they performed, thus demonstrating awareness of procedures with symbols according to fixed rules. These students show a certain level of "operational conception" (Sfard, 1991, p.4) or "process conception" (Dubinsky & McDonald, 1991, p.3), however they do not necessarily reveal "structural conception" (Sfard, 1991, p.4) or "object conception" (Dubinsky & McDonald, 1991, p.3). It is well known and documented (Herscovics, 1996; Herscovics & Linchevski, 1994; Hiebert, 1988; Hiebert & Carpenter, 1992; Kieran, 1992; Kieran & Chalouh, 1993; Langrall & Swafford, 1997) that correct and seemingly fluent procedural skills are not always substantiated by conceptual understanding, or as Skemp (1976) suggested "relational understanding" (p. 21). Panasuk (2010) argued that when the students demonstrate the ability to recognize "structurally the same" (Dreyfus & Eisenberg, 1996, p. 268) relationship/concept presented in different modalities (verbal, diagrammatic and symbolic), it is likely they have developed conceptual understanding of the relationship/concept and advanced from procedural skills ("process conception") to structural skills ("object conception"). Having conceptual understanding enables students to meaningfully operate upon rules and procedures, and provides a strong basis for effective problems solving. Neimi (1996) asserted that "superior representational knowledge" is likely to be associated "with higher performance on complex tasks requiring principled understanding" of mathematical concepts (p. 353).

The purpose of this study was to investigate the association between middle school students' achievement level based on mathematics standardized test and their ability to recognize and solve problems involving "structurally the same" linear relationship with one unknown posed different representations (i.e., words, diagrams and symbols).

The study was conducted with a relatively large sample size (N = 443) of 7th and 8th grade students from one underperforming urban district. The student achievement level was measured by the mathematics portion of the statewide Comprehensive Assessment System (sCAS), which has been proven to be a consistent and reliable assessment.

BACKGROUND

The role of multiple representations, in probing understanding of mathematics learning, development of algebraic reasoning, and solving problems posed via different representational modalities are the foundation of this research study. This section briefly outlines the major theoretical positions and research that pertain to and serve as a background for this study.

Algebraic reasoning and conceptual understanding in algebra

The term *algebraic reasoning* has been used to describe mathematical processes of generalizing a pattern and modeling problems with various representations (Driscoll, 1999; Herbert & Brown, 1997; NCTM, 2000). Driscoll (1999) defined algebraic reasoning as the "capacity to represent quantitative situations so that relations among variables become apparent" (p. 1). For Swafford and Langrall (2000) algebraic reasoning is "the ability to operate on an unknown quantity as if the quantity is known" (p.2). Vance (1998) characterizes algebraic reasoning as a way of reasoning involving variables, generalizations, different modes of representation, and abstracting from computations. These definitions provide the foundation for drawing conclusions and explanations about conceptual understanding in algebra in this paper.

Conceptual understanding in algebra is characterized by the ability to recognize functional relationships between *known,* and *unknown, independent* and *dependent variable*s, and to distinguish between and interpret different representations of the algebraic concepts. It is manifested by competency in reading, writing, and manipulating both number symbols and algebraic symbols. Those who reveal a conceptual understanding grasp the full meaning of the concept, and can discern, interpret, compare and contrast related ideas of the subtle distinctions among a variety of situations. Fluency in the language of algebra demonstrated by confident use of its vocabulary and meanings and flexible operation upon mathematical properties and conventions are indicative of conceptual understanding in algebra, as well.

Representations

Representations and symbol systems are fundamental to mathematics as a discipline since mathematics is "inherently representational in its intentions and methods" (Kaput, 1989, p. 169).

The process of representation or representing involves identification, selection and presenting one idea through something else (Seeger, 1998). It might be referred to as a structurally equivalent presentation through pictures, symbols and signs (notations). Vergnaud (1997) suggested viewing representation as an attribute of mathematical concepts, which are defined by three variables: the situation that makes the concept useful and meaningful, the operation that can be used to deal with the situation, and the set of symbolic, linguistic, and graphic representation that can be used to represent situations and procedures.

Several ideas related to the concept of representation are pertinent to this research. Bruner (1966) proposed to distinguish three different modes of mental representation – the sensory-motor (physical action upon objects), the iconic (creating mental images) and the symbolic (mathematical language and symbols). Pape and Tchoshanov (2001) described mathematics representation as an internal abstraction of mathematical ideas or cognitive schemata, that according to Hiebert and Carpenter (1992) the learner constructs to establish internal mental network or representational system.

The notion of multiple representations in mathematics education has evolved considerably in recent years, and different theories of representations utilize different terminology (Goldin & Shteingold, 2001). Pirie (1998) associated representations with mathematical language which she classified as ordinary language, mathematical verbal language, symbolic language, visual representation, unspoken but shared assumptions and quasi-mathematical language. She asserted that the function of any type of representation is to communicate mathematical ideas, and that each representational system adds to effective communication and helps to convey different meanings of a single mathematical concept.

The research in the area of representation has been focused on student generated representations and subsequent impact of these representations on learning mathematical concepts (e.g., Ainsworth, et. al., 2002, Boulton-Lewis & Tait, 1993; Diezmann, 1999; Diezmann & English, 2001; Lowrie, 2001; Outhred & Saradelich, 1997; Swafford & Langrall, 2000). Pape and Tochanov (2001) observed that when students generate representations of a concept or while solving problems (as a means of mathematical communication) they naturally tend to reduce the level of abstraction (given by the problem) to a level that is compatible with their existing cognitive structure.

Drawn from Piaget's (1970) idea that children first learn about an object by acting upon it and through interaction they eventually understand its nature, theories (Sfard, 1991, 1992; Dubinsky, 1991; Dubinsky & McDonald, 1991) distinguish between a *process conception* or *operational conception* and an *object conception* or *structural conception* of mathematical principles and notions, and agree that when a mathematical concept is learned, its conception as a process precedes its conception as an object. These theories also suggest that the *process conception* (e.g. simply following or performing the steps shown by teacher) is less abstract than an object conception (the nature of the concept with its properties, rules and understanding of how and why the rules work). Thus, the process conception of a mathematical concept can be interpreted as being on a lower (reduced) level of abstraction than its conception as an object. When students need to reduce the level of abstraction, it is likely that they have not yet developed conceptual understanding.

METHOD

A non-experimental mixed methodology design involved a multi-component survey and interviews with selected students.

Sample and Process

Four hundred and forty three (N = 443) 7th and 8th grade students from 21 classes in three schools from an urban low performing school district took the survey. The students in the district had regularly performed in the bottom third on the annual state standardized mathematics test. In the year prior to the data collection, the 8th grade failure rate was 43% as compared to 24% in the state. Due to a very transient district population some of the participating students had not been in the school the previous year, thus the data of the corresponding sCAS score were available for three hundred ninety students (N = 390). The state Comprehensive Assessment System (sCAS) reports the scores in six categories, which include Low Warning (LW), High Warning (HW), Low Needs Improvement (LNI), High Needs Improvement (HNI), Proficient (Prof), and Advanced (Adv). Table 1 shows the scores of the participating students in each category of the sCAS.

Table 1

Participating Students' Standardized Test Scores

Tier	sCAS Classification		Frequency/Percent
Lower	Low Warning	(LW)	4/1.0
	High Warning	(HW)	89/22
	Low Needs Improvement	(LNI)	60/15.4
	High Needs Improvement	(HNI)	73/18.7
Upper	Proficient	(Prof)	119/30.5
	Advanced	(Adv)	45/11.5

The district used the Connected Mathematics Program (CMP), which claimed to promote multiple representations, critical thinking and problem solving. The curriculum introduced the concept of linear relationship with one unknown in 7th grade, and it is further developed in 8th grade. The survey was administered after the completion of the units on linear relationship with one unknown. Students' state identification number was available in order to distinguish each student's standardized test score.

We do recognize that there are other ways to classify students' achievement levels (e.g., report cards, benchmark tests), and that high- or low-achieving categorization based on a single standardized test limits the study's conclusions and generalizability. However, given the relatively large sample size, the sCAS scores were the most attainable and controllable source of reliable and consistent data.

Instrument

The original instrument, designed by the researchers (Panasuk, 2006, 2010; Beyranevand, 2010) for a multiyear large scale study consisted of several interrelated parts. This paper reports on the data collected from two parts. Part R (Recognition) and Part P (Problems) of the instrument were focused on and measured both students' ability to recognize structurally the same linear relationship and to solve problems that involve linear relationships with one unknown posed via multiple representations. The data collected from both parts were correlated with the students' achievement level based on their sCAS scores. As Neimi (1996) suggested, "Commonly used achievement tests provide at best only indirect and highly limited information on students' understanding of specific conceptual domains" (p. 351). Since conceptual understanding in algebra was one of the foci of this study, testing and measuring both the ability to recognize and to solve problems involving structurally the same linear relationship with one unknown posed in different modalities strengthen the analysis of the data and validated the findings.

Figure 1 illustrates Part R which involved structurally the same linear relationship with one unknown posed in three different representations.

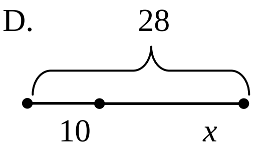
W. Mark is 10 years older than his sister. He is now 28. How old is his sister now?	D. 	S. $x + 10 = 28$

Figure 1. Part R of the instrument

The students were asked to observe (not solve the problems) and explain in writing if they recognize some similarities between the problems presented in three different modes. Ideally, they were expected to state that each problem represents structurally the same linear relationship, i.e., the sum of 10 and an unknown quantity is 28.

Part P had three sets each of which consisted of three problems (total nine problems): Problem Set W (words), Problem Set D (diagrams), and Problem Set S (symbols) similar to the problems shown on the Figure 1. The first two problems in each Problem Set could have been solved in one-two steps; the third problem in each set required more then two steps. The problems in each set have their counterparts in other sets presented in different modalities. The students were asked to solve each problem. For each Problem Set a coding system was created.

Followed by the analysis of the data, fifteen most representative surveys were selected and the students were invited for interviews. The selection was based on the responses that had a potential to provide information related to the different achievement level students' ability to recognize structurally the same linear relationship and solve problems. Of those fifteen, nine agreed to be interviewed.

RESULTS

Quantitative Component

All collected surveys were coded on a nominal or ordinal scale, and frequencies, means, and standard deviations were calculated for each part of the instrument; regression analysis was completed to quantify the relationship between the variables.

The scoring frequencies of the students' responses were measured as ordinal data, grouped in four categories and compared to the corresponding standardized test score for each student (Table 2). Students who did not recognize that three problems in the Part R represented structurally the same relationship (answered 'no') or left the space blank, were grouped together into category 0. The next group of students (category 1) indicated neither 'yes' nor 'no' but used vague unspecific language attempting to provide some description that could have been considered relevant. The category 2 included students who answered 'yes' and made an attempt to describe similar features of the problems. They either indicated that "the problems had the same numbers", or described the operation needed to solve the problems, or recognized that "all the problems had the same numerical solution." The category 3 was formed with students who clearly identified and explained that all three problems represent structurally the same relationship.

We must comment here that the responses that fell into the categories 0 and 3 were relatively easy to distinguish and classified. However, scoring of the responses in the categories 1 and 2 was particularly difficult due to students' vague and unspecific language. To ensure certain level of scoring trustworthiness, each response was reexamined for consistency. What actually mattered was whether the students did not recognized (category 0), distinctively recognized and explained (category 3), and/or produced some explanations, yet unambiguous (category 1 and 2). Thus the categories 1 and 2 were combined together for analysis purposes.

Table 2

sCAS categories and Part R frequency

Coding categories		Category 0: Did not recognize the same relationship	Category 1: Used vague unspecific language	Category 2: Recognized similarities, but not explicitly identified the same relationship	Category 3: Recognized the same relationship and explained
sCAS Score		Frequency/Percent			
Lower tier	LW	0 / 0	4 / 100	0 / 0	0 / 0
	HW	6 / 7	62 / 70	16 / 18	5 / 6
	LNI	6 / 10	38 / 63	11 / 18	5 / 8
	HNI	3 / 4	43 / 59	17 / 23	10 / 14
Upper tier	Prof	1 / 1	78 / 56	20 / 11	20 / 11
	Adv	0 / 0	29 / 64	12 / 26	4 / 9

Interesting enough that almost all (except one) students in the category 0 were from the sCAS lower tier, and only 10% of the upper tier students were in the category 3. The majority of the category 1 and 2 students were from the lower tier and few from the upper tier. Only about one quarter of all participating students formed the category 3, which an indication that not all upper tier students were able to recognize that the structurally the same relationship was presented in different modalities, thus not showing conceptual understanding of the concept.

Pearson and Spearman correlations between the ability to recognize and achievement level yielded a weak, yet statistically significant positive correlation (Table 3).

Table 3

Recognizing the Same Linear Relationship: Pearson and Spearman Correlations for

Variable independent	Variable dependent	Pearson	Sig.	Spearman	Sig.
sCAS Score	Identified the same relationship posed in different modalities	.150	.003 *	.142	.005*

Note: * indicates a significance value < 0.05.

Table 4 shows the mean, standard deviation and frequencies of the students' correct solutions for each problem within the Part P. Each student's score is based on the number of responses answered correctly for each of the three problems in each Problem Set.

Table 4

Part R: Problem Set Frequencies

Problem Set	Mean of all three problems	Standard deviation	Problem correct		
			#1	#2	#3
W: Verbal representation	2.34	.634	86.7%	98.2%	49.2%
D: Diagrammatic representation	2.36	.879	88.0%	87.7%	60.6%
S: Symbolic representation	2.62	.731	92.9%	88.7%	80.7%

The data revealed that students were most successful solving the equations represented symbolically and less successful finding the unknown length of the line segment.

To compare the students' test scores with their ability to solve problems posed in three different modalities, Pearson and Spearman correlations were calculated (see Table 5).

Table 5

Pearson and Spearman Correlation for solving all problems in the Problem Sets

Variable independent	Variable dependent	Pearson	Sig.	Spearman	Sig.
sCAS Score	Solving problems in all Problem Sets	.471	.000 *	.478	.000 *

Note: * indicates a significance value < 0.05.

The results yielded a strong positive correlation with a significance of $p < 0.001$. The students who were able to correctly solve problems in Part P posed in the three different representations were significantly more likely to be in the sCAS upper tier.

To understand the association between the students' ability to recognize structurally the same relationship and solve problems in each of the same representations, Pearson and Spearman were calculated (see Table 6).

Table 6

Correlations for Recognition of the Same Relationship and Solving Equations

Variable independent	Variable dependent	Pearson	Sig.	Spearman	Sig.
Recognized the same relationship posed in different modalities	Solved problems posed in different modalities	.124	.010 *	.104	.031*

Note: * indicates a significance value < 0.05.

While the correlation is weak, it is statistically significant to conclude that the students who recognized structurally the same relationship were likely to be able to solve problems involving linear relationship with one unknown posed in different modalities.

In summary, the results showed that there is an association between students' achievement based on the standardized tests and their ability to recognize and solve problems involving structurally the same linear relationship with one unknown presented in different modes (words, diagrams, and symbols).

Qualitative Component

The interviews with nine selected followed a semi-structured format and provided another layer of evidence. Watching how the students solved the problems posed in different representations informed our understanding of the students' ability to recognize different representations of structurally the same concept, translate between the representations and to communicate their reasoning in a perceptible mathematics language.

First, the students were encouraged to look at their Part R's responses and verbalize the relationship in each of the three problems (Figure 1). Then, the students we encouraged to create diagrams for the problem #2 from both Set W (words) and Set S (symbols), and write *algebraic* equations for same problem from both the Set W (words) and Set D (diagram).

Lower tier.

The lower tier students tended to use trial and error method to solve problems from the Set W and Set S. It is likely that these algebra students have not transitioned from iconic to the symbolic mode of thinking (Bruner, 1966). When encouraged to draw a diagram that would represent linear relationship described in words, they produced the drawings that showed rather the solution process to assist their calculations (e.g., see Figure 2) than algebraic structure of the relationship.

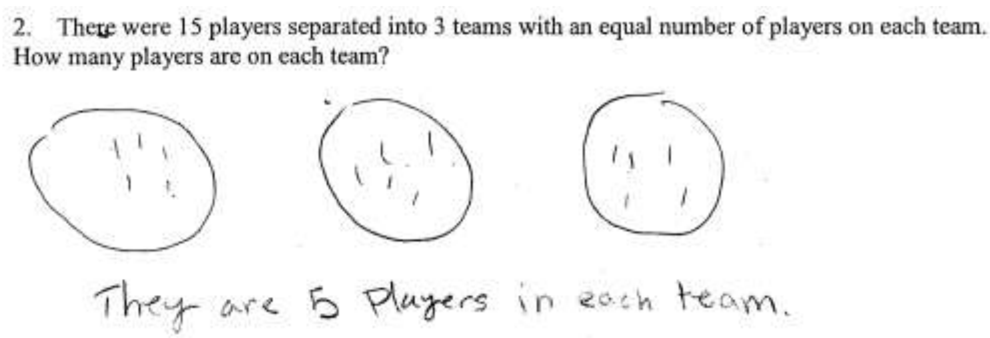


Figure 2. An example of a lower tier student's picture

While the picture on the Figure 2 seemingly assists calculations, it does not exactly represent *the linear relationship* the word problem describes. The picture shows that there are three groups of 5 objects, which is in numerical mode, 3×5. It would probably be acceptable for elementary students to think in terms of numbers and operations. Algebra level students were expected to produce more sophisticated algebraic statements that would show the structural properties of the relationship where a letter stands for unknown number (e.g., $15 \div 3 = n$, or $3 \times n = 15$). Clearly, these students had difficulties to abstracting from computations and showed a persistent tendency to reduce the level of symbolic abstraction to the level of numerical (lower level) abstraction (Pape & Tochanov, 2001). When solving algebraic equations, they were still thinking in arithmetic terms by undoing the chain of operations mainly showing calculation in a column format or in the form of the numerical equation (e.g., 28 - 10 = 18). Since these students had not developed reasoning abilities that were necessary for structural comprehension of linear relationship with one unknown, they were not able to recognize the structure of the relationship between quantities at a more abstract level than numerical.

Some of the students in this tier were able to find the correct solutions to the equations presented in symbols and words by using instantiations and/or manipulating with numerical and/or algebraic symbols (Problem Set W and S), however it was quite obvious that they were having difficulty or were unable to operate upon unknown quantity presented in a diagrammatic mode (Problem Set D). This explains why only few could recognize the relationships presented via diagram in the problem Set D. As a result, the lack of the diagrammatic skill (Diezmann, 1999, 2002) created a barrier for their meaningful learning (Kieran, 1992) and development of conceptual understanding.

Upper tier.

The major characteristic of the upper tier students who recognized and explained using a relatively clear written language that three different modes represented structurally the same linear relationship, was the ability to fluently transit between representations. These students solved all nine Part P problems correctly using algebraic methods and showed flexible use of each mode of representation. During the interviews, they made connections between the representations, logically interpreted and translated among representations. According to Cifarelli (1988) these students operated at the higher level of reflective abstraction, i.e., structural awareness. They had blended the process conception and the object conception by showing the ability to think and act upon the problems' structures (Sfard, 1991, 1992; Tall, 2008).

However, not all upper tier students were able to recognize structurally the same relationship with one unknown in the Part R. The most telling comment was written on the survey by the Advanced student shown on the Figure 3.

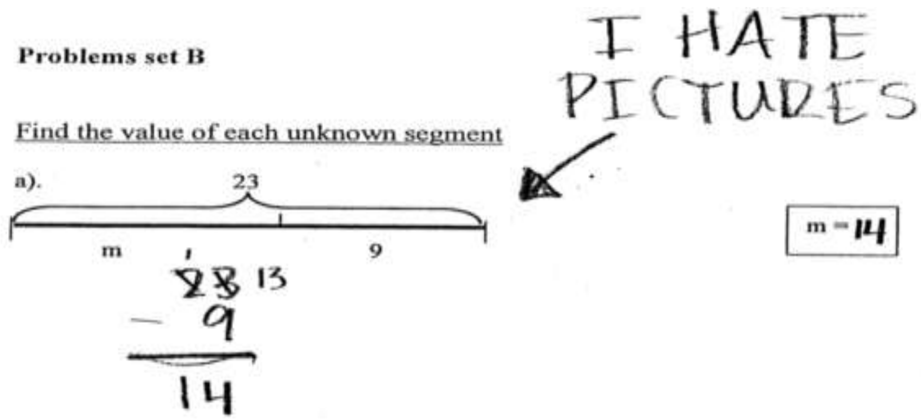


Figure 5. "I hate pictures"

This student articulated that the pictures confused him and that he strongly preferred using the symbolic mode of representation. Another Proficient student said that "equations make it easier because you can just put in the numbers and solve for it." Such responses were alarming, and when probed with questions, these students revealed well-trained procedural skills and minimal flexibility in thinking while dealing with the linear relationship. They justified their preference for symbolic representation saying that "it is easier for me to look at." They explained that they "got confused with the pictures" because they did not know "what to do" and "how to do it." "How to do it" would involve the extraction of and analysis of the structure of the relationship which is not immediately obvious and explicit when presented in words and/or pictures.

Through the interviews, it was noticeable that these upper tier students were lacking solid understanding of the deep structure of the linear relationship, the nature of unknown, and presumably mechanically used the rules and followed the steps.

DISCUSSION AND CONCLUSION

This study reveals that the students who were able to recognize structurally the same relationship posed in the different representational modes, i.e., showed conceptual understanding (Panasuk, 2010), were most likely to perform at a higher level on the sCAS standardized test.
As suggested by Panasuk (2010), these students are able to interpret, connect and translate with confidence among the representations of the same relationship, which in turn is likely to lead to higher achievement. However, we assert that the converse is not automatically true. We found that the achievement level is not a strong indicator of the conceptual understanding. In this study, some of the students who were able to solve linear equations with one unknown correctly did not necessarily show the ability to recognize the same relationship presented in words, as a diagram, and/or in symbols. The interviews showed that students could manipulate symbols but revealed little conceptual understanding. It is likely that these students could have little exposure to multiple representations, in general and diagrammatic training, in particular. Thus, our findings support previous research (e.g., Moseley, 2005; Niemi, 1996) that claimed the benefits of encouraging students to use multiple representations when learning concepts and solving problems. Moseley (2005) suggested "an early exposure to more diverse perspectives (representations) of rational numbers assists students in developing more interconnected and viable representation knowledge for rational numbers" (p. 37). Niemi (1996) made similar conclusions stating that "the more varied the students' representational knowledge was, the more likely they were able to solve the problems correctly" (p. 360). Diezmann and English (2001) called for diagrammatic literacy and suggested that the students must be exposed to systematic instruction which would help them to move beyond

the surface or artistic representation of information to a more structural and "sophisticated representation" of the problem information (p. 83). van Essen and Hamaker (1990) asserted that pictures provide an external sketchpad where students can represent and connect pieces of information and that generating pictorial representations facilitate the conceptualization of the problem structure and form the basis for a solution. Larkin and Simon (1987) documented that pictures facilitate the reorganization of information and help in making implicit problem information explicit.

Based on the above finding, we deduce that students would highly benefit from systematic and consistent learning how to recognize the same relationship and solve problems posed in different representational modes. The information to be learned in the classroom must be consistently and explicitly presented to the students in multiple ways in order for them to be able to build up a variety of thinking methods and techniques and to enhance their cognitive structures. This study supports the assertion that the more diverse the students' representational knowledge was, the more likely they were able to produce correct solutions to problems and that exposure to numerous representations assists students in developing their mathematical knowledge.

Final rumination

Neither the purpose nor the methodology of the study intended to establish cause-effect association between the achievement and the ability to recognize structurally the same relationship presented via multiple modes. However, the fact of the existence of such association is significant enough to acknowledge it. If the high-achieving students are able to observe the connections between different representations of the same concept/problem, they reveal conceptual understanding (Panasuk, 2010). It is reasonable to assume and/or question that those high achievers who did not recognize the same mathematical structure in multiple representations, either have not achieved conceptual understanding, or have difficulties explaining due to a lack of diagrammatic skills. These factors must be studied as they are critical to educators who wish to learn more how to effectively facilitate the development of students' conceptual understanding in algebra. Multiple representations need no advocacy; however more research will help to understand effective instructional methods that utilize multiple representations that facilitate mathematics students' conceptual understanding.

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