Transforming pedagogical practice in mathematics: Moving from telling to listening

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Abstract

The research reported in this article is part of a larger study that examines an initiative to expand teacher expertise in facilitating mathematical problem solving within the framework of developing and field-testing pedagogical resources. We focus on one year of the study and report on the complex process of professional development as teachers move from traditional pedagogies of teacher explanation of mathematical operations followed by student practice to a pedagogy of teacher and student exploration of number operations within a problem-solving environment. We see that, in this initiative, teachers, consultants, and professional developers explored new mathematical and pedagogical ideas, creating a collective that provided a supportive environment that served as a model for classroom settings where teachers and students, in turn, investigated mathematical ideas.

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> "The depressing thing about arithmetic badly taught is that it destroys a child's intellect and, to some extent, his integrity. Before they are taught arithmetic, children will not give their assent to utter nonsense; afterwards they will. Instead of looking at things and thinking about them, they will make wild guesses in the hopes of pleasing a teacher." (W.W. Sawyer, *Mathematician's Delight*)

For more than three decades perspectives in mathematics education have strongly promoted developing an understanding of mathematical concepts, procedures, connections, and applications through problem solving (National Council of Teachers of Mathematics [NCTM], 1989, 2000; National Research Council, 1989; Silver, Ghousseini, Gosen, Charalambous, & Font Strawhun, 2005). However, facilitating the development of mathematical understanding through problem solving remains a challenge for teachers. The study reported in this article discusses the complex task of working with elementary teachers as they move from traditional pedagogies of teacher explanation to a pedagogy of teacher and student exploration of mathematical ideas within a problem-solving environment.

Introduction

Current approaches to mathematics education emphasize the development of mathematical understanding through students solving problems and sharing solutions and strategies (NCTM, 2000). In terms of understanding number sense and operation, while operational procedures are still important, developing an understanding of the operations that those procedures represent is seen as essential and is best developed through students developing their own strategies for solving problems (Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Forman & Ansell, 2001; Kamii, 1985, 1989, 1994, 2008).

This view is supported by evidence that mathematical procedures have often been imposed on students in ways that do not necessarily develop mathematical thinking or understanding (Carpenter et al., 1993; Huinker, 1998). Further, despite hours of instruction and practice, students often fail to master basic school algorithms or to apply them correctly in problem solving situations (Brown & VanLehn, 1980; Carroll, 1996; Silver, Shapiro, & Deutsch, 1993; VanLehn, 1986). Burns (1994) suggests that imposing the standard algorithms on children gives students the idea that mathematics is a collection of mysterious and often magical rules and procedures that need to be memorized and practiced.

Students' multiple strategies

Accompanying this perspective is support and research (Baek, 1998; Boufi & Skaftourou, 2004; Huinker, 1998; Kamii, 1985, 1989, 1994; Kamii & Lewis, 1993) for more attention being paid to students' development of their own strategies for arithmetic operations and less attention given to the teaching and practicing of traditional school algorithms. Studies suggest that children develop a deeper and more flexible understanding of operations through doing their own thinking and developing procedures that have meaning for them (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Fuson et al. 1997). Further, when students develop their own strategies, they are more aware of place value and there are fewer conceptual errors (Carpenter et al., 1998; Carroll & Porter, 1998; Kamii & Lewis, 1993). The understanding developed through creating their own strategies is also extended to solve unfamiliar or non-routine problems more readily than strictly a procedural knowledge (Hiebert, 1986; Hiebert & Carpenter, 1992; Hiebert & Wearne, 1996). Gravemeijer and van Galen (2003) suggest that whether these semi-informal methods develop into the conventional algorithms, the quality of the students' understanding is much greater than if merely taught conventional algorithms.

However, students do not develop strategies in isolation. Methods of solution are constructed in a social context in which students share their strategies with one another. In Baek's study (1998) of student development of algorithms for multi-digit multiplication, students develop their own procedures through sharing different ways of solving problems and discussing the mathematical meaning underlying their inventions. Students may use mental procedures, manipulatives (play money or bingo chips), pictures, or written procedures to solve their problems. Students work on problems individually or in small groups, and then share their personal strategies or algorithms, discuss similarities or differences among them, and explore underlying mathematical concepts. Further, as students share their strategies they provide teachers with a window into how they think about the operational composition of numbers (Baek, 1998; Carroll, 1996; Willis & Fuson, 1988).

The Role of the Teacher

Teachers have an important role in guiding students' mathematical development by engaging them in problems, facilitating the sharing of their solutions, observing and listening carefully to their ideas and explanations, and discerning and making explicit the mathematical ideas presented in the solutions (Ball, 1993; Lampert, 2001; NCTM, 2000). Several research projects (see for example, Cobb, Wood, & Yackel, 1990; Fennema et al., 1996; Franke & Kazemi, 2001; Simon & Schifter, 1991) have found that when teachers attend to their students' mathematical thinking there are many benefits which include higher levels of conceptual understanding by students and more positive attitudes held by both teachers and students towards mathematics.

In particular, the encouragement of students' methods of solution requires that the teacher develops a listening orientation. Such an orientation promotes a learning environment conducive to and respectful of students' own sense making and intellectual autonomy (Davis, 1996; Kamii, 1989). Listening to students' mathematical thinking is one of the central tasks of mathematics teaching (NCTM, 1991). However, listening to students' thinking is hard work, especially when students' ideas sound and look different from standard mathematics (Ball, 1993; Morrow, 1998; Wallach & Even, 2005). Davis (1996) suggests that there are various ways that teachers listen to students' mathematical ideas and not all forms of listening are conducive and respectful of students' thinking.

Implementing this vision of mathematics classes where students' autonomous sense making and problem solving are facilitated challenges previous held notions of what it means to teach mathematics (Silver et al., 2005; Manouchehri & Goodman, 2000). The notion of teaching as telling (speaking, explaining, demonstrating) rather than listening (hearing, seeing, interpreting) still pervades most mathematics classrooms. Despite the many benefits seen by listening to students' mathematical thinking, focusing on students' thinking is challenging for several reasons (Ball, 2001; Schifter, 2001; Wallach & Even, 2005). One of the reasons is that students present a variety of ways of thinking about a mathematical problem and teachers may worry whether they will recognize mathematical understanding in all of the representations presented. Although a student may not appear to a teacher to understand a concept, there may actually be sense in their thinking and explanation. When teachers do not attend to student thinking they tend to dismiss what students bring to the mathematical community and instead impose traditional formalized procedures on students.

Professional Development

While there is evidence of the value of new teaching practices in mathematics, there is also evidence that mathematics teachers have not easily adapted to these changes and in many cases, little has changed (Manouchehri & Goodman, 2000; Silver, et. al., 2005). Designing professional development that helps teachers develop the practices that promote problem solving and students' multiple solutions poses several challenges. Most teachers have experienced traditional school mathematics programs and in their minds, mathematics largely consists of meaningless memorization of mathematical facts, rules, and procedures and they see their role as delivering such procedures (Carroll, 1996; Carroll & Morrow, 1998). It takes time for teachers to see the importance of posing problems, providing opportunities for students to explore the problems, and listening to their solutions. Even when they are convinced of the importance of these practices, it takes time for teachers to learn to incorporate them.

There are several components to professional development that have been viewed as positively supporting teachers in shifting practice. Carroll, Fuson, and Diamond (2000) highlight the importance of sustained professional development when teachers are working towards developing their skills at listening and interacting with students' mathematical thinking. Some researchers (Ball, 1988; Crespo, 2000) have noted that introducing teachers to unfamiliar mathematical tasks ensures that teachers genuinely engage with mathematical content. By introducing unfamiliar problems to teachers and engaging teachers in exploring new mathematical ideas themselves, teachers shift from looking at students' work in merely an evaluative manner to listening for changes in meaning (Ball, 1988; Crespo, 2000). Engaging teachers in examining student work and (Simonsen & Teppo, 1999), connecting professional development to actual classrooms through reflection and analysis of concrete classroom episodes (Scherer & Steinbring, 2006) and supporting teachers as they attend to their own students' thinking (Steinberg, Empson, & Carpenter, 2004) helps teachers develop a better understanding of children's mathematical learning processes, and mathematics teaching and learning in general.

In this article, we examine a professional development initiative that is designed to help teachers develop classroom practices that promote problem solving, the sharing of student strategies, and attention to students' thinking. We focus on the experiences of the teachers and examine what they have learned, what changes they have made in their practice, and what obstacles they faced. We also are able to provide insights into the particular aspects of the professional development initiative that were key to developing new mathematical and pedagogical understandings and provided support.

Our Study

Our study focuses on a model of professional development designed to provide teachers with experiences to enhance their mathematical knowledge and pedagogical expertise and to support teachers as they connect their professional development experiences with classroom practice. The study helps to describe the experiences of elementary teachers as they engage in new reform-oriented classroom practices to support student understanding of mathematical operations through the development of their own strategies. In examining teachers' experiences in this situation we are drawing on a social-constructivist framework (Confrey, 1990; Davis, Mayer & Noddings, 1990; Forman, 2003; Yackel & Cobb, 1996). From this perspective, new understanding is constructed and emerges from the interactions of people and activity environments. This shifts the traditional view of learning from the acquisition of something to a formulation of learning as participatory (Sfard, 1998) which is important for conceptualizing professional development as situations for building communities for inquiry into practice.

Context

This study and the professional development initiative are set within the community of francophone educators in the province of Ontario, Canada. The francophone community represents 4.8% of the population of Ontario and has its own set of publicly-funded school boards. The provincial government sets the curriculum and curriculum documents are provided in both French and English. However, resources for teaching mathematics that align with the curriculum are much more readily available in English than in French. In one attempt to address this problem, the Ministry of Education financed a team of mathematics educators within the francophone community to develop pedagogical resources intended for teaching mathematics in the French-language schools. As well as developing materials, this initiative included field-testing the resources with francophone teachers from different grade levels and supporting them as they used the resources. The team of resource developers thus also became a professional development team who we will refer to as Resource/Professional Development (RPD) team.

Support came in a variety of forms. First, groups of teachers from schools participated so that they could support one another at the school-level. Second, for the school boards involved, board-level consultants were included in the initiative to provide support and feedback to teachers and developers. Third, the teachers and consultants had regularly scheduled meetings throughout the school year with the resource developers who also provide the professional development. Thus, the participants in this professional development initiative formed a collective that includes classroom teachers, mathematics consultants at the school board level, and the resource developers themselves. These participants were also all participants in our study. It should be noted that our involvement in this initiative was solely as researchers. In other words, the professional development was designed and carried out by the professional development team that did not include the authors of this article.

The professional development initiative

The professional development initiative spanned three years. The first year focused on the development of primary mathematics and included Kindergarten to Grade 3 teachers; the second year was aimed at teachers of Grades 4 and 5; and the third year focused on Grades 6, 7 and 8. Each year followed a similar pattern and engaged the group of teachers for one full school year, with professional development meetings spread throughout the year so that teachers had the opportunity to try new ideas in their classroom and discuss them with their colleagues. The sessions focused primarily on the teachers' understanding of mathematics and on ways to help students develop mathematical understanding. In the first meeting, the professional development team shared examples of current research in math education to provide a rationale for changing some of the traditional approaches to mathematics so that students develop a better understanding of mathematical concepts (Cobb et al., 1991; Carpenter et al., 1993). Teachers discussed topics such as teaching through problem solving and connecting school mathematics with the mathematics in students' lives. All sessions engaged the participants in mathematical problem solving and solution sharing, working with manipulatives, viewing videos of classroom activities, and discussion. After the first session, time was also devoted to sharing experiences of using some of the new resources in classrooms.

Teachers worked in collectives where an entire group of teachers from the same school took part in the initiative and were supported by their own school board consultant as well s the RPD team. The school board consultants made regular visits to the teachers' classrooms throughout the year, as well as attending the formal professional development meetings. These visits took a variety of forms such as observation, co-teaching, or assistance with preparation of materials. The teachers were also able to contact the RPD team directly if they had questions or need assistance. The immediate classroom needs of the teachers were met by providing appropriate grade level resources and feedback on classroom practice.

The resource and the training focused on the domain of numeration and number sense. Particular attention was given to the development of operational sense and students' own algorithms and procedures for mathematical operations. Teachers were prompted to look at numbers in a variety of ways in much the same way that they would be prompting their students.

For instance, they were asked "What does 10,000 mean and what are the different ways that it can be represented?". They used concrete materials such as base ten blocks or images such as grids of grids of 100. The representations then helped to model number relationships such as 10,000 can be represented by two groups of 5000 or four groups of 2500 or 10 groups of 1000 or by 9,999 + 1.

Another important strategy was to use contextual problems that could be solved in different ways. The variety of solutions and the sharing of these solutions prompted teachers to identify the relationships between mathematical operations. For instance, the teachers were presented with a problem that asked "How many cars it would take to bring 19 children to the zoo if each car holds 5 children?". Although this problem was easy for the teachers to solve, it was used to see the variety of operations that it could incorporate. Its solution can be linked to division ($19 \div 5$), multiplication (5 x ? = 19), addition (5 + 5 + … = 19), or subtraction (19 -5 - 5 - …) depending on how it is solved. Further, it posed dilemmas in terms of uneven groupings. Participants tried the problem, shared solutions, and then the problem was discussed in light of research on using this problem with children in Kindergarten and Grade 1 (Carpenter et al., 1993). Teachers discussed the different ways that students might solve this problem and the kinds of materials that they might use to solve the problem.

The RPD team also led discussions regarding ways to help students develop and share their strategies. The teachers discussed how to organize students to work together, share their solutions in pairs, and then explain their reasoning to others in the class. The importance of children representing their reasoning with mathematical language and symbols so that others can see what they are thinking were discussed which clearly connected to issues of assessment.

Participants

Although our research includes all three years of this project, our focus in this article is on the development of teachers' facilitation of mathematics in Grade 4 and 5 classrooms. Thus we focus on the second year of the project which included five consultants and 39 Grade 4 and 5 teachers from three different school boards. The teachers were evenly spread across these two grades, with 18 teachers teaching Grade 4, 16 teaching Grade 5, one teacher teaching a combination class of Grades 3 and 4 (Grade 3/ 4 split), two teachers teaching a Grade 4/5 split, and two teachers teaching a Grade 5/6 split. The teachers had a variety of experiences ranging from a few beginning teachers to over 50% with more than 9 years of teaching experience. The resource/professional development team for this group was made up of six mathematics educators with different experiences and areas of expertise.

Method

We use a case study design that includes multiple sources of data. This project is a specific example of a group of teachers being involved in a unique initiative to learn more about teaching mathematics through the field-testing of pedagogical resources within a supportive environment. As Merriam and Simpson (1995) suggest, a case study approach helps to provide an intense description of a group or community to describe the interplay of various factors and complexity of human situations.

Several sources of data were used including questionnaires, interviews, and training materials. The first questionnaire was administered to the teachers in November 2005, allowing researchers to gather information from the teachers such as grade level, experience in teaching mathematics, level of confidence about mathematics teaching, and reasons for their involvement in the project. A second questionnaire was administered in March 2006 to gather information about several aspects of the project as well as the teachers' classroom experiences and ways that they were supported and challenged.

Data were also collected through interviews with participating teachers, consultants, and the professional development team. At the end of June 2006, teachers met with one researcher in six small focus groups to discuss different aspects of their experience with the project. A focus group with the professional development team and personal interviews with the school board mathematics consultants involved in the project provided data on their perspectives of the project. All interviews were audio-taped and transcribed.

We had access to training materials that included binders with the agendas and activities of the training days. We also had access to the full pedagogical resource package that was created for each grade that includes a discussion of the philosophy of the resource, an overview of the year's plan, the lesson plans, activities and assessments that teachers would use in their classes, as well a DVD created from the field test classrooms to show teachers what this would look like in a classroom.

A qualitative analysis of the data was completed through content analysis of the open-ended questions from the teachers' questionnaire and the transcribed discussions of the focus groups and personal interviews. Our analysis was inductive to uncover emergent categories, themes and concepts (Guba & Lincoln, 1981). As this study was done with a francophone community, all of our data is in French. As a bilingual research team, we worked with the original French data in our analysis and have translated quotations, and classroom episodes from French to English for the preparation of this article.

Discussion of Results

In this section, we report on teachers' classroom experiences of facilitating students' development of their own strategies by first presenting a description of a teaching episode that is fairly representative of teachers' classroom experiences. This is followed by an analysis of the themes that emerged from the focus group interviews as teachers discussed their experiences of attending to students' multiple strategies in problem solving settings.

Description of Classroom Episode

The following episode took place in a Grade 4 classroom of one of the participating teachers. The teacher introduced a problem, students worked in pairs, and then students presented the solutions on the board with the whole class discussing and comparing strategies and solutions.

Before introducing the problem the teacher briefly discussed the context of saving money, putting savings in a bank, and the difference between different types of banks. The teacher then introduced the problem:

> Ms. Sansoucis had saved $8300 in the bank. She needs to take out $2625 to buy an exercise bicycle. How much money does she have left?

After the introduction of the problem, pairs of students were provided with a sheet that restated the problem, had a space for students to estimate the answer, and space for students to show how to solve the problem in two different ways (see figure 1).

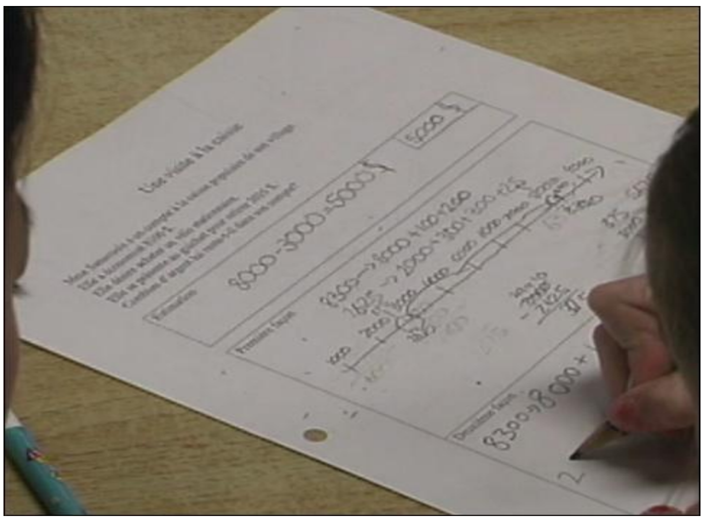

Figure 1. Students working in pairs on the problem with provided worksheet.[1]

Working in pairs, students made an estimate. The teacher asked the class to share their estimates as a whole class with estimates varying between $5000 and $6000. As students shared their estimates, the teacher places these values on a partial number line that allowed for their range of values. Students then went back to working in pairs to solve the problem. The teacher circulated the class and consulted with pairs as they worked. When the pairs had recorded two different ways to solve the problem, it was time to share solutions as a whole class. The teacher started the discussion with the following question:

> Do you think that this question is a question with many possible answers or just one answer?

The class realized that although there were many ways to solve this problem, there would be only one answer. Then different pairs presented their solutions on the board by having one of the students from the pair go to the board to show their solution. While the student presented

[1] From "*Guide pédagogique 4e année: Numération et sens du nombre et mesure – Les mathématiques… un peu, beaucoup, à la folie!*" by J. Gaudreault, N. Gervais, C. Renaud-Charette, and P. Moisan, 2007. Copyright by the Centre franco-ontarien de ressources pédagogiques. Reprinted with permission.

their solution on the board, the teacher simultaneously recorded their method of solution on chart paper that was taped to the board. In this way, the teacher was helping the student to record their reasoning while also producing a permanent record of the solution that could be posted next to other solutions that subsequent pairs proposed.

The first pair presented the following solution on the board and explained it as they went:

On the board	What was said
2625 + 75 = 3000 → 3000 + 2000 = 5000	"2625 plus 75 is 3000"
	"3000 plus 2000 is 5000"
5000 + 300 = 5300 → 5300 + 3000 = 8300	"5000 plus 300 is 8300"

Therefore, the answer is 5675

The class was a bit perplexed. Although the answer agreed with most of the students' answers, the teacher's recording (on a number line) of the parts that were being added [75 + 2000 +300 + 3000] would result in the answer 5375. However, rather than the class (or teacher) proclaiming "no, it's wrong", they all sought to find the missing 300. One student in the class spotted the error [in the first line 2625 + 75 is not 3000] and the class discussed the mistake with the error being corrected. Before they moved on to another pair's solution, the teacher suggested that whole class use the calculator to verify the answer. The teacher also discussed the answer in light of the original estimation.

Another pair of students presented their solution next. The student presenting drew a number line and put 2675 at one end and 8300 at the other. She then jumped down the number line using "easy" numbers (- 300, - 5000 and – 375) to move from 8300 down to 2675.

A third pair used similar reasoning of repeated subtraction but did not use a number line.

On the board	What was said
8300 – 600 = 7700	"removing 600 leaves 2025 to remove"
7700 – 2000 = 5700	"removing 2000 leaves 25 to remove"
5700 – 25 = 5625	

A fourth pair presented their strategy, which the teacher again recorded. The whole class then looked at the variety of solutions posted on the pieces of chart paper with the teacher leading a discussion about the similarities and differences of the various ways that students solved the problem.

Analysis of teaching episode

This episode shows several teaching strategies that helped to facilitate students' thinking. First, the teacher introduced the problem in a way that set the context and introduced the vocabulary of the problem so that the problem was understandable and accessible to all students. In addition, having students estimate an answer at the beginning of the problem and revisiting the estimate midway through the solutions helped the students examine the reasonableness of their answers.

The teacher's use of questioning throughout the presentations helped to include all of the students in focusing on the different ways to solve the problems. Students also were all included in the verification of the solution and responded when there were errors or misunderstandings. As well, the teacher helped students see the value in learning from errors and did not shy away from exploring errors. He handled student mistakes as opportunities for learning by recognizing that a strategy can be valid even though the answer could be incorrect. The teacher was able to turn an error into a teachable moment.

The worksheet that provided a statement of the problem, space for an estimation and space for two different solutions prompted the students to seek a variety of ways to solve problems. This format of problem sheets was used by most teachers in the initiative and one teacher in a focus group commented that some of her students had been taught only traditional algorithms in the previous year and the format of the sheet was helpful because her "students would first provide the traditional algorithm but then because I wanted two, they would go back and work on a second one. And with the second one, they had to think about what they are doing" (Focus group K).

The teacher's chart paper reproduction of student solutions may have several benefits. First, in a way, the recording of the student solution may have helped the teacher think through the solution in the same way that the student did. Second, the teacher's reproduction presented a slightly different representation of the same strategy and this may help students make connections between different representations. For instance, sometimes the teacher would represent the student's subtraction or addition as movement on a number line. Third, the recording provided a permanent record so that the similarities and differences of solutions can be discussed.

Analysis of the Focus Group Data with Respect to Attending to Student Thinking

The focus group data presented several themes that were connected to the facilitation of students' multiple strategies and attention to student thinking. First, many focus group participants said that they thought it would be impossible to learn to facilitate the development and sharing of students' solutions without this training and support. For instance, one participant suggested that the most important idea that she had learned during the year of professional development was "to be able to see that students can add, subtract, and multiply in different ways without using the standard algorithm" (Focus group F). Teachers also talked about the enhancement of their own understanding of mathematics, the changes in their classroom practice, the ways they were challenged and supported in changing classroom practice, and the observation of increased student understanding in mathematics through these new approaches.

Increased student understanding

Teachers overwhelmingly reported a change in their students' learning and attitude towards mathematics. At first, the teachers needed to place some trust in the notion that students can develop their own procedures. Once that trust was in place, the teachers saw that the students developed an understanding that they realized did not vanish over time in the way that the memorized procedures did. Students seemed more positive, mathematics seemed more interesting to them and the students were involved in more mathematical activity. Some teachers mentioned that it appeared that the students who were benefiting the most from the project were those who had been having difficulties in mathematics. Teachers explained that the openness about different ways of thinking about mathematics problems encouraged all students to be successful in mathematics as this teacher explains:

> The students who were less strong, became more strong. It was Wow! I was really surprised. Some of the students who were having trouble succeeding, out of the blue, became stronger. Sometimes it was hard for me to understand how they were doing it but because it was a personal algorithm, if he understood what he was doing he was able to get the right answer. And he could explain his method. It was a good algorithm for him. (Teacher, Focus group G)

Teachers' understanding of math

Teachers reported an increase in their own understanding of mathematics. Many teachers expressed that they had difficulty with mathematics when they were in school but they now felt more at ease and had more confidence based upon their professional development experience and work with students in the classroom.

> The big change this year is to see the level of understanding of students and to see how my own understanding of math has changed when we look at things in different ways. (Teacher, Focus group K)

They stated that working with different ways of using mathematical operations allowed them to understand some of the standard algorithms that they had been using for years. Many expressed that they wish they had been taught using these methods as these teachers explain.

> Teacher: I thought that I understood numeration, such as work with powers of 10, but I realize that I had just memorized when you add or remove a 0. (Focus group H)

> Teacher: I always had difficulties with math in school. But now, doing this with students, I understand the math much better and I wish that this was the way I had learned it. (Focus group H)

They were enjoying doing mathematics and found that they were learning from their students' strategies. Some said that they used to hate teaching mathematics and would avoid it or stick to a very traditional approach.

While some mentioned that in doing mathematics themselves, they sometimes have difficulty setting aside the traditional algorithms, there were moments when they found themselves using some of the invented strategies their students used. For instance, a teacher in Focus Group K described a scenario of working with a colleague and coming to a point where she needed to multiply multi-digit numbers. She realized that rather than reach for her calculator she used an area model to do the multiplication, as students in her class had done. She produced the accurate result and although her colleague reached for the calculator to check the answer, she had no reason to doubt her own accuracy. She reflected on what a change this was for her in her approach to solving problems.

Changes in teaching practice

One of the more important points observed through the data analysis of teacher focus groups was the transformation of their teaching strategies. Most teachers said that they are using questioning more effectively and letting students respond to questions in a variety of ways. Multiple ways of looking at a task or solving a problem were encouraged. The teachers also reported promoting a different learning environment, encouraging more student communication and risk-taking. Teachers reported feeling more confident that they know what they are doing and why they are doing it. Some teachers reported that their new way of looking at mathematics teaching has also had an impact on their teaching of other subjects. Teachers from a variety of groups reported that it changed their way of teaching:

> Teacher: It changed our way of teaching. No, more than that, it changed our way of thinking. (Focus Group G)

> Teacher: After this training, it would be hard to go back to teach the same way. We've changed our whole perception of math. (Focus group F)

> Teacher: Before this training, I would show one method and I would ask the students to model that method. Now, I let them be more creative. It has even spilled over in different disciplines such as science and language. (Focus group K)

Teachers noted that they were developing more communication in their classrooms and in particular, more mathematical talk between students. They also recognized the value of such discussion and observed that when students work in pairs they seem to learn more as they share their strategies. The following discussion between four teachers highlights these points.

> Teacher 1: Communication, that's the big thing now. Before they all worked individually but now they work in pairs.

> Teacher 2: Yes, it has improved our students' communication. I think it was where I needed to improve as a teacher - to help my students communicate more. They're anxious to share and participate now. They go to the board in pairs and explain what they did.

> Teacher 3: Yes, before we practically had to force them to go to the board. Now they all want to go up and explain how they solved a problem.

> Teacher 4: Even students that used to have difficulties want to share as well. For instance, one of my students who used to struggle has changed completely. He now

> wants to get involved, go to the board, and explain his thinking. Sometimes he's the only one who can figure out a problem. (Focus Group G)

Teachers also stated that the increased student communication made the teachers more aware and responsive to students' thinking

> Before, students just gave the answer. . . . And if it was right, we'd say "good boy". But now I realize that just because the answer is right, you have no idea of where it is coming from, the thinking behind it. Yeah, you're right, you've got the answer. But others could have had the wrong answer and still been on the right path. Now I know that if you take the time to look at a student's solution, the answer could be wrong but most of the process is good, they maybe made a small mistake towards the end. (Teacher, Focus Group F)

Teachers reported that they used manipulatives in a more meaningful way. Previously they had used manipulatives to make math fun and interesting for students but they now recognize that manipulatives present a different representation of mathematical ideas and enhance student understanding. Whereas before some of the teachers provided manipulatives only to struggling students, they now recognize that manipulatives are beneficial to all students.

> I have learned that students need to have access to manipulatives. Before I believed that it was only for students with difficulties. I knew that kids needed to touch, play, and manipulate and that's the way they learn. (Teacher, Focus group H)

New strategies for classroom management and the use of manipulatives were also developed. Previously, teachers were frustrated because students would play with manipulatives, but they now realize that students need more exposure to manipulatives and more time to play before they start to use them in meaningful ways. The participants were more open to students using a variety of manipulatives for any one problem. This is confirmed through data from classroom observations as we saw students using a variety of manipulatives such as popsicle sticks, cardboard rolls, linking cubes, or coloured squares in working on a particular mathematical task. And while this initiative focused on number sense and numeration, teachers also saw their use of manipulatives increase in other areas of mathematics.

Teachers also have developed ways of listening and responding to student solutions and in particular, of responding to student errors. They recognized that students are able to correct their own errors through the sharing of solutions. For instance, if three groups of students had three different answers, the teacher opened a discussion to explore the solutions and come to a consensus. Even when students were at the board, there was no stigma attached to making an error. Students paid attention to the errors of others and used them as a way to examine their own solutions and understanding.

Support

Teachers reported that the support from their colleagues, coordinators, and the RPD team was an important part of the project and provided an opportunity to discuss classroom challenges. Including all teachers of the same grades within a school supported teacher implementation of new ideas. One teacher commented that when she could not understand a student's written solution she would consult with the teacher next door and together they would figure it out. She suggested that previously she might have given up and marked it wrong.

Teachers also reported that the visits from the consultants were useful as they provided feedback, modeled activities, or helped in assessing student understanding through conferencing. The consultants also reported positively about their own experience in the project and described the meaningful feedback they have received from the teachers. Through their visits and discussions with the teachers they saw the impact of the project on teachers' practices.

The interactions between the teachers and the RPD team were viewed as dynamic and extremely valuable. During professional development days, teachers were able to share their experiences, and felt that their comments about the field-testing activities in their classroom were well received by the RPD team. This collaboration was seen as an important motivational aspect of the project.

Challenges

The teachers reported several challenges to their work. One is the challenge of changing practice that is well-documented in the research literature and is evident in this study. As one teacher reported "Your brain has been used to doing it the old way for 20 years." The teachers also recognized that although they had support for their teaching from their colleagues and board-level coordinator, support from the principal and parents was also a necessary element. The principal was often the gatekeeper to needed resources such as a budget for photocopying the many student activities or for the purchase of manipulatives. Principal support was also necessary for communicating information to parents about some of the changes in practice, addressing parent's concerns about students generating their own ways of performing mathematical operations.

Concluding Remarks

The results presented demonstrate that the teachers are reporting a number of changes: changes in teachers' classroom practices, changes in teachers' understanding of mathematics, and changes in students' understanding of mathematics. Much of this change is connected to listening to student thinking. It is also seen in the classroom episodes, such as the one reported in this article. Students are working on problems and sharing solutions, while teachers are focusing on understanding students' mathematical reasoning.

The classroom environment itself has changed with an emphasis on greater communication, including student-to-student communication as well as communication between student and teacher. Students are provided with a variety of tools to explore and express their ideas with more meaningful use of manipulative materials. Both teachers and students are more open to taking risks and discussing a wide variety of solutions rather than just one single method. The teachers have the confidence to allow a variety of solutions to be put forward and understand why it is important to do so. Teachers even use student errors as ways to explore ideas that lead to better understanding, rather than merely signals of misunderstanding.

The teachers report that there were several aspects of the professional development initiative that supported their new mathematical and pedagogical understandings. The multiple components of the initiative such as the sessions that included problem solving, current research, and videos of classroom practice; the sustained professional development over a year; working together as a community within the school; and the support of the coordinators and the RPD team all contributed to teachers' new learning. The teachers, along with consultants and the RPD team had an opportunity to discuss classroom experiences and challenges. The collaboration achieved by including all teachers of the same grades within a school, the dynamic of the teachers moving back and forth from organized meetings to classroom practice, and consultants visiting classrooms supported the development of new ideas and classroom practices. The professional development initiative, itself, with teachers, consultants, and the RPD team listening and responding to one another provided a model for teachers in the classroom.

> I think I can compare how kids feel to what we have done because we felt the same way. Because we had the experience of doing this ourselves in the training, our confidence to proceed was based on experience rather than theory. (Teacher, Focus group H)

Listening to one another became a collective action, not a technique or strategy, that was the essential factor in their learning, both in the classroom and in the professional development activity.

The professional development initiative did not tell the teachers what they should be doing in their classrooms but allowed the teachers to experience new ways of teaching and learning and to talk about those experiences. The teachers, in turn, moved this model of collaboration and attentive listening and responding to their classrooms where students experienced new ways of exploring mathematics. In both settings, new pedagogical and mathematical understandings emerged.

References

Baek, J.-M. (1998). Children's invented algorithms for multidigit multiplication problems. In L. J. Morrow & M. J. Kenney (Eds.), *The teaching and learning of algorithms in school mathematics – 1998 Yearbook* (pp. 151-160). Reston, VA: National Council of Teachers of Mathematics.

Ball, D. L. (1988). Unlearning to teach mathematics. *For the Learning of Mathematics, 8*(1), 40-48.

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal, 93*(4), 373-397.

Ball, D. L. (2001). Teaching, with respect to mathematics and students. In T. Wood, B.S. Nelson, & J. Warfield (Eds.), *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 11-22). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Boufi, A., & Skaftourou, F. (2004, July). *From formal to semi-informal algorithms: The passage of a classroom into a new mathematical reality*. Paper presented at the meeting of the International Group for the Psychology of Mathematics Education, Bergen, Norway.

Brown, J. S., & VanLehn, K. (1980). Repair theory: A generative theory of bugs in procedural skills. *Cognitive Science, 4*, 379-426.

Burns, M. (1994). Arithmetic: The last holdout. *Phi Delta Kappan, 75,* 471-476.

Carpenter, T. P., Ansell, E., Franke, M. L., Fennema, E., & Weisbeck, L. (1993). Models of problem solving: A study of kindergarten children's problem-solving processes. *Journal for Research in Mathematics Education, 24*(5), 427-440.

Carpenter, T. P., Franke, M. L., Jacobs, V. R., Fennema, E., & Empson, S. B. (1998). A longitudinal study of invention and understanding in children's multidigit addition and subtraction. *Journal for Research in Mathematics Education, 29*(1), 3-20.

Carroll, W. M. (1996). Use of invented algorithms by second graders in a reform mathematics curriculum. *Journal of Mathematical Behavior, 15*(2), 137-150.

Carroll, W., Fuson, K., & Diamond, A. (2000). Use of student-constructed number stories in a reform-based curriculum. *Journal of Mathematical Behavior, 19*, 49-62.

Carroll, W. M., & Porter, D. (1998). Alternative algorithms for whole-number operations. In L. J. Morrow & M. J. Kenney (Eds.), *The teaching and learning of algorithms in school mathematics – 1998 Yearbook* (pp. 106-114). Reston, VA: National Council of Teachers of Mathematics.

Cobb, P., Wood, T., & Yackel, E. (1990). Classrooms as learning environments for teachers and researchers. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views of the teaching and learning of mathematics: Journal for Research in Mathematics Education – Monograph 4* (pp. 125-144). Reston, VA: National Council of Teachers of Mathematics.

Cobb, P., Wood, E., Yackel, J., Nicholls, G., Wheatley, B., Trigatti, B., et al. (1991). Assessment of a problem-centered second-grade mathematics project. *Journal for Research in Mathematics Education, 22*(1), 3-29.

Confrey, J. (1990). What constructivism implies for teaching. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views of the teaching and learning of mathematics: Journal for Research in Mathematics Education – Monograph 4* (pp. 107-122). Reston, VA: National Council of Teachers of Mathematics.

Crespo, S. (2000). Seeing more than right and wrong answers: Prospective teachers' interpretations of students' mathematical work. *Journal of Mathematics Teacher Education, 3*, 155-181.

Davis, B. (1996). *Teaching mathematics: Towards a sound alternative*. New York: Garland Publishing.

Davis, R. B., Maher, C. A., & Noddings, N. (1990). Introduction: Constructivist views on the teaching and learning of mathematics. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views of the teaching and learning of mathematics: Journal for Research in Mathematics Education – Monograph 4* (pp. 1-3). Reston, VA: National Council of Teachers of Mathematics.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27*(4), 403-434.

Franke, M., & Kazemi, E. (2001). Teaching as learning within a community of practice: Characterizing generative growth. In T. Woods, B. S. Nelson, & J. Warfield (Eds.), *Beyond classic pedagogy: Teaching elementary school mathematics* (pp. 47-74). Mahwah, NJ: Lawrence Erlbaum Associates.

Forman, E. A. (2003). A sociocultural approach to mathematics reform: Speaking, inscribing, and doing mathematics within communities of practice. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 333-352). Reston, VA: National Council of Teachers of Mathematics.

Forman, E. A., & Ansell, E. (2001). The multiple voices of a mathematics community. *Educational Studies in Mathematics, 46*, 115-142.

Fuson, K. C., Wearne, D., Hiebert, J. C., Murray, H. G., Human, P. G., Olivier, A. I., et al. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education, 28*, 130-162.

Guba, E. G., & Lincoln, Y. S. (1981). *Effective evaluation*. San Francisco, CA: Jossey-Bass.

Gravemeijer, K., & van Galen, F. (2003). Facts and algorithms as products of students' own mathematical activity. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 114-122). Reston, VA: National Council of Teachers of Mathematics.

Hiebert, J. (Ed.). (1986). Conceptual and procedural knowledge: The case of mathematics. Hillsdale, MI: Erlbaum.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Crouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: Macmillan.

Hiebert, J., & Wearne, D. (1996). Instruction, understanding, and skill in multidigit addition and subtraction. *Cognition and Instruction, 14*, 251-283.

Huinker, D. (1998). Letting fraction algorithms emerge through problem solving. In L. J. Morrow & M. J. Kenney (Eds.), *The teaching and learning of algorithms in school mathematics – 1998 Yearbook* (pp. 170-182). Reston, VA: National Council of Teachers of Mathematics.

Kamii, C. K. (1985). Young children reinvent arithmetic: Implications of Piaget's theory. New York: Teachers College Press.

Kamii, C. K. (1989). Young children continue to reinvent arithmetic: Implications of Piaget's theory. New York: Teachers College Press

Kamii, C. K. (1994). Young children continue to reinvent arithmetic - 3rd grade: Implications of Piaget's theory. New York: Teachers College Press.

Kamii, C.K. (2008, January). *Encouraging children to do their own thinking*. Paper presented at the Math SUMmit of the Ottawa Carleton Catholic School Board, Ottawa, Ontario, Canada.

Kamii, C.K. , & Lewis, B. A., (1993). The harmful effects of algorithms in primary arithmetic. *Teaching Pre K-8, 23*, 36-38.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University Press.

Manouchehri, A., & Goodman, T. (2000). Implementing mathematics reform: The challenge within. *Educational Studies in Mathematics, 42*, 1-34.

Merriam, S. B., & Simpson, E. L. (1995). *A guide to research for educators and trainers of* adults (2nd ed). Malabar, FL: Krieger Publishing.

Morrow, L. J. (1998). Whither algorithms? Mathematics educators express their views. In L. J. Morrow & M. J. Kenney (Eds.), *The teaching and learning of algorithms in school mathematics – 1998 Yearbook* (pp. 1-6). Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (NCTM). (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics. (NCTM). (1991). *Professional standards for teaching mathematics.* Reston, VA: NCTM.

National Council of Teachers of Mathematics. (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

National Research Council. (1989). *Everybody counts.* Washington, DC: National Academy Press.

Sawyer, W.W. (1991). *Mathematician's delight*. New York: Penguin Press.

Scherer, P., & Steinbring, H. (2006). Noticing children's learning processes – Teachers jointly reflect on their own classroom interaction for improving mathematics teaching. *Journal of Mathematics Teacher Education, 9*, 157-185.

Schifter, D. (2001). Learning to see the invisible: What skills and knowledge are needed to engage with students' mathematical ideas?. In T. Wood, B. S. Nelson, & J. Warfield. (Eds.), *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 109-134). Mahwah, NJ: Lawrence Erlbaum.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher, 27*(2), 4-13.

Silver, E. A., Ghousseini, H., Gosen, D., Charalambous, C., & Font Strawhun, B. T. (2005). Moving from rhetoric to praxis: Issues faced by teachers in having students consider multiple solutions for problems in the mathematics classroom. *Journal of Mathematical Behavior, 24,* 287-301.

Silver, E. A., Shapiro, L. J., & Deutsch, A. (1993). Sense making and the solution of division problems involving remainders: An examination of middle school students' solution processes and their interpretations of solutions. *Journal for Research in Mathematics Education*, *24*(2), 117-135.

Simon, M., & Schifter, D. (1991). Towards a constructivist perspective: An intervention study of mathematics teacher development. *Educational Studies in Mathematics*, *22*, 309–331.

Simonsen, L. M., & Teppo, A. R. (1999). Using alternative algorithms with preservice teachers. *Teaching Children Mathematics*, *5*(9), 516-519.

Steinberg, R. M., Empson, S. B., Carpenter, T.P. (2004). Inquiry into children's mathematical thinking as a means to teacher change. *Journal of Mathematics Teacher Education, 7*, 237-267.

VanLehn, K. (1986). Arithmetic procedures are induced from examples. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 133-180). Hillsdale, NJ: Lawrence Erlbaum.

Wallach, T., & Even, R. (2005). Hearing students: The complexity of understanding what they are saying, showing, and doing. *Journal of Mathematics Teacher Education, 8*, 393-417.

Willis, G. B., & Fuson, K. C. (1988). Teaching children to use schematic drawings to solve addition and subtraction word problems. *Journal of Educational Psychology, 80*(2), 192-201.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27,* 458-477.