STRAND I: Geometry and Trigonometry

Unit 37 *Further Transformations*

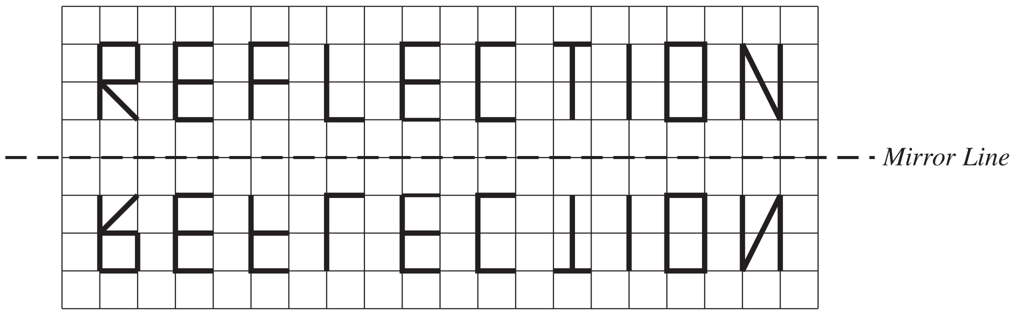
Student Text

Contents

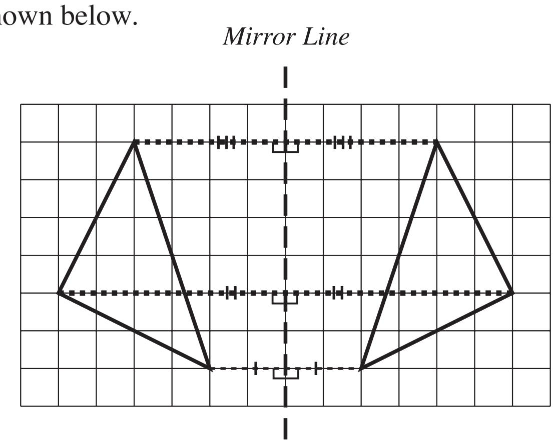
37 Further Transformations

37.1 Reflections

Reflections are obtained when you draw the image that would be obtained in a mirror.



Every point on a reflected image is always the same distance from the *mirror line* as the original. This is shown below.

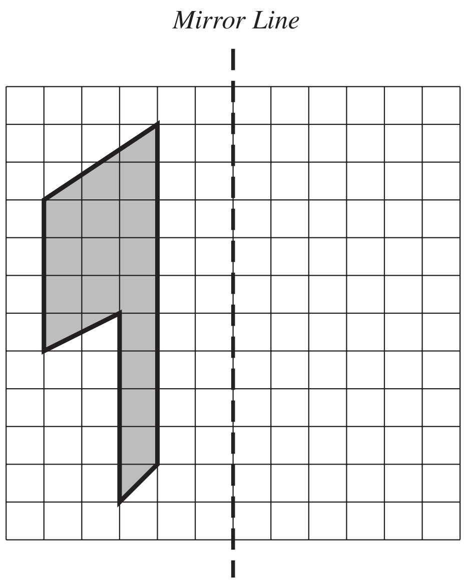


Note

Distances are always measured at right angles to the *mirror line*.

Worked Example 1

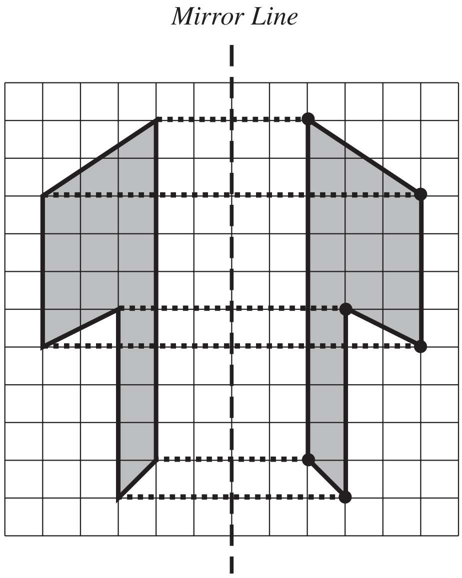
Draw the reflection of the shape in the mirror line shown.

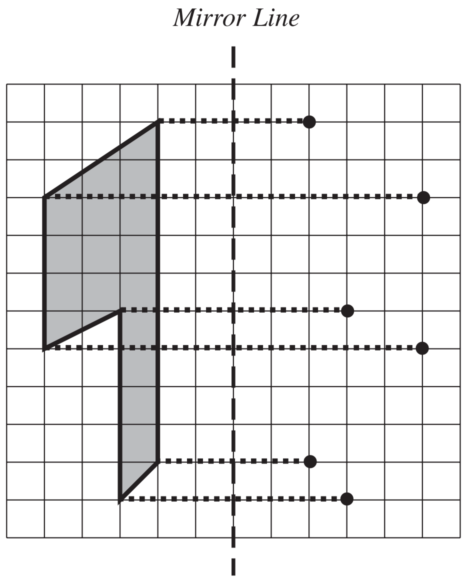


Solution

The lines added to the diagram show how to find the position of each point after it has been reflected.

Remember that the image of each point is the same distance from the mirror line as the original.



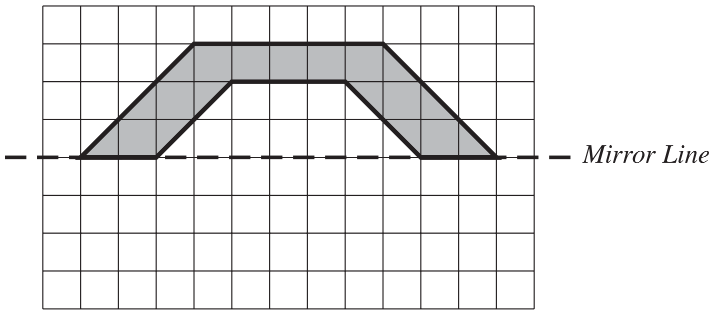


The points can then be joined to give the reflected image.

If the construction lines have been drawn in pencil they can be rubbed out.

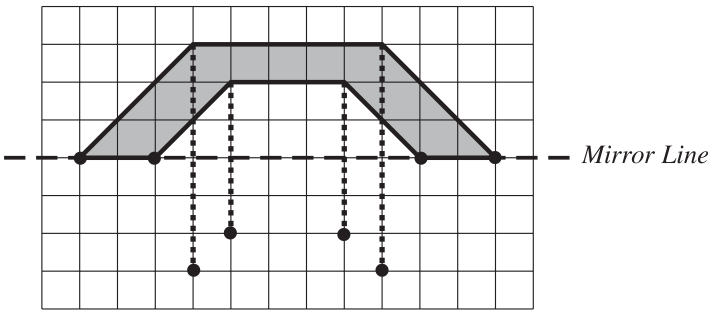
Worked Example 2

Reflect this shape in the mirror line shown in the diagram.

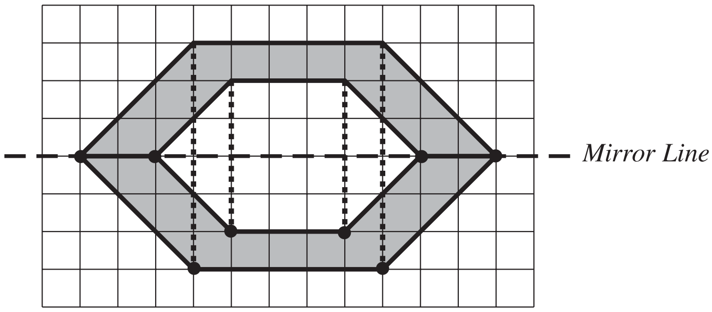


Solution

The lines are drawn at right angles to the mirror line. The points which form the image must be the same distance from the mirror lines as the original points. The points which were on the mirror line remain there.



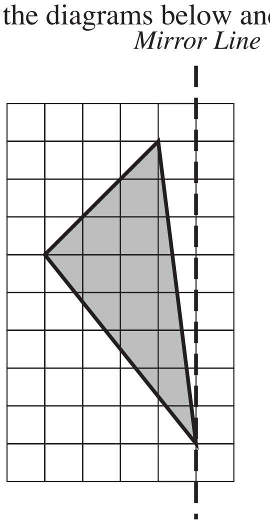
The points can then be joined to give the reflected image.



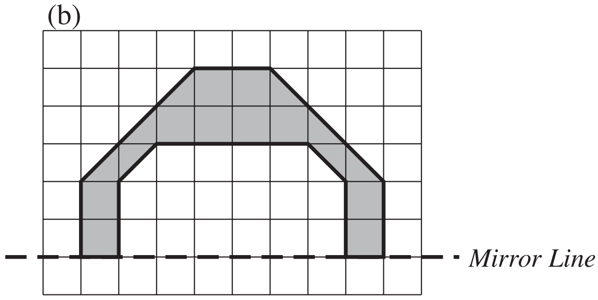
Exercises

1. Copy the diagrams below and draw the reflection of each object.

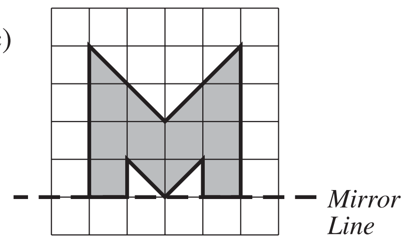
(a)



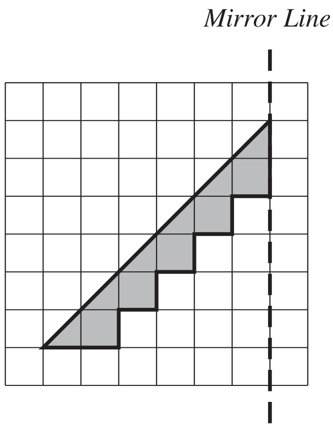
(b)



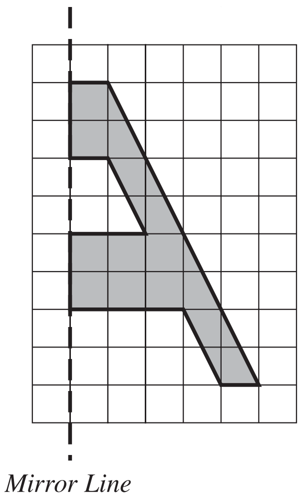
(c)



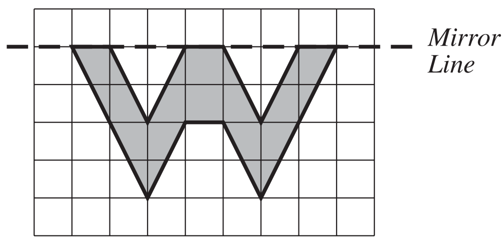
(d)



(e)

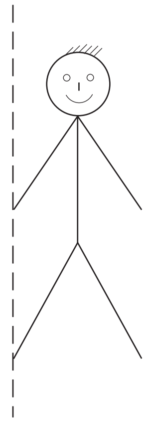


(f)

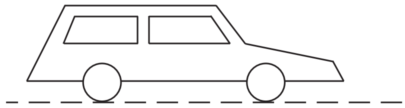


2. Copy each diagram and draw the reflection of each shape in the mirror line shown.

(a)



(b)



(c)

(d)

(e)

(f)

3. (a) Draw a set of axes with x and y values from -5 to 5.

(b) Plot the points with coordinates

$$(1, 1),\ (1, 5),\ (4, 5),\ (4, 3),\ (2, 3),\ (2, 1).$$

Join the points in that order to form a shape.

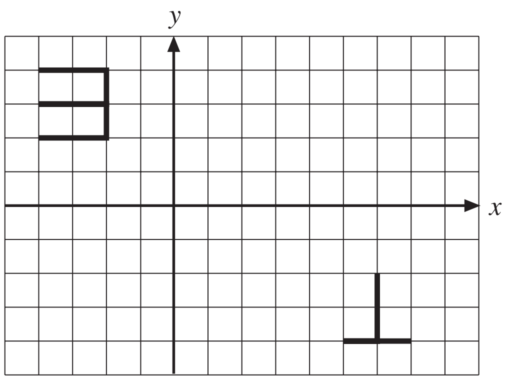
(c) Reflect the image in the y-axis. Write down the coordinates of the corners of this shape.

(d) Reflect the image obtained in (c) in the x-axis. List the coordinates of the corners.

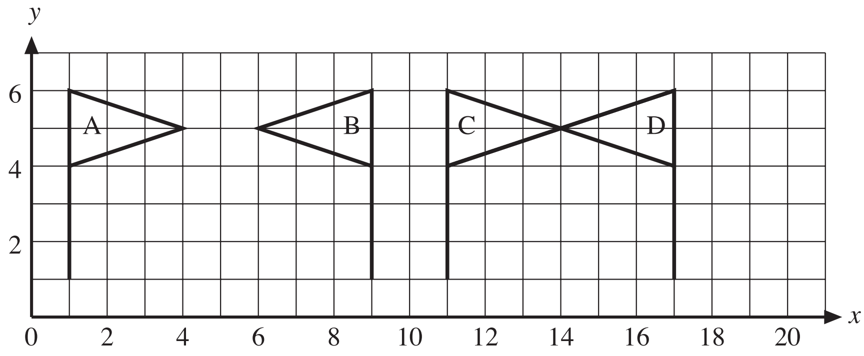
(e) Reflect the image obtained in (d) in the y-axis. Describe how this shape could have been obtained directly from the original shape.

4. A student reflected his two initials, the first in the y-axis and the second in the x-axis, to obtain the image opposite.

Copy the diagram and show the original position of the initials.



5. Copy the diagram below.



Draw in the mirror line for each reflection described.

(a) A → B (b) B → C (c) C → D (d) A → D

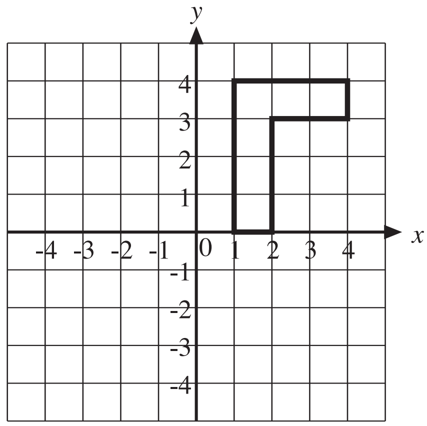
6. (a) Copy the axes and shape shown.

(b) (i) Draw the reflection of the shape in the *y*-axis.

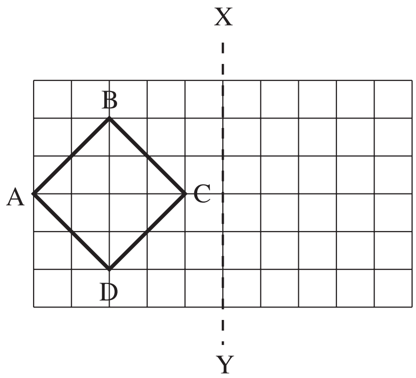
(ii) Compare the coordinates of each shape.

(iii) Describe what happens to the coordinates of a point when it is reflected in the *y*-axis.

(c) Repeat (b) using the *x*-axis.



7.

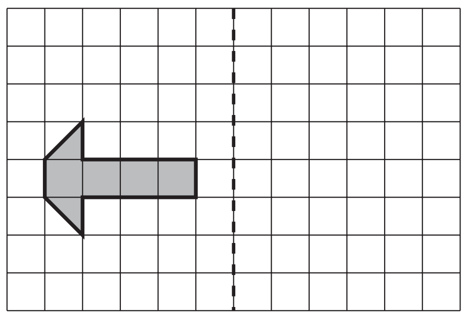


(a) Copy the diagram and draw the reflection of ABCD in the mirror line XY.

(b) ABCD has rotational symmetry. Mark with a cross its centre of rotation.

8. (a) Find the area of the shaded shape.

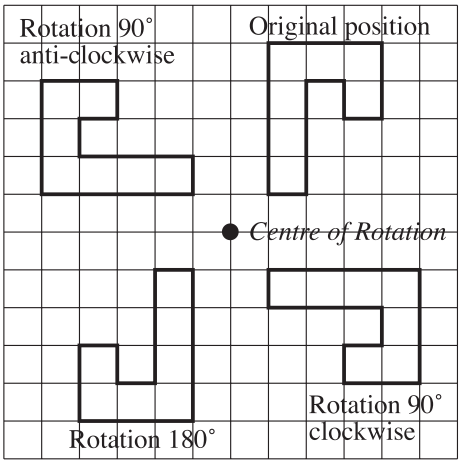
(b) Copy the diagram and draw the reflection of the shaded shape in the mirror line.



Mirror Line

37.2 Rotations

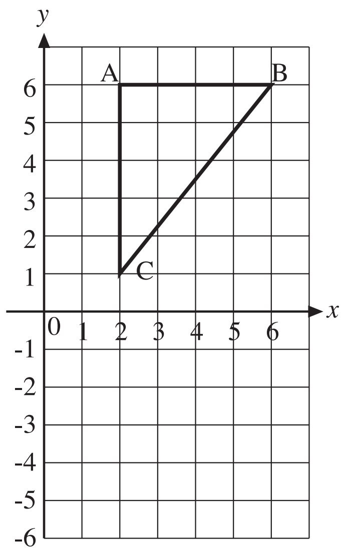
Rotations are obtained when a shape is rotated about a fixed point, called the *centre of rotation*, through a specified angle. The diagram shows a number of rotations.



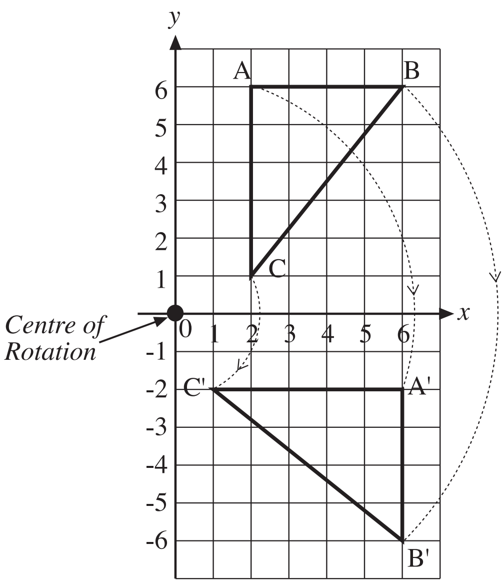
It is often helpful to use tracing paper to find the position of a shape after a rotation.

Worked Example 1

Rotate the triangle ABC shown in the diagram through $90°$ clockwise about the point with coordinates $(0, 0)$.



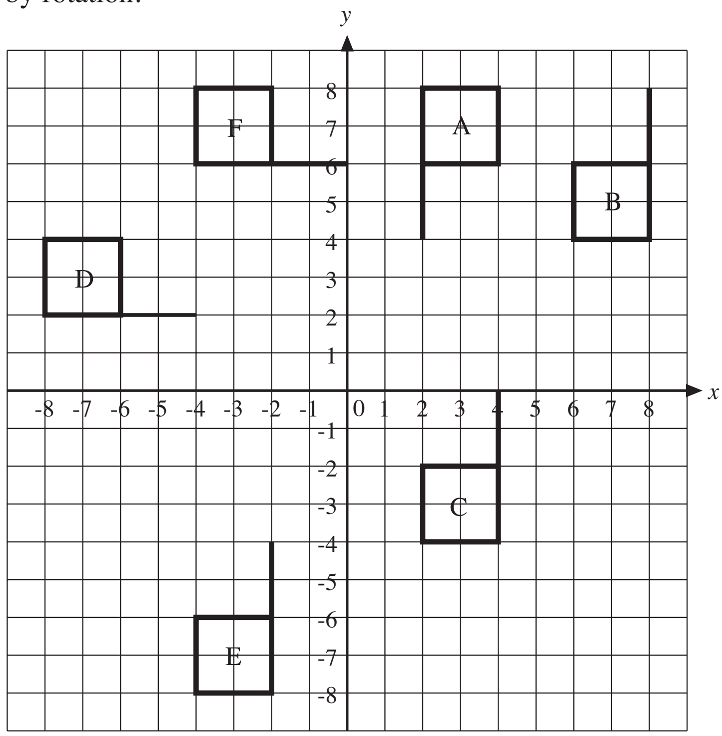
Solution



The diagram opposite shows how each vertex can be rotated through $90°$ to give the position of the new triangle.

Worked Example 2

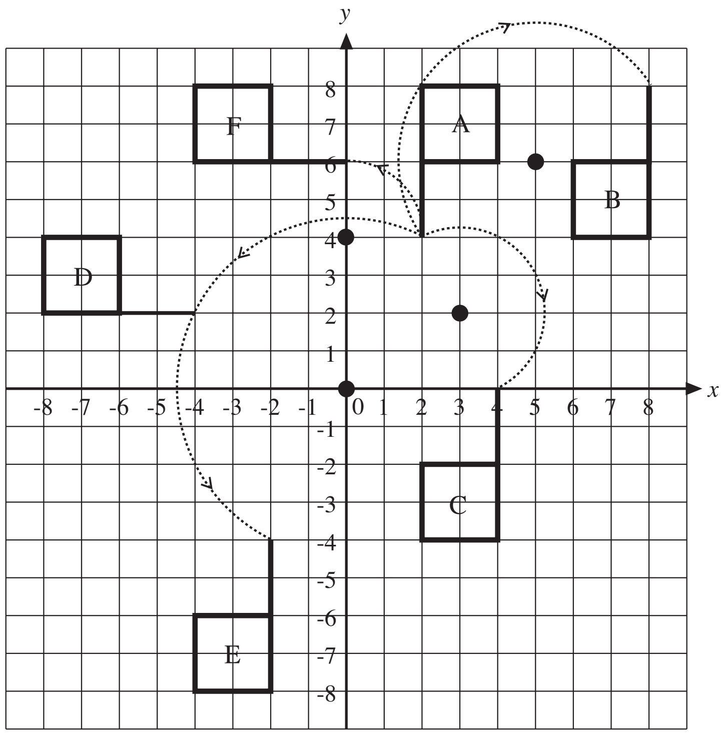
The diagram shows the position of a shape A and the shapes, B, C, D, E and F which are obtained from A by rotation.



Describe the rotation which moves A onto each other shape.

Solution

The diagram shows the centres of rotation and how one vertex of the shape A was rotated.



Each rotation is now described.

- A to B: Rotation of $180°$ about the point $(5, 6)$.
- A to C: Rotation of $180°$ about the point $(3, 2)$.
- A to D: Rotation of $90°$ anti-clockwise about the point $(0, 0)$.
- A to E: Rotation of $180°$ about the point $(0, 0)$.
- A to F: Rotation of $90°$ anti-clockwise about the point $(0, 4)$.

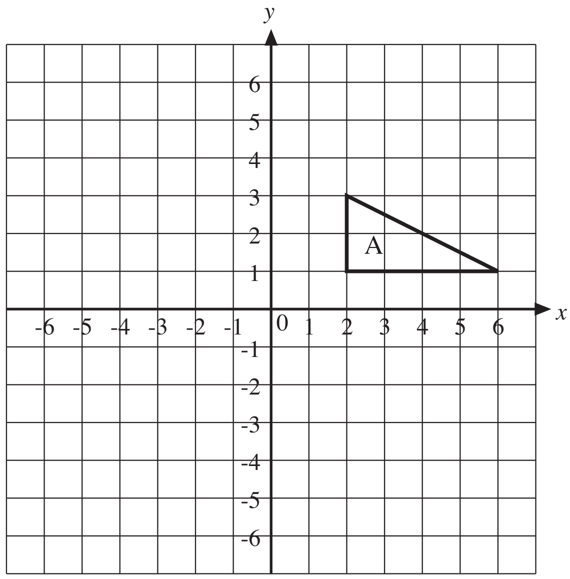
Exercises

1. Copy the axes and triangle shown opposite.

   (a) Rotate A through $90°$ clockwise around $(0, 0)$ to obtain B.

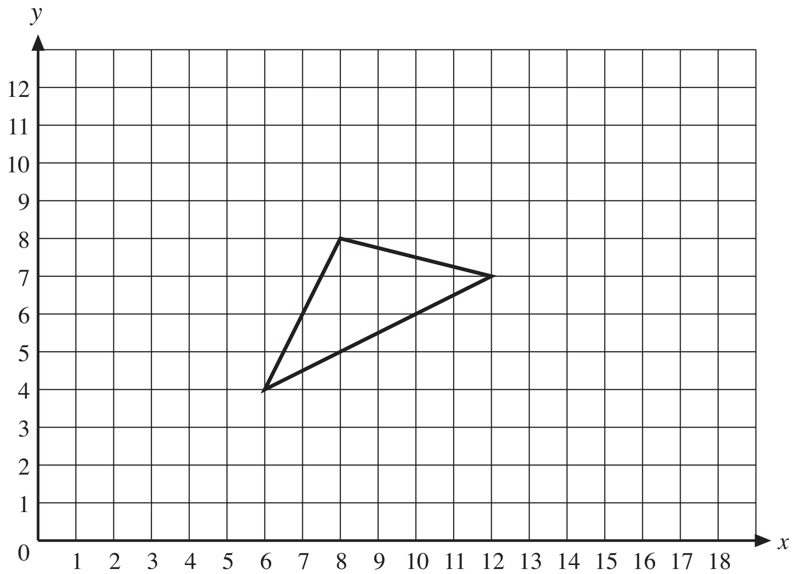
   (b) Rotate A through $90°$ anticlockwise around $(0, 0)$ to obtain C.

   (c) Rotate A through $180°$ around $(0, 0)$ to obtain D.



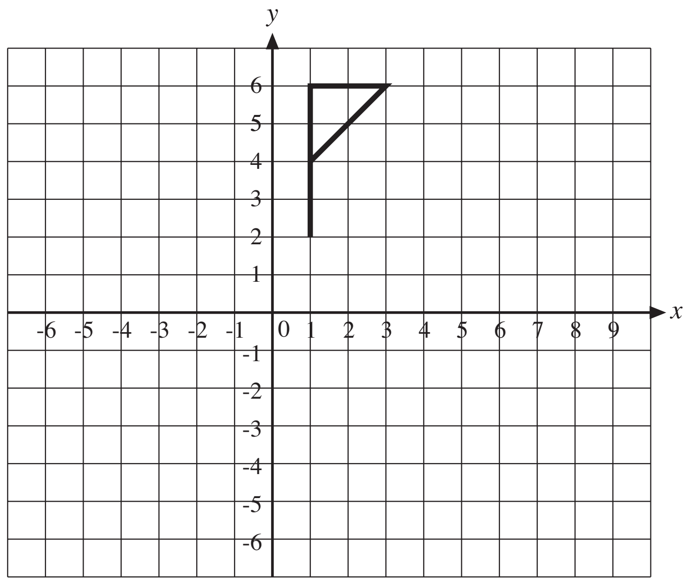
2. Repeat *Question 1* for the triangle with coordinates $(3, 1)$, $(6, 2)$ and $(0, 4)$.

3. Copy the axes and triangle shown below.



Rotate the triangle through $180°$ using each of its vertices as the centre of rotation.

4. Copy the axes and shape shown below.



(a) Rotate the original shape through $90°$ clockwise around the point $(1, 2)$.

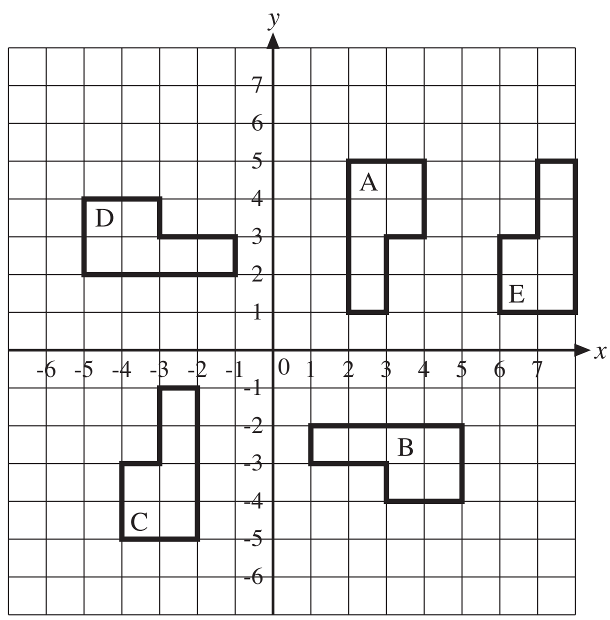
(b) Rotate the original shape through $180°$ around the point $(3, 4)$.

(c) Rotate the original shape through $90°$ clockwise around the point $(1, -2)$.

(d) Rotate the original shape through $90°$ anti-clockwise around the point $(0, 1)$.

5. Repeat *Question 4* for the triangle with vertices at $(2, 2)$, $(1, 3)$ and $(3, 5)$.

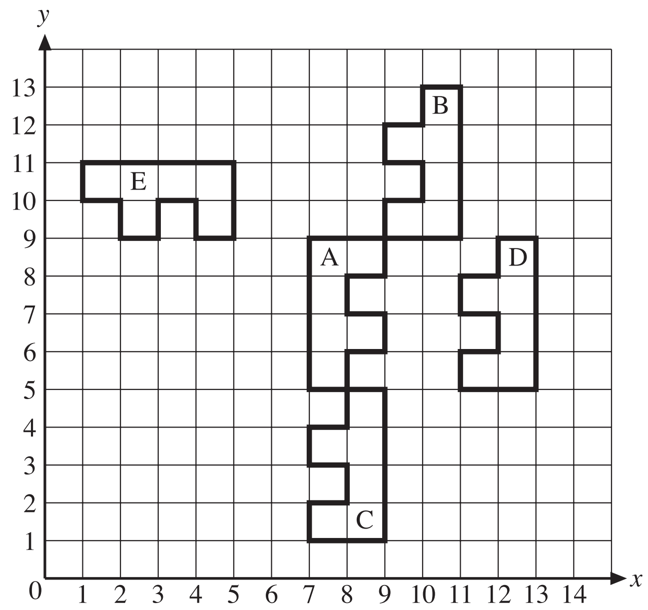
6. The diagram shows the position of a shape labelled A and other shapes which were obtained by rotating A.



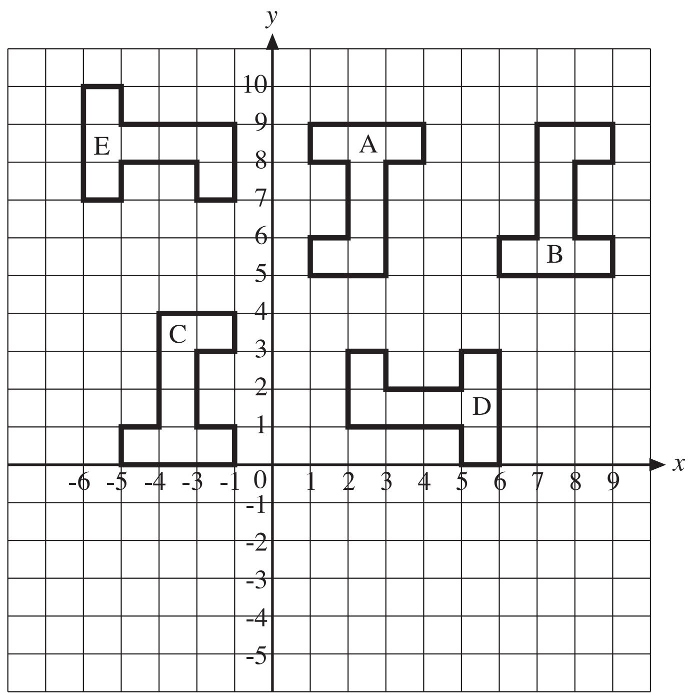
(a) Describe how each shape can be obtained from A by a rotation.

(b) Which shapes can be obtained by rotating the shape E?

7. The shape A has been rotated to give each of the other shapes shown. For each shape, find the centre of rotation.



8. (a) Describe how each shape shown below can be obtained from A by a rotation.



(b) Which shapes cannot be obtained from C by a rotation?

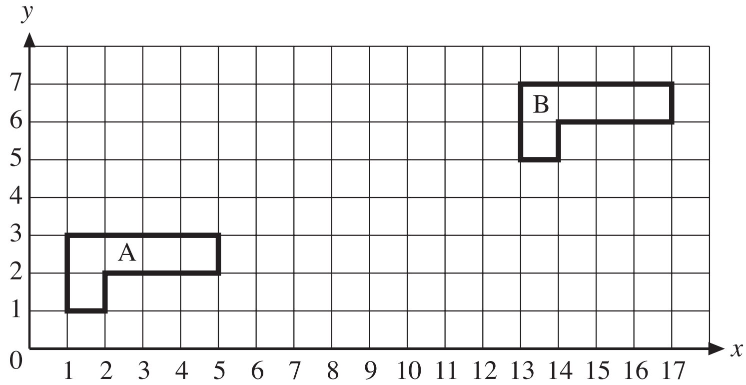
9. On a set of axes with x and y values from -2 to 12, draw the triangle with vertices at the points $(0, 0)$, $(4, 5)$ and $(1, 4)$.

   (a) Rotate the triangle through $90°$ clockwise about the point $(5, 6)$.

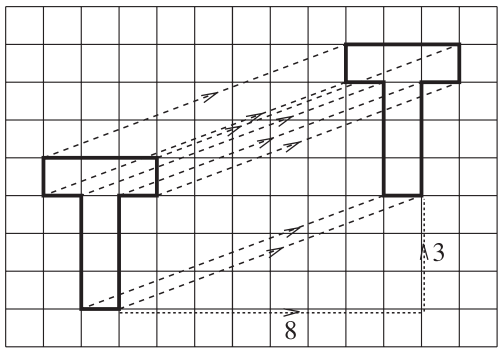
   (b) Rotate the second triangle through $90°$ clockwise about the point $(4, 3)$.

   (c) Describe how to obtain the third triangle from the original triangle by a single rotation.

10. The shape B can be obtained from A by two rotations. Describe these rotations.



37.3 Translations



A *translation* moves all the points of an object in the same direction and the same distance. The diagram shows a translation.

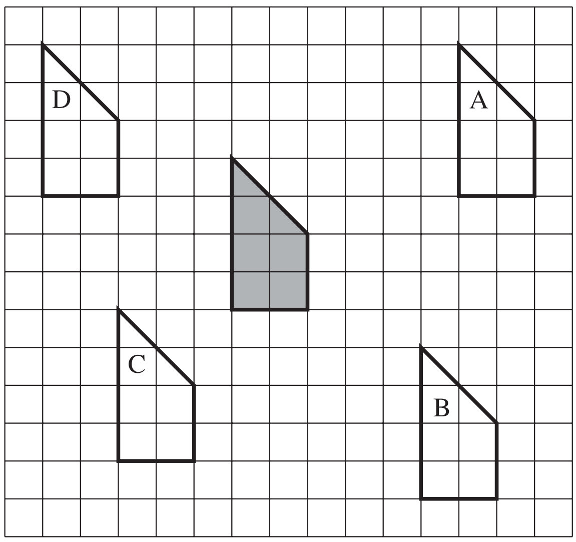
Here every point has been moved 8 units to the right and 3 units up.

This translation is described by what is called a vector $\begin{pmatrix} 8 \\ 3 \end{pmatrix}$

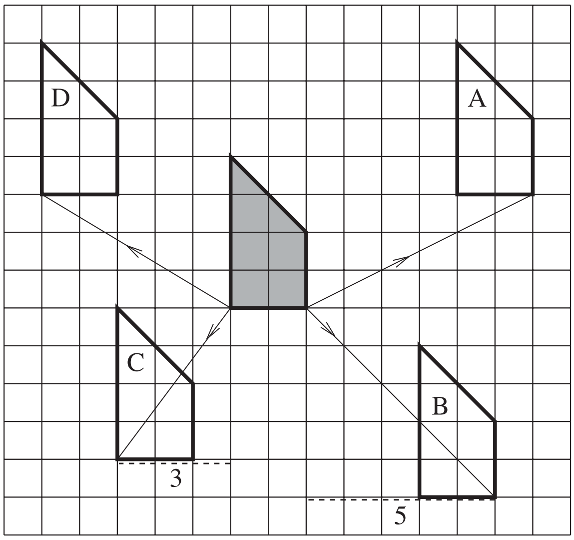
Further work on vectors is in Strand J, Unit 38.

Worked Example 1

Describe the translation which moves the shaded shape to each of the other shapes shown.



Solution



To move to A, the shaded shape is moved 6 units to the right (horizontally) and 3 units up (vertically).

This is described by the vector $\begin{pmatrix} 6 \\ 3 \end{pmatrix}$.

To obtain B, the shaded shape is moved 5 units to the right and 5 units down.

This is described by the vector $\begin{pmatrix} 5 \\ -5 \end{pmatrix}$.

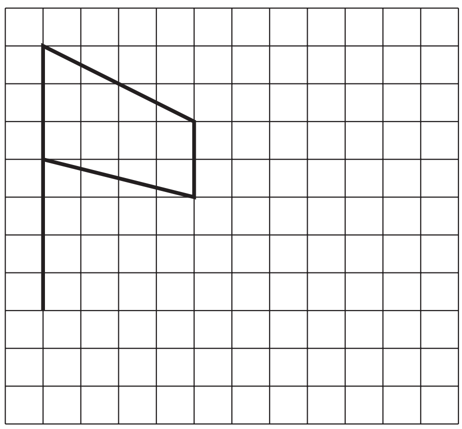
To obtain C, the shaded shape is moved 3 units to the left and 4 units down.

This is described by the vector $\begin{pmatrix} -3 \\ -4 \end{pmatrix}$.

To obtain D, the shaded shape is moved 5 units to the left and then 3 units up.

This is described by the vector $\begin{pmatrix} -5 \\ 3 \end{pmatrix}$.

Worked Example 2

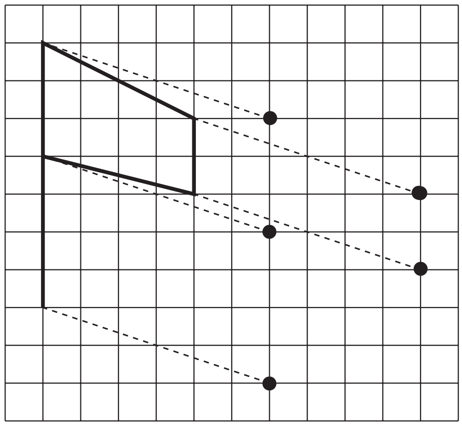


The shape shown in the diagram is to be translated using the vector $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$.

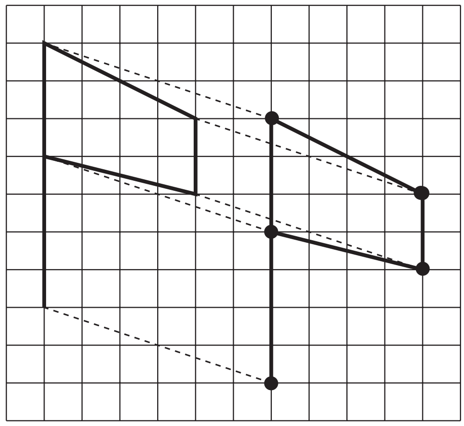
Draw the image obtained using this translation.

Solution

The vector $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$ describes a translation which moves an object 6 units to the right and 2 units down. This translation can be applied to each point of the original.

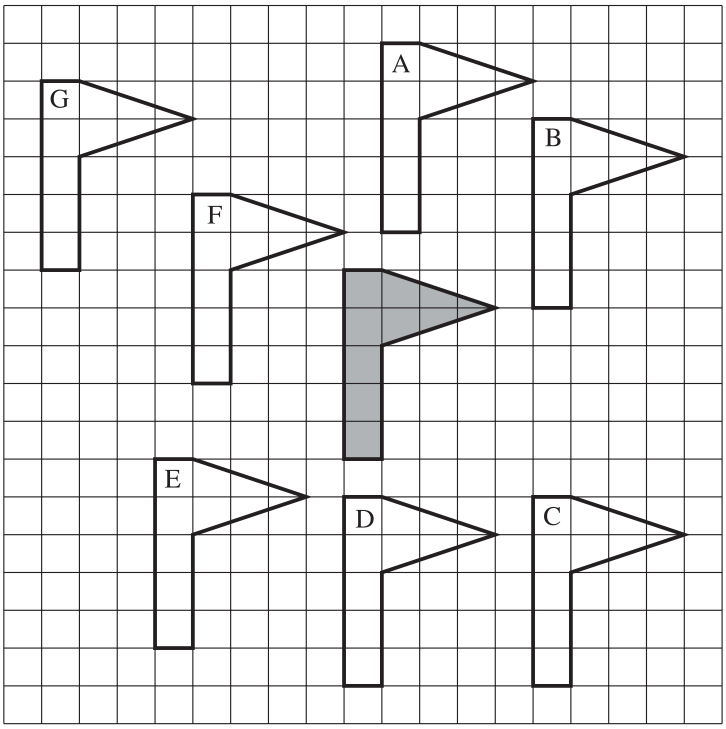


The points can then be joined to give the translated image.

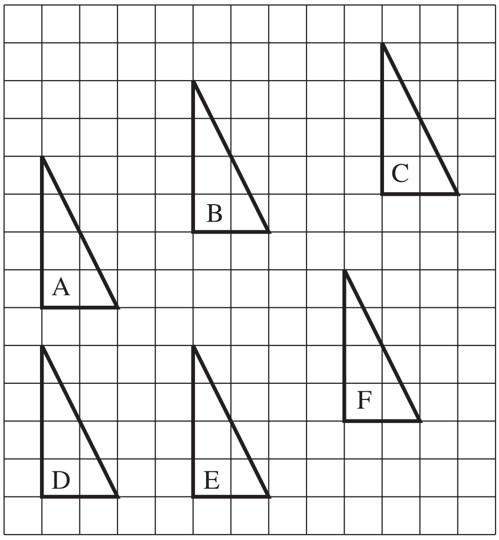


Exercises

1. The shaded shape has been moved to each of the other positions shown by a translation. Give the vector used for each translation.



2. Describe the translation which moves:



(a) A $\rightarrow$ C

(b) C $\rightarrow$ B

(c) F $\rightarrow$ E

(d) B $\rightarrow$ D

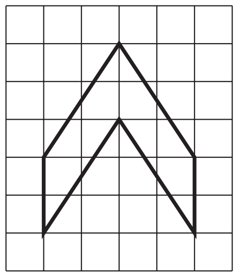
(e) D $\rightarrow$ B

(f) E $\rightarrow$ C

(g) C $\rightarrow$ D

(h) A $\rightarrow$ F

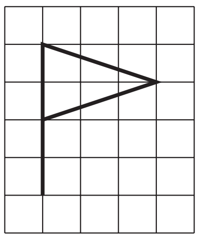
3. Draw the shape shown and its image when translated using each of the following vectors.



(a) $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ (b) $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

(c) $\begin{pmatrix} -2 \\ -5 \end{pmatrix}$ (d) $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

4.



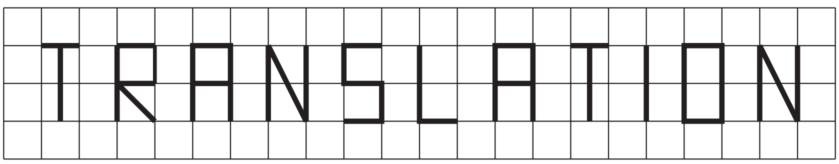
(a) Draw the shape shown.

(b) Translate using the vector $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

(c) Translate the image using the vector $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

(d) Which vector would be needed to translate the final image back to the position of the original?

5.

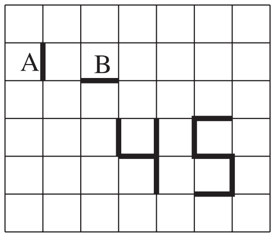


(a) Describe a translation which would move one A onto another A.

(b) Describe any other translations which would move a letter onto the same letter in a different position.

6. The number 45 can be formed by translation of the lines A and B.

   Describe the translations which need to be applied to A and B to form the number 45.



7. (a) Draw a simple shape.

   (b) Write down the coordinates of each corner of your shape.

   (c) Translate the shape using the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and write down the coordinates of the new shape.

   (d) Compare the coordinates obtained in (b) and (c). How do they change as a result of the translation?

   (e) Repeat (c) and (d) with a translation using the vector $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

8. (a) Draw a simple shape and translate it using the vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

   Then translate the image using the vector $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

   (b) Which single translation would map the original shape to its final position?

   (c) Translate your shape using the vector $\begin{pmatrix} 3 \\ 7 \end{pmatrix}$. Then translate the image using the vector $\begin{pmatrix} -3 \\ -4 \end{pmatrix}$. Which single translation would move the original shape to its final position?

   (d) If a shape was translated using the vector $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and then the vector $\begin{pmatrix} -6 \\ -8 \end{pmatrix}$, which single translation would be equivalent?

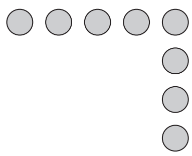
9. The points A, B, C and D have coordinates $(4, 7)$, $(2, -6)$, $(-3, -6)$ and $(0, 7)$. Find the vector which would be used to translate:

   (a) A to B (b) C to D (c) D to A (d) A to D.

Challenge!

*1. By moving only **one** coin in the pattern shown, make one row and one column, each containing 5 coins.*

2. Rearrange the 8 coins to form a square with 3 coins on each side. By rearranging 4 coins, make a square with 4 coins on each side.

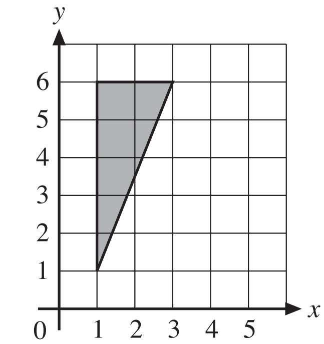


37.4 Combined Transformations

An object can be subjected to more than one transformation, so when describing how a shape is moved from one position to another it may be necessary to use two different transformations.

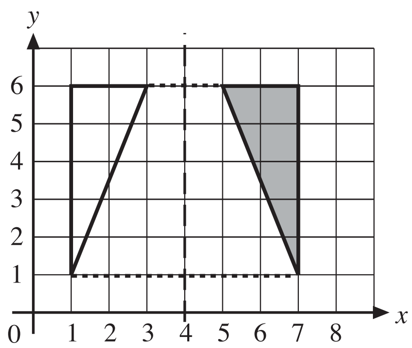
Worked Example 1

Draw the image of the triangle shown if it is first reflected in the line $x = 4$ and then rotated clockwise about the point $(4, 0)$.

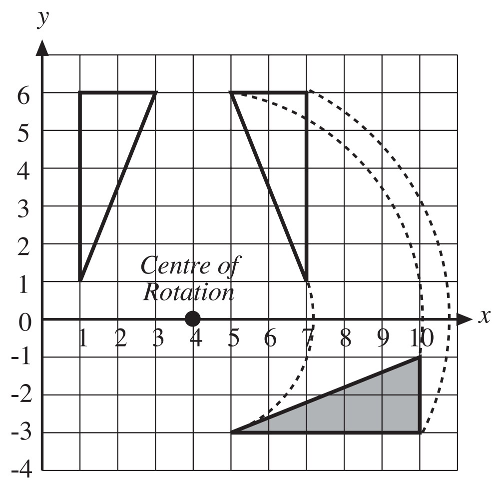


Solution

The diagram below shows the line $x = 4$ and the image of the triangle when it has been reflected in this line.

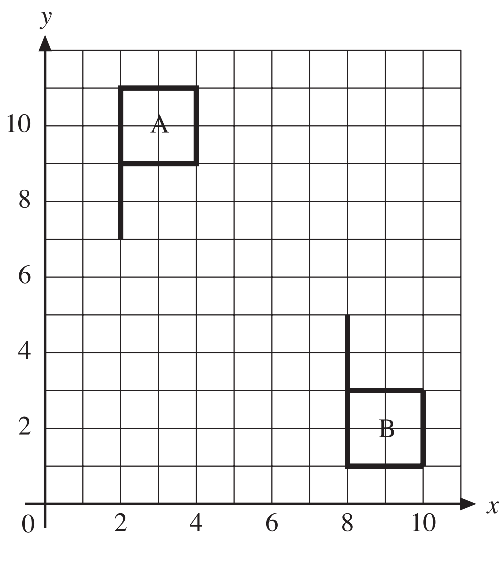


The new image can then be rotated about the point $(4, 0)$, as below.



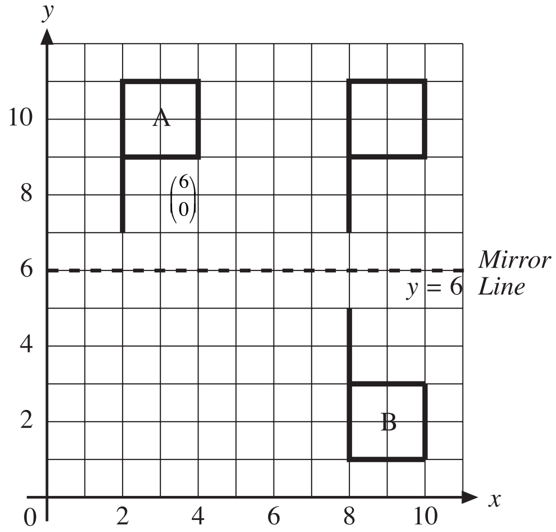
Worked Example 2

Describe two different ways in which the shape marked A can be moved to the position shown at B.



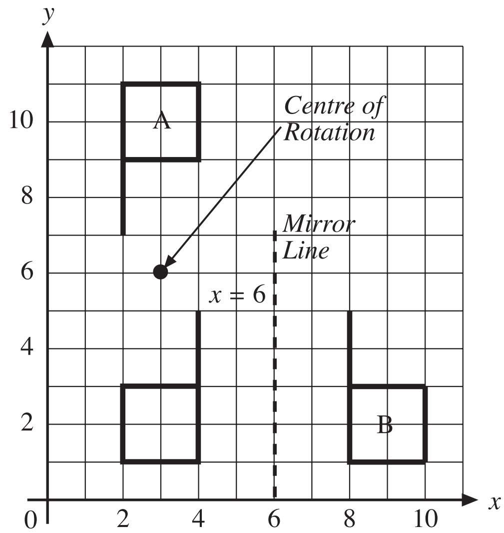
Solution

One way, shown below, is to first translate A using the vector $\begin{pmatrix}6\\0\end{pmatrix}$, and then reflect in the line $y = 6$.

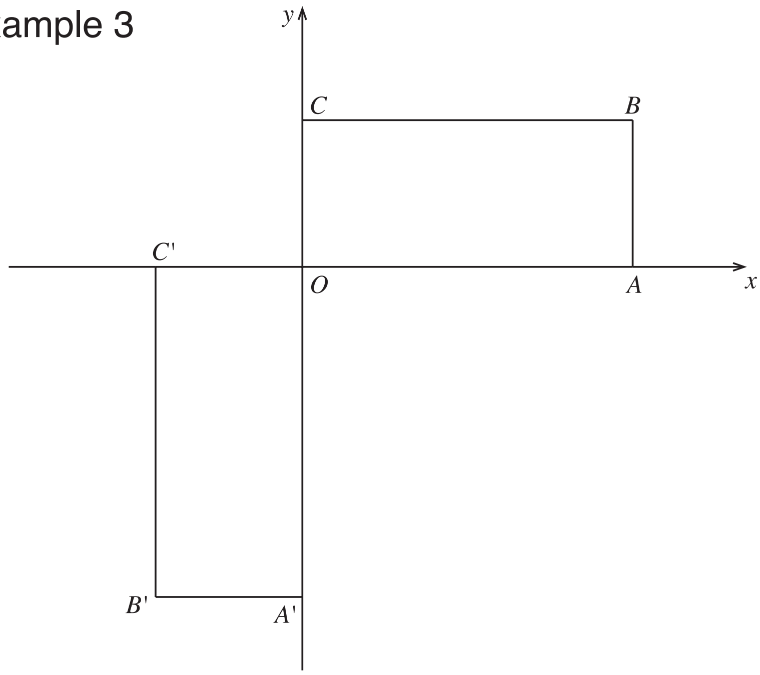


An alternative approach is to rotate shape A through $180°$ around point (2, 6).

This can then be reflected in the line $x = 6$ to obtain B, as shown below.



Worked Example 3



In the diagram above, $OC = OC'$, $BC = B'C'$ and all angles are right angles. $OABC$ can be mapped onto $OA'B'C'$ by a transformation, J, followed by another transformation, K.

Describe fully the transformations

(a) J

(b) K. *(CXC)*

Solution

(a) Rotate *OABC* by $90°$ clockwise, centre O.

(b) Reflect new shape in the y-axis.

Worked Example 4

On graph paper, taking 1 cm to represent 1 unit on both the x and y axes, draw

(a) the triangle *ABC* formed by joining the points A (0, 0), B (2, 3) and C (4, 2).

(b) the triangle $A'\,B'\,C'$, the image of triangle *ABC*, under a reflection in the y-axis.

A transformation Q maps the image of triangle *ABC* onto triangle $A''\,B''\,C''$ such that

$$A(0, 0) \rightarrow A''\,(3, 4)$$
$$B(2, 3) \rightarrow B''\,(5, 7)$$
$$C(4, 2) \rightarrow C''\,(7, 6)$$

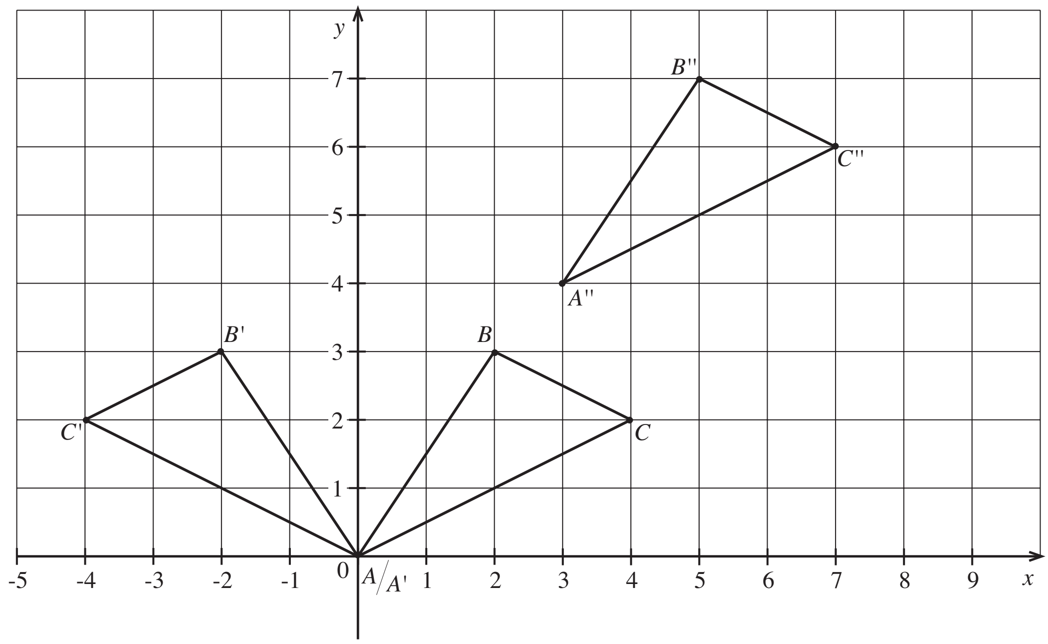
(c) Draw the triangle $A''\,B''\,C''$

(d) Describe the transformation Q in TWO different ways.

(CXC)

Solution

(a), (b), (c) as shown on the diagram below.



(d) For example, translate *ABC* by the vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ followed by $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$.

Translate *ABC* by the vector $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ followed by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Exercises

1. (a) Draw a set of axes with x and y values from 0 to 9. Plot the points $(5, 1)$, $(7, 4)$, $(9, 4)$ and $(7, 1)$. Join them to form a single shape.

   (b) Reflect the shape in the line $y = 4$.

   (c) Translate the shape obtained in (b) using the vector $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$.

   (d) Rotate the original shape through $180°$ about the point with coordinates $(5, 4)$

2. (a) Draw a set of axes with x-values from 0 to 14 and y-values from 0 to 4. Join the points $(1, 4)$ and $(4, 1)$ to form a straight line. Rotate this line through $90°$ clockwise around the point $(4, 1)$.

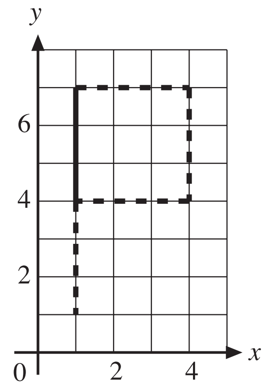
   (b) Describe two ways in which the shape you have obtained could be transformed into a 'W' shape.

3. The letter P is to be formed by applying a number of transformations to the solid line. Each transformation maps the solid line onto one of the dashed lines.

   Describe how this could be done using:

   (a) only rotations

   (b) only reflections.



4. (a) Draw a set of axes with x values from 0 to 16 and y values from -12 to 8.

   (b) Join the points with coordinates $(4, 1)$, $(6, 1)$ and $(6, 4)$ to form a triangle.

   (c) Enlarge this triangle with scale factor 2 using the point $(0, 1)$ as the centre of enlargement.

   (d) Rotate the new triangle through $180°$ about the point $(9, -2)$.

   (e) Describe fully the transformations which map the final triangle back onto the original.

5. (a) Draw a set of axes with x and y values from 0 to 8. Plot and join the points $(1, 5)$, $(3, 5)$, $(3, 8)$ and $(1, 7)$.

   (b) Rotate this shape through $90°$ clockwise around the point $(3, 5)$.

   (c) Then translate the new image using the vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

   (d) Reflect this image in the line $y = 4$.

   (e) How could the final image be mapped back to the position of the original with a single transformation?

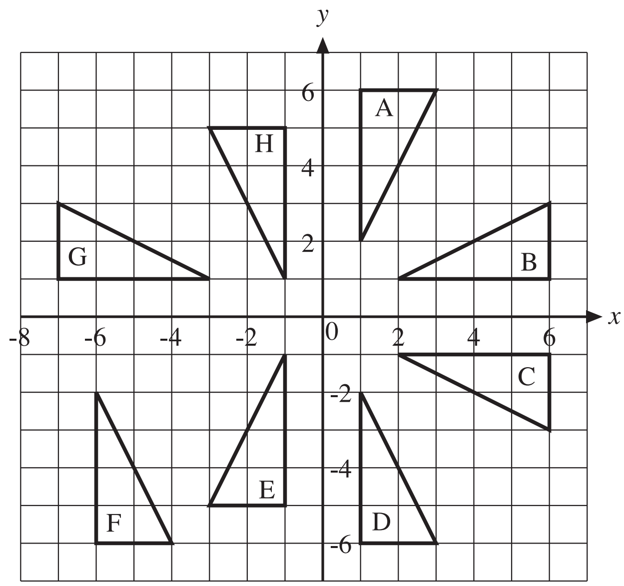
6. (a) Draw a set of axes with x and y values from 0 to 10. Plot the points with coordinates (2, 1), (4, 1) and (2, 4) and join them to form a triangle.

   (b) Reflect this triangle in the line $x = 4$ and then reflect the image in the line $x = 7$.

   (c) Which single transformation would map the original triangle onto its final position?

   (d) Reflect the original triangle in the line $y = 6 - x$ and then reflect this image in the line $y = 11 - x$. Which single transformation would map the final triangle back to its original position?

7.



   (a) The triangle A can be mapped onto B, C and D using single transformations. Describe fully each transformation.

   (b) The triangle A can be mapped onto E, F, G and H using two transformations. Describe fully each pair of transformations.

8. (a) (i) Draw a simple shape and reflect it in any vertical line.

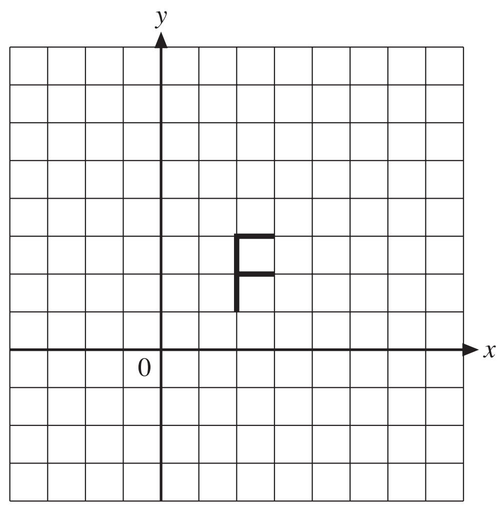
   (ii) Reflect the image in any horizontal line.

   (b) Describe two other ways in which the original image could have been mapped onto the final image.

   (c) Repeat (a) and (b) using any two lines which are perpendicular.

   (d) Do you obtain the same result in each case?

9. Copy the diagram below and then show the answers to the questions on your copy of the diagram. (You are advised to use a pencil.)



(a) Draw the reflection of the F in the *x*-axis.

(b) Rotate the original F through $90°$ anticlockwise, with O as the centre of rotation. Draw the image.

(c) Enlarge the original F with centre of enlargement O and scale factor 2.

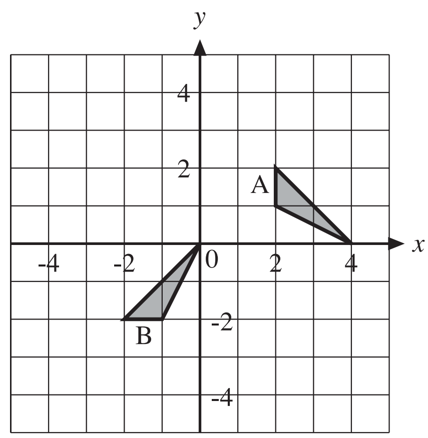
10. (a) Triangle A is mapped onto triangle B by means of an anticlockwise rotation, centre the origin, followed by another translation.

(i) Write down the angle of rotation.

(ii) Find the column vector of the translation.

(b) Triangle A may be mapped onto triangle B by means of a single rotation.

Find the coordinates of the centre of rotation.



(c) Triangle B is reflected in the line $y = -2$ to form triangle C. Describe the single transformation which would map triangle A onto triangle C.