8 FURTHER CALCULUS

Objectives

After studying this chapter you should

- be able to differentiate expressions defined implicitly;
- be able to use approximate methods for integration such as the trapezium rule and Simpson's rule;
- understand how to calculate volumes of revolution;
- be able to find arc lengths and areas of surfaces of revolution;
- be able to derive and use simple reduction formulae in integration.

8.0 Introduction

You should have already covered the material in *Pure Mathematics,* Chapters 8, 11, 12, 14, 17 and 18 on calculus. This chapter will enable you to see applications of the ideas of integration you have already met and help you to find derivatives and values of integrals when previously you had no method available.

8.1 Implicit functions

When a curve is defined by a relation of the form $y = f(x)$ we say that y is an **explicit function** of x and we can usually apply one of the standard procedures to find $\frac{dy}{dx}$.

However, the expression $y^3 - 8xy + x^2 = 4$ is an **implicit function** of x. It also defines a curve, but because we cannot easily make y the subject we need to adopt a different strategy if we wish to find the gradient at a particular point. Let us consider a more simple curve first of all.

Activity 1

Sketch the circle with equation $x^2 + y^2 = 5$. The point P(1, 2) lies on the circle and O is the origin. Write down the gradient of OP. Deduce the gradient of the tangent to the circle at P.

Suppose $z = y^2$, then by the Chain Rule

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$$

$$= 2y\,\frac{dy}{dx}$$

Activity 2

By differentiating each term with respect to *x*, show that when

$$x^2 + y^2 = 5, \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Hence deduce the value of the gradient at the tangent to the circle $x^2 + y^2 = 5$ at the point (1, 2).

Check your answer with that from Activity 1.

Example

Find the gradient of the curve with equation $y^3 - 8xy + x^2 = 4$ at the point (1, 3).

Solution

By the product rule,

$$\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$$

Also $$\frac{d}{dx}\left(y^3\right) = 3y^2\,\frac{dy}{dx}$$

So differentiating $y^3 - 8xy + x^2 = 4$ with respect to *x* gives

$$3y^2\,\frac{dy}{dx} - 8y - 8x\frac{dy}{dx} + 2x = 0$$

When $x = 1$ and $y = 3$,

$$27\frac{dy}{dx} - 24 - 8\frac{dy}{dx} + 2 = 0$$

$$\Rightarrow \quad 19\frac{dy}{dx} - 22 = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{22}{19}$$

Example

A curve has equation $x^3 + xy + y^3 + 29 = 0$ and P is the point (1, –3).

(a) Show that P lies on the curve.

(b) Show that the curve has a stationary point at P.

(c) Find the value of $\frac{d^2y}{dx^2}$ at P and hence determine whether the curve has a maximum or minimum point at P.

Solution

(a) Substituting $x = 1,\ y = -3$ into the right-hand side of the equation gives

$$1 - 3 - 27 + 29 = 0.$$

Hence P lies on the curve.

(b) Differentiating implicitly with respect to x gives

$$3x^2 + y + x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

Substituting $x = 1,\ y = -3$ gives

$$3 - 3 + \frac{dy}{dx} + 27\frac{dy}{dx} = 0$$

So $\frac{dy}{dx} = 0$ at the point P.

(c) You need to differentiate the first equation in the solution to (b) implicitly with respect to x. This gives

$$6x + \frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right) + 3y^2\frac{d^2y}{dx^2} = 0$$

At P, $x = 1,\ y = -3$ and $\frac{dy}{dx} = 0$, so

$$6 + \frac{d^2y}{dx^2} + 27\frac{d^2y}{dx^2} = 0$$

$$\frac{d^2y}{dx^2} = -\frac{3}{14}$$

This negative value of $\frac{d^2y}{dx^2}$ at the stationary point P tells us we have a maximum point at P.

Exercise 8A

Find the gradient of the following curves at the points indicated in Questions 1 to 6.

1. $x^2 + y^2 = 3$ at $(2, -1)$
2. $2x^3 + 4xy - y^2 = 5$ at $(1, 3)$
3. $xy^2 + 3x^2y = 6$ at $(-3, 2)$
4. $\cos(x+y) + xy = 0$ at $\left(\frac{1}{2}\pi, 0\right)$
5. $e^{3x-y} + y^2 = 10$ at $(1, 3)$
6. $\sqrt{x} + \sqrt{y} = 3$ at $(4, 1)$
7. Given that $e^y = e^x + e^{-x}$, show that

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - 1 = 0 .$$ (AEB)

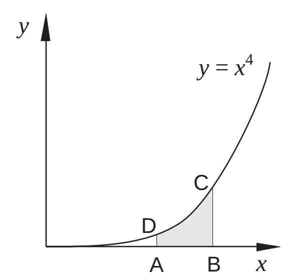
8. Find the value of $\frac{dy}{dx}$ and the value of $\frac{d^2y}{dx^2}$ at the point (2, −1) on the curve with equation

$$x^2 - 3x^2y + y^4 = 17 .$$

8.2 Approximate integration – trapezium rule

Activity 3

Evaluate $\int_2^3 x^4 dx$.



Now consider the curve with equation $y = x^4$. The points A and B on the x-axis are where $x = 2$ and $x = 3$ respectively. The lines $x = 2$ and $x = 3$ cut the curve at D and C respectively. Write down the y-coordinates of D and C.

Find the area of the trapezium ABCD and compare your answer with the integral you evaluated.

Why is the approximation quite good?

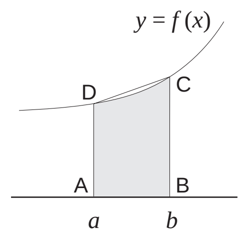
Activity 4

Repeat the idea of Activity 3 for the integral $\int_1^2 x^4 dx$ and the corresponding trapezium.

Do you get a good approximation for the integral this time by considering a trapezium?

Can you explain why?

In general, when the graph of $y = f(x)$ is approximately linear for $a \le x \le b$, the value of the integral $\int_a^b f(x)\,dx$ can be approximated by the area of a trapezium. This can be seen easily from the diagram opposite for the case when $f(x) \ge 0$ for $a \le x \le b$.



The area of the trapezium ABCD is

$$\frac{1}{2}(b-a)\{f(a)+\mathrm{f}(b)\}$$

Hence $$\int_a^b f(x) \approx \frac{(b-a)}{2}\{f(a)+f(b)\}$$

The curve can usually be approximated to a straight line if the interval considered is very small. To obtain an approximation over a larger interval, the interval is usually split into smaller ones. For ease of computation, the interval of integration is usually divided into strips of equal width.

Activity 5

By considering four strips of equal width and considering the approximate area to be that of four trapezia, estimate the value of

$$\int_0^1 \cos\sqrt{x}\,dx,$$

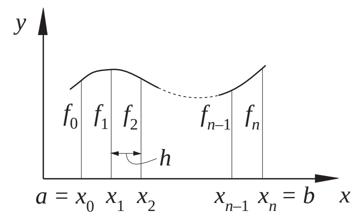
working with four decimal places and giving your final answer to three significant figures. (Remember to use a *radian* setting on your calculator!)

How close were you to the exact value

$$2\cos 1 + 2\sin 1 - 2 \approx 0.763546?$$

In general, if you wish to find the approximate value of $\int_a^b f(x)\,dx$, you divide the interval $a \le x \le b$ into n strips of equal width h. Let $x_0(=a),\ x_1,\ x_2,\ \ldots,\ x_n(=b)$ be the equally spaced x-coordinates. Then the values $f(x_0),\ f(x_1),\ \ldots,\ f(x_n)$ can be written more conveniently as $f_0,\ f_1,\ f_2,\ \ldots,\ f_n$ and these values are called the **ordinates**.

Activity 6



Given that $f(x) \geq 0$ for $a \leq x \leq b$, the integral $\int_a^b f(x)\,dx$ is an area. By considering the strips on the diagram opposite to be approximately trapezia, show that the value of the integral above is approximately

$$\frac{h}{2}(f_0 + f_1) + \frac{h}{2}(f_1 + f_2) + \ldots + \frac{h}{2}(f_{n-1} + f_n).$$

Hence show that

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left\{f_0 + 2(f_1 + f_2 + \ldots + f_{n-1}) + f_n\right\}$$

This result is known as the **trapezium rule**.

Sometimes an integral is very difficult or impossible to evaluate exactly and so an approximate method is used.

Example

Use the trapezium rule with 5 ordinates to find the approximate value of

$$\int_1^3 \frac{1}{\sqrt{(1+x^3)}}\,dx,$$

giving your answer to three decimal places.

Solution

We must remember that 5 ordinates means 4 strips (rather like 5 fence posts and 4 strips of fencing between them). In this case $x_0(=a)=1$ and $x_4(=b)=3$. Each strip is of width 0.5 and so $h=0.5$; $x_1=1.5$; $x_2=2.0$; $x_3=2.5$. Therefore $f_0 \approx 0.7071$; $f_1 \approx 0.4781$; $f_2 \approx 0.3333$; $f_3 \approx 0.2453$; $f_4 \approx 0.1890$.

Using the trapezium rule,

$$\text{integral} \approx \frac{0.5}{2}\left\{f_0 + 2(f_1 + f_2 + f_3) + f_4\right\}$$

$$\approx 0.752$$

Activity 7

If you have a graphics calculator with the facility to graph and integrate, find the value given for

$$\int_1^3 \frac{1}{\sqrt{(1+x^3)}}\,dx$$

and compare your answer with that in the example above.

Suggest a way that a more accurate approximation to the integral in the above example could be found.

Write a computer program to find the approximate value of the integral using the trapezium rule where you input the number of strips and consider how many strips are necessary in order to give the correct value to a certain number of decimal places.

8.3 Simpson's rule

Activity 8

Let $f(x)$ be approximated by the quadratic function ax^2+bx+c over the interval $-h \le x \le h$.

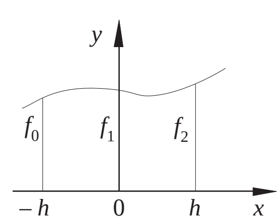
Show that

$$\int_{-h}^{h} f(x)\,dx \approx \frac{h}{3}\left(2ah^2+6c\right)$$

Activity 9

Given that the quadratic function with equation $y=ax^2+bx+c$ passes through the points $(-h,\ f_0)$, $(0,\ f_1)$ and $(h,\ f_2)$, show that

$$f_0+4f_1+f_2=2ah^2+6c$$



Suppose that the region bounded by the curve $y=f(x)$, the lines $x=-h$ and $x=h$ and the x-axis is divided into two strips as shown in the diagram. You can use the the results from Activities 8 and 9 to show that, provided $f(x)$ is approximately a quadratic,

$$\int_{-h}^{h} f(x)\,dx \approx \frac{h}{3}\left(f_0 + 4f_1 + f_2\right)$$

Since the curve $y = f(x)$ has been approximated by a quadratic curve the result above is usually a better approximation than that given by the trapezium rule which approximates the curve $y = f(x)$ by a straight line.

Activity 10

Suppose that f_0, f_1 and f_2 are the ordinates at $x = a$, $x = a + h$ and $x = a + 2h$ respectively. Explain, using the previous result and a translation of the axes, why

$$\int_{a}^{a+2h} f(x)\,dx \approx \frac{h}{3}\left(f_0 + 4f_1 + f_2\right).$$

By taking several **pairs** of strips, the result from Activity 10 can now be extended . If you divide the interval $a \le x \le b$ into n equal strips of width h as before, and again let $f_0, f_1, \ldots, f_n,$ be the corresponding ordinates, the above rule applied to each pair of strips in turn gives

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left\{\left(f_0 + 4f_1 + f_2\right) + \left(f_2 + 4f_3 + f_4\right) + \left(f_4 + 4f_5 + f_6\right) + \ldots + \left(f_{n-2} + 4f_{n-1} + f_n\right)\right\}$$

or, as the formula is more usually written,

$$\int_{x_0}^{x_n} f(x)\,dx \approx \frac{1}{3}h\left[f_0 + f_n + 4\left(f_1 + f_3 + \ldots + f_{n-1}\right) + 2\left(f_2 + f_4 + \ldots + f_{n-2}\right)\right]$$

or, as some find it easier to remember

$$\frac{1}{3}h\,[\text{'ends'} + 4 \times \text{'odds'} + 2 \times \text{'evens'}].$$

This result is know as **Simpson's rule**.

Note: n must be even, giving an odd number of ordinates.

It is sometimes interesting to compare the accuracy using Simpson's rule with that from the corresponding application of the trapezium rule. Earlier, the trapezium rule with 5 ordinates was used to find the approximate value of

$$\int_1^3 \frac{1}{\sqrt{\left(1+x^3\right)}}\, dx,$$

and the value 0.752 was obtained.

Example

Use Simpson's rule with 5 ordinates to find the approximate value of

$$\int_1^3 \frac{1}{\sqrt{\left(1+x^3\right)}}\, dx,$$

giving your answer to three decimal places.

Solution

As in the previous example, $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2.0$, $x_3 = 2.5$, $x_4 = 3.0$ and $h = 0.5$.

Also, $f_0 = 0.7071$, $f_1 = 0.4781$, $f_2 = 0.3333$, $f_3 = 0.2453$, $f_4 = 0.1890$, working to four decimal places.

Using Simpson's rule, the approximate value of the integral is

$$\frac{1}{3}\times 0.5\left[f_0 + f_4 + 4\left(f_1 + f_3\right) + 2f_2\right]$$

$$= \frac{1}{6}[0.8961 + 1.9124 + 0.9812 + 0.6666]$$

$$\approx 0.743.$$

This answer is in fact correct to three decimal places.

Exercise 8B

1. Use the trapezium rule with 3 ordinates to estimate $\int_1^5 (2x+7)\,dx$.

   Evaluate by integration $\int_1^5 (2x+7)\,dx$.

   Explain your findings.

2. Estimate the value of each of the following definite integrals (1) to (4) using

   (a) the trapezium rule;

   (b) Simpson's rule
   each with

   (i) 3 ordinates, (ii) 5 ordinates.

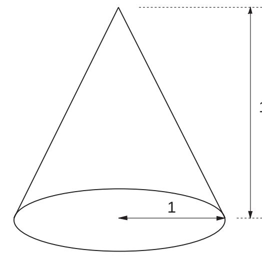
   (1) $\int_2^6 \frac{120}{x^3}\,dx$ (2) $\int_1^2 \left(x^3 - 3x^2\right)dx$

   (3) $\int_2^4 \ln 2x\, dx$ (4) $\int_0^1 \frac{1}{\sqrt{\left(4-x^2\right)}}\,dx$

3. Evaluate each of the integrals in Question 2 exactly and comment on the accuracy of the approximations.

8.4 Volumes of revolution

Integration is a very powerful tool for finding quantities like areas, volumes and arc lengths. This section will illustrate how integration is used for one particular application, namely the determination of volumes. First, though, some approximate methods will be used.

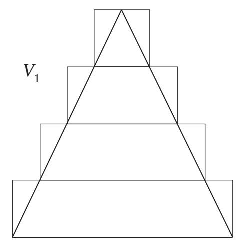


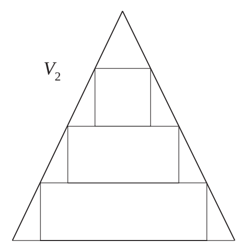
The diagram opposite shows a cone with radius 1 unit, height 1 unit (not to scale). What is its volume?

To find its volume you would probably use the formula

$$V = \tfrac{1}{3}\pi r^2 h$$

but how do you know this formula is correct and why the one third factor? One way to approximate the volume of the cone is to consider it to be made of a series of cylinders.



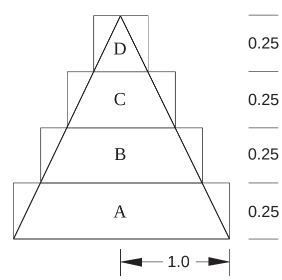


Two methods are shown opposite. V_1 clearly over-estimates the volume, whereas V_2 clearly under-estimates the volume.

However we do know that the true volume is 'trapped' between the volume V_1 and the volume V_2.

Activity 11

(a) Calculate an approximation of the volume of a cube by 'averaging' the volumes of V_1 and V_2 where $V_1 = \text{A} + \text{B} + \text{C} + \text{D}$ and where $V_2 = \text{A}' + \text{B}' + \text{C}'$.



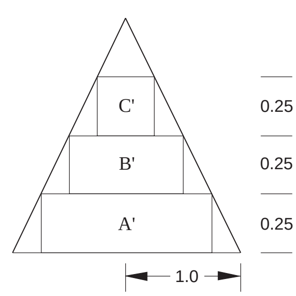
(b) What is the ratio

$$\frac{\pi}{\text{average } V} = \frac{\pi}{\tfrac{1}{2}\left(V_1 + V_2\right)} = ?$$

Now repeat with cylinders of height:

(i) 0.2 (ii) 0.1 (iii) 0.05

In each case, calculate the ratio



$$\frac{\pi}{\tfrac{1}{2}\left(V_1 + V_2\right)}$$

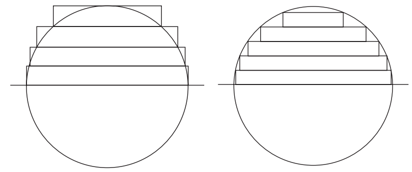
As the cylinder height approaches zero what does the ratio

$$\frac{\pi}{\tfrac{1}{2}\left(V_1 + V_2\right)}$$

approach?

Activity 12

Find an approximation of the volume of a sphere by considering it to be made of a series of cylinders as shown opposite.



Consider the cone again. *Diagram 1* shows the line $y = x$ from $x = 0$ to $x = 1$.

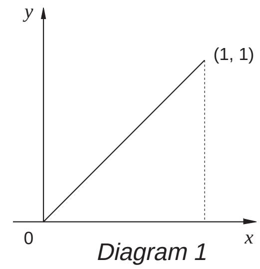


Diagram 1

If you rotate this area about the *x*-axis through 2π or $360°$ you will generate a cone with radius 1 and length (or height) 1 (*diagram 2*).

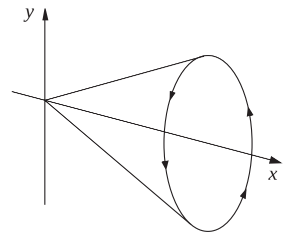


Diagram 2

Now consider a thin slice of the cone of radius *y* and length δx. This will be the slice obtained by rotating the shaded strip shown in *diagram 3* .

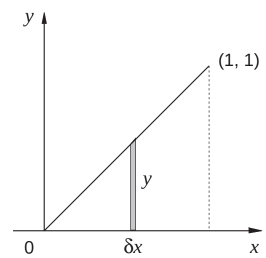


Diagram 3

The volume of this slice is approximately that of a cylinder with volume $\pi y^2 \delta x$ and the smaller we make δx the better the approximation becomes.

The volume of the cone therefore is

$$\approx \Sigma \pi y^2 \delta x$$

where the Σ sign means summing up over all such slices.

Another way of expressing this is to write

$$\delta V \approx \pi y^2 \delta x$$

where δV is the volume corresponding to the small cylindrical disc of height δx. Hence

$$\frac{\delta V}{\delta x} \approx \pi y^2$$

and letting $\delta x \to 0$

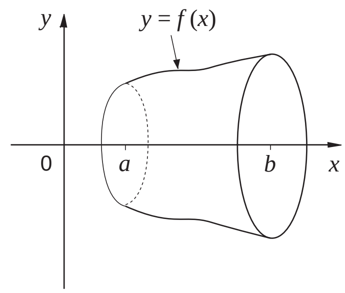
gives $\dfrac{dV}{dx} = \pi y^2$.

Integrating gives an expression for the volume

$$\boxed{V = \int_0^1 \pi y^2 \, dx}$$

Activity 13

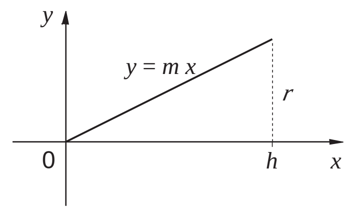
Evaluate the expression for V given above, noting that $y = x$ in this case.



The formula for the volume of revolution of any curve with equation $y = f(x)$ about the x-axis between $x = a$ and $x = b$ is given by

$$V = \int_a^b \pi y^2 dx$$

Activity 14



The diagram opposite shows part of the line $y = mx$ between $x = 0$ and $x = h$. The region bounded by the lines $y = mx$, $y = 0$ and $x = h$ is to be rotated about the x-axis to generate a cone.

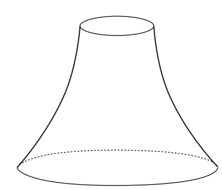
(a) Express m in terms of h and r (radius of the cone).

(b) By evaluating $V = \int_0^h \pi y^2 dx$, show that the volume of a cone is given by

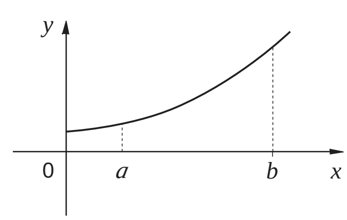
$$V = \tfrac{1}{3}\pi r^2 h$$

What would happen if the line $y = mx$ in the above diagram was rotated about the y-axis?

Activity 15



The diagram opposite is a model of a power station cooling tower. It is necessary to find its volume. The diagram below it shows part of the curve $y = e^x$.



If the region bounded by the curve $y = e^x$ and the lines $x = a$ and $x = b$ is rotated about the x-axis, a shape which is an approximation to the model of the cooling tower will be generated.

Show that the volume generated is given by

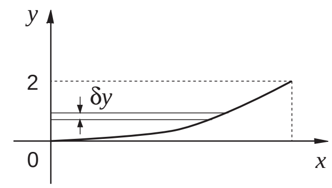
$$\frac{\pi}{2}\left[e^{2b} - e^{2a}\right]$$

Hence calculate the volume when $a = 2$ and $b = 5$.

Activity 16

(a) In order to estimate the amount of liquid that a saucer will hold, a student decides to use the function $y = \frac{x^2}{32}$.

She rotates the area bounded by the line $y = \frac{x^2}{32}$ and the line $y = 2$ about the y-axis. Calculate the volume generated.

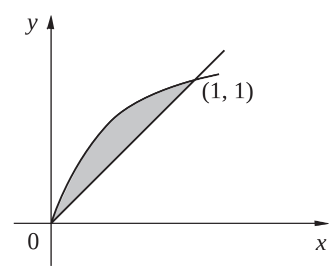


(b) Obtain a 'saucer' of your own and suggest possible improvements to the mathematical model used by this student.

(c) Obtain a 'cup' of your own and, by producing a mathematical model, estimate its volume. Compare your estimated volume with the actual volume. Modify your model to give a better approximation.

Example

The area given by $y^2 \le x$ and $y \ge x$ is to be rotated about the x-axis to form a 'bowl'. Find the volume of the material in the bowl.



Solution

The points of intersection are (0, 0) and (1, 1). You can think of the bowl as the solid formed by the outer curve y_1 with the inner curve y_2 taken away.

Therefore

$$V = \pi \int y_1^2 dx - \pi \int y_2^2 dx$$

$$= \pi \int_0^1 x\, dx - \pi \int_0^1 x^2 dx$$

$$= \pi \int_0^1 \left(x - x^2\right) dx$$

$$= \pi \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{\pi}{6} \text{ cubic units.}$$

Exercise 8C

1. Find the volume of the solid generated when the region bounded by the x-axis, the lines $x=1$ and $x=2$ and the curve with equation $y=x^2$, is rotated once about the x-axis.

2. Sketch the curve with equation $y=1-\frac{4}{x^2}$.

   The region R is bounded by the curve $y=1-\frac{4}{x^2}$, the x-axis and the lines $x=\frac{1}{2}$ and $x=1$. Find the volume generated when R is rotated completely about the x-axis.

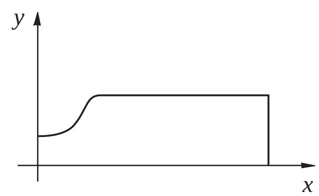
3. The area bounded by the curve with equation $y=\tan x$, the x-axis and the line $x=\frac{\pi}{3}$ is rotated about the x-axis. Calculate the volume of the solid of revolution so formed.

4. Draw a rough sketch of the circle with equation $x^2+y^2=100$ and the curve with equation $9y=2x^2$. Find the coordinates of the points A and B where they meet. Calculate the area of the region R bounded by the minor arc AB of the circle and the other curve. Find also the volume obtained by rotating R about the y-axis.

5. The region bounded by the curve with equation $y=\sqrt{x}+\frac{1}{\sqrt{x}}$, the x-axis and the lines $x=1$ and $x=4$, is R. Determine
   (a) the area of R,
   (b) the volume of the solid formed when R is rotated through 2π radians about the x-axis. (AEB)

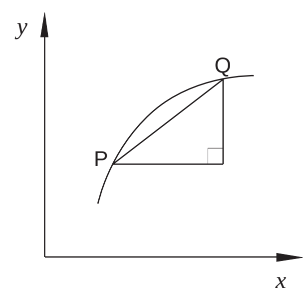
6. A curve has equation $y=\cosh x$ and the points P and Q on the curve have x-coordinates 0 and ln2 respectively. The region bounded by the arc PQ of the curve, the coordinate axes and the line $x=\ln 2$, is rotated through 2π radians about the x-axis to form a solid of revolution. Show that the volume of the solid is $\pi\left(\frac{1}{2}\ln 2+\frac{15}{32}\right)$. (AEB)

7. By using suitable approximations for the function to represent the shape for a milk bottle, use integration to find the volume of revolution. Check this result with the expected value.



8. The finite region bounded by the curves with equations $y=x^3$ and $y=\sqrt{x}$ is R.
   (a) Determine the area of R.
   (b) Calculate the volume generated when R is rotated through 2π radians about
   (i) the x-axis,
   (ii) the y-axis.

8.5 Lengths of arcs of curves



Suppose P and Q are two points fairly close together which lie on the curve with equation $y=f(x)$, so that P has coordinates (x, y) and Q has coordinates $(x+\delta x, y+\delta y)$. Let the length of the curve between P and Q be δs. This length must be approximately equal to the length of the line segment PQ. Thus

$$\delta s \approx \sqrt{\left((\delta x)^2+(\delta y)^2\right)}$$

So $$\frac{\delta s}{\delta x} \approx \sqrt{\left(1+\left(\frac{\delta y}{\delta x}\right)^2\right)}$$

Taking the limit as $\delta x \to 0$,

$$\frac{ds}{dx} \approx \sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}$$

Hence the length of arc of curve from the point where $x = x_1$ to the point where $x = x_2$ is

$$s = \int_{x_1}^{x_2} \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\, dx$$

Example

Find the length of the arc of the curve with equation $y = \frac{4}{3}x^{\frac{3}{2}}$

from the point where $x = \frac{3}{4}$ to the point where $x = 2$.

Solution

Here $\quad \frac{dy}{dx} = 2x^{\frac{1}{2}},$

so $\quad 1+\left(\frac{dy}{dx}\right)^2 = 1+4x$

Arc length $\quad s = \int_{\frac{3}{4}}^{2} (1+4x)^{\frac{1}{2}}\, dx$

$$= \left[\frac{1}{4} \times \frac{2}{3}(1+4x)^{\frac{3}{2}}\right]_{\frac{3}{4}}^{2}$$

$$= \frac{1}{6}(27-8) = 3\frac{1}{6}$$

Activity 17

Show that if x and y are expressed in terms of a parameter t, then the arc length between the points on the curve where $t = t_1$ and $t = t_2$ is given by

$$s = \int_{t_1}^{t_2} \sqrt{\left(\dot{x}^2 + \dot{y}^2\right)}\, dt$$

where $\quad \dot{x} = \frac{dx}{dt}$ and $\dot{y} = \frac{dy}{dt}$

Example

A curve is defined by

$$x = t - \sin t; \quad y = 1 - \cos t$$

where t is a parameter. Calculate the length of the arc of the curve from the point where $t = 0$ to the point where $t = 2\pi$.

Solution

$$x = \frac{dx}{dt} = 1 - \cos t; \ \dot{y} = \frac{dy}{dt} = \sin t$$

$$\dot{x}^2 + \dot{y}^2 = 1 - 2\cos t + \cos^2 t + \sin^2 t$$

But $\quad \cos^2 t + \sin^2 t = 1$

giving $\quad \dot{x}^2 + \dot{y}^2 = 2 - 2\cos t$

$$= 4\sin^2 \tfrac{1}{2} t$$

As $2\sin\tfrac{1}{2}t \geq 0$ for $0 \leq t \leq 2\pi$,

$$\sqrt{\left(\dot{x}^2 + \dot{y}^2\right)} = 2\sin\tfrac{1}{2}t$$

and $\quad s = \int_0^{2\pi} \sqrt{\left(\dot{x}^2 + \dot{y}^2\right)}\, dt$

$$= \int_0^{2\pi} 2\sin\tfrac{1}{2}t\, dt$$

$$= \left[-4\cos\tfrac{1}{2}t\right]_0^{2\pi}$$

$$= -4\cos\pi + 4\cos 0$$

$$= 4 + 4 = 8$$

Exercise 8D

1. Use integration to find the length of the arc of the curve with equation $y = \sqrt{\left(1 - x^2\right)}$ from the point where $x = 0$ to the point where $x = \tfrac{1}{2}$. By identifying the curve, verify your answer.
2. A curve is defined parametrically by $x = 2\cos t - \cos 2t$, $y = 2\sin t - \sin 2t$, where t is a parameter. Find the length of the arc of the curve from the point where $t = 0$ to the point where $t = \dfrac{\pi}{4}$.
3. Sketch the curve defined by the parametric equations $x = \cos^3 t$; $y = -\sin^3 t$ $(0 \leq t \leq 2\pi)$ using a graphics calculator or software package.

   Show that the total length of the curve is 6 units, by making use of symmetry properties of the curve.

   Why can you not find the arc length by integrating directly from $t = 0$ to $t = 2\pi$?

4. Given that $y = \ln\sec x$, where $\frac{-\pi}{2} < x < \frac{\pi}{2}$, show that $\frac{dy}{dx} = \tan x$. Hence find the length of the curve with equation $y = \ln\sec x$ from the point where $x = -\frac{\pi}{4}$ to the point where $x = \frac{\pi}{6}$.

5. Show that the length of arc of the parabola with parametric equations $x = at^2$, $y = 2at$ (where $a>0$), from the point where $t = t_1$ to the point where $t = t_2$, is given by

$$2a\int_{t_1}^{t_2} \sqrt{\left(1+t^2\right)}\, dt .$$

By means of the substitution $t = \sinh\theta$, or otherwise, show that the arc length from the point where $t = 0$ to the point where $t = 3$ is

$$a\left(\sinh^{-1} 3 + 3\sqrt{10}\right).$$

6. Calculate the length of the curve defined by the equations $x = \tanh t$, $y = \operatorname{sech} t$, from the point where $t = -1$ to the point where $t = 2$.

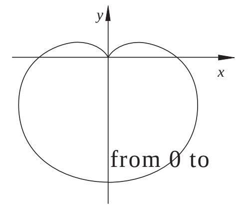
7. Given that a is a positive constant, find the length of the curve defined parametrically by

$$x = at\cos t; \quad y = at\sin t \quad (0 \le t \le 3\pi).$$

8. The curve defined parametrically by

$$x = (1-\sin t)\cos t, \quad y = (1-\sin t)\sin t$$

is sketched for $0 \le t \le 2\pi$.

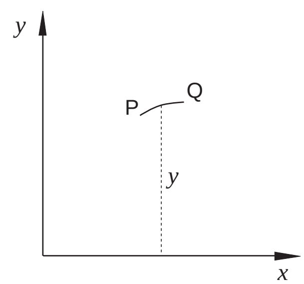


It is called a cardioid.

What happens when you try to find the total perimeter by integrating from 0 to 2π?

Use symmetry properties to find the perimeter of the cardioid.

8.6 Curved surface areas of revolution



Consider the small arc PQ of length δs of a curve distance y from the x-axis. When the arc is rotated through 2π radians about the x-axis it generates a band with surface area δA approximately equal to $2\pi y \times \delta s$.

Therefore

$$\frac{\delta A}{\delta x} \approx 2\pi y \frac{\delta s}{\delta x}$$

Taking the limit as $\delta x \to 0$ and using the formula for $\frac{ds}{dx}$ obtained in the previous section gives

$$\frac{dA}{dx} = 2\pi y \frac{ds}{dx} = 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

Hence the curved suface area of revolution when the arc of a curve from $x = x_1$ to $x = x_2$ is rotated through 2π radians about the x-axis is

$$\boxed{\int_{x_1}^{x_2} 2\pi y \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\, dx}$$

Example

The arc of the parabola with equation $y^2 = 2x$ from the origin to the point $(4,\ 2\sqrt{2})$ is rotated through 2π radians about the x-axis.

Calculate the area of the curved surface generated.

Solution

$$y^2 = 2x \Rightarrow 2y\frac{dy}{dx} = 2$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{1}{y^2} = 1+\frac{1}{2x} = \frac{2x+1}{2x}$$

Since the arc is on the branch where y is positive, $y = (2x)^{\frac{1}{2}}$ and so curved surface area

$$= \int_0^4 2\pi y\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx$$

$$= \int_0^4 2\pi(2x)^{\frac{1}{2}}\left(\frac{2x+1}{2x}\right)^{\frac{1}{2}}dx$$

$$= 2\pi\int_0^4 (2x+1)^{\frac{1}{2}}\,dx$$

$$= 2\pi\left[\tfrac{1}{2}\times\tfrac{2}{3}(2x+1)^{\frac{3}{2}}\right]_0^4$$

$$= 2\pi\times\frac{1}{3}\times(27-1) = \frac{52}{3}\pi$$

The results from Activity 17 give a corresponding result when a curve is expressed in parametric form with parameter t, as below.

Curved surface area is

$$\int_{t_1}^{t_2} 2\pi y\sqrt{\left[\dot{x}^2+\dot{y}^2\right]}\,dt$$

Activity 18

A curve is defined parametrically by $x = t - \sin t, \ \ y = 1 - \cos t.$

Show that

$$\dot{x}^2 + \dot{y}^2 = 4\sin^2 \tfrac{1}{2} t\,.$$

The part of the curve from $t = 0$ to $t = \pi$ is rotated through 2π radians about the x-axis to form a curved surface.

Show that the surface area can be expressed in the form

$$8\pi\int_0^\pi \sin^3\left(\frac{t}{2}\right) dt$$

Use the substitution $u = \cos \tfrac{1}{2} t$, or otherwise, to show that this surface area has value $\dfrac{32}{3}\pi$.

Exercise 8E

1. The arc of the curve described by $x = \cos^3 t$, $y = \sin^3 t$, from the point where $t = 0$ to the point where $t = \dfrac{\pi}{2}$, is rotated through 2π radians about the x-axis.

   Calculate the area of the surface generated.

2. Calculate the area of the surface generated when each of the following arcs of curves are rotated through 2π radians about the x-axis.

   (a) the part of the line $y = \tfrac{1}{4}x$ from $x = 1$ to $x = 5$;

   (b) the arc of the curve $y = \cosh x$ from $x = 0$ to $x = 2$;

   (c) the portion of the curve $y = e^x$ from $x = 0$ to $x = \ln 3$.

3. The parametric equations of a curve are

   $$x = t - \tanh t, \quad y = \operatorname{sech} t$$

   The arc of the curve between the points with parameters 0 and ln2 is rotated about the x-axis. Calculate in terms of π the area of the curved surface formed. (AEB)

4. A curve is defined by

   $$x = t^3 \ , \ \ y = 3t^2$$

   where t is a parameter.

   Calculate the area of the surface generated when the portion of the curve from $t = 0$ to $t = 2$ is rotated through 2π radians about the x-axis.

8.7 Reduction formulae in integration

Activity 19

Use integration by parts to show how $I = \int_0^1 x^2 e^{-x} dx$ can be expressed in terms of $\int_0^1 x e^{-x} dx$.

Evaluate $\int_0^1 x e^{-x} dx$ and hence find the value of I.

If you consider the more general form of this integral

$$I_n = \int_0^1 x^n e^{-x} dx$$

you can use integration by parts to express I_n in terms of I_{n-1}. Now putting $u = x^n$ and $\frac{dv}{dx} = e^{-x}$ gives

$$I_n = \left[-x^n e^{-x}\right]_0^1 + n\int_0^1 x^{n-1} e^{-x} dx$$

Assuming $n > 0$,

$$I_n = -e^{-1} + n\, I_{n-1}$$

This is an example of a **reduction formula**. Suppose you wish to find the value of the integral in Activity 19. You need I_2. But

$$I_2 = -e^{-1} + 2\, I_1$$

and $$I_1 = -e^{-1} + I_0$$

But

$$I_0 = \int_0^1 e^{-x} dx = \left[-e^{-x}\right]_0^1 = 1 - e^{-1}$$

Hence

$$I_1 = -e^{-1} + I_0 = 1 - 2e^{-1}$$

and

$$I_2 = -e^{-1} + 2\left(1 - 2e^{-1}\right) = 2 - 5e^{-1}$$

Activity 20

Use the reduction formula and value of I_2 to obtain the values of

$$\int_0^1 x^3 e^{-x} dx \text{ and } \int_0^1 x^4 e^{-x} dx$$

Example

Given that $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$, express I_n in terms of

I_{n-2} for $n \geq 2$. Hence evaluate $\int_0^{\frac{\pi}{2}} \sin^5 x \, dx$ and $\int_0^{\frac{\pi}{2}} \sin^6 x \, dx$.

Solution

Using the formula for integration by parts

$$\int u \frac{dv}{dx} dx = uv - \int \frac{du}{dx} v \, dx$$

let $\quad u = \sin^{n-1} x, \quad \frac{dv}{dx} = \sin x,$

giving $\quad \frac{du}{dx} = (n-1)\sin^{n-2} x \cos x, \quad v = -\cos x.$

So for $n \geq 2$

$$I_n = \left[-\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx$$

$$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\left(1 - \sin^2 x\right) dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x \, dx$$

$$= (n-1)I_{n-2} - (n-1)I_n$$

Therefore for $n \geq 2$

$$n \, I_n = (n-1)I_{n-2}$$

or $\quad I_n = \frac{(n-1)}{n} I_{n-2}.$

You need $I_5 = \frac{4}{5} I_3$ and $I_3 = \frac{2}{3} I_1$

Also $I_1 = \int_0^{\frac{\pi}{2}} \sin x \, dx = [-\cos x]_0^{\frac{\pi}{2}} = 1$

and $I_5 = \frac{4}{5} \times \frac{2}{3} \times 1 = \frac{8}{15}$.

Similarly,

$$I_6 = \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times I_0 = \frac{15\pi}{96}.$$

Activity 21

(a) Differentiate $\sec^{n-2} \tan x$ with respect to x and express your answer in terms of powers of $\sec x$. By integrating the result with respect to x, establish the reduction formula

$$(n-1)I_n - (n-2)I_{n-2} = \sec^{n-2} x \tan x + \text{constant}$$

where $I_n = \int \sec^n x \, dx$.

(b) Use the reduction formula in (a) to evaluate

$$\int_0^{\frac{\pi}{4}} \sec^4 x \, dx \quad \text{and} \quad \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^5 x \, dx.$$

Exercise 8F

1. Given that $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$, prove that
   $I_n = \frac{n-1}{n} I_{n-2}$. Hence find I_4 and I_7.

2. Given that $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$, establish the reduction formula
   $$\left(n^2+4\right) I_n = n(n-1) I_{n-2} + 2e^{\pi}.$$
   Evaluate I_4 and I_5.

3. If $I_n = \int_0^{\frac{\pi}{2}} e^{2x} \sin^n x \, dx$, $n > 1$, show that
   $$\left(n^2+4\right) I_n = n(n-1) I_{n-2} + 2e^{\pi}$$
   Hence, or otherwise, find $\int_0^{\frac{\pi}{2}} e^{2x} \sin^3 x \, dx$. (AEB)

4. If $I_n = \int \tan^n x \, dx$, obtain a reduction formula for I_n. Hence, or otherwise, show that
   $$\int_0^{\frac{\pi}{4}} \tan^4 x \, dx = \frac{3\pi - 8}{12}.$$ (AEB)

5. Simplify $\frac{\sin 2n\theta - \sin 2(n-1)\theta}{\sin \theta}$.
   If $I_n = \int \frac{\sin 2n\theta}{\sin \theta} \, d\theta$, prove that
   $I_n = I_{n-1} + \frac{2}{(2n-1)} \sin(2n-1)\theta$. Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin 5\theta}{\sin \theta} \, d\theta$. (AEB)

6. Given that $p = \ln 2$, write down the values of $\cosh p$ and $\sinh p$. Find a reduction formula relating I_n and I_{n-2} where $I_n = \int_0^p \cosh^n \theta \, d\theta$.
   Hence find I_3 and I_4.

8.8 Miscellaneous Exercises

1. Sketch the curve given by the parametric equations $x = t^3; y = 3t^2$. Find the length of the arc of the curve from $t = 0$ to $t = 2$. (AEB)

2. Given that a is a positive constant, find the length of the arc of the curve with equation $y = a\cosh\left(\frac{x}{a}\right)$ between $x = 0$ and $x = k$, where k is a constant. Hence show that, as k varies, the arc length is proportional to the area of the region bounded by the arc, the coordinate axes and the line $x = k$.

3. Given that $xy = 3x^2 + y^2$, find $\frac{dy}{dx}$, giving your answer in terms of x and y. (AEB)

4. Find the length of the arc l of the curve with equation $y = \ln(\cos x)$ from the point at which $x = 0$ to the point at which $x = \frac{1}{6}\pi$. The arc l is rotated completely about the x-axis to form a surface, S. Show that the area of S is

$$\left| 2\pi\int_0^{\frac{\pi}{6}} \frac{\ln(\cos x)}{\cos x}\, dx \right|$$

Use Simpson's rule with three ordinates to estimate this area, giving your answer to 2 decimal places. (AEB)

5. Given that $I = \int_0^1 \frac{x^2}{\sqrt{(4-x^2)}}\, dx$,

   (a) using Simpson's rule with four equal intervals, working with 4 decimal places, obtain an approximation to I, giving your answer to three significant figures;

   (b) by putting $x = 2\sin\theta$, show that

$$I = 4\int_0^{\frac{\pi}{6}} \sin^2\theta\, d\theta .$$

   Hence show that the exact value of I is

$$\frac{\pi}{3} - \frac{\sqrt{3}}{2} .$$ (AEB)

6. Find the gradient at the point (2, 3) on the curve with equation $3x^2 + 6xy - 2y^3 + 6 = 0$. (AEB)

7. A curve has equation $y = x - 1 + \frac{1}{x+1}$.

   Calculate the coordinates of the turning points of the curve and determine their nature.

   The finite region bounded by the curve, the x-axis from $x = 0$ to $x = 4$ and the line $x = 4$ is R. Express the area of R as an integral and show that its exact value is $4 + \ln 5$. Use Simpson's rule with four equal intervals to find an estimate of the same integral, giving your answer to three decimal places. Hence find an approximation for ln5, giving your answer to two decimal places. (AEB)

8. Shade on a sketch the finite region R in the first quadrant bounded by the x-axis, the curve with equation $y = \ln x$ and the line $x = 5$. By means of integration, calculate the area of R. The region R is rotated completely about the x-axis to form a solid of revolution S.

x	1	2	3	4	5
$(\ln x)^2$	0	0.480	1.207	1.922	2.590

   Use the given table of values and apply the trapezium rule to find an estimate of the volume of S, giving your answer to one decimal place. (AEB)

9. Showing your working in the form of a table, use Simpson's rule with 4 equal intervals to estimate $\int_0^2 \ln(1+x^2)\,dx$, giving your answer to 3 decimal places. Deduce an approximate value for $\int_0^2 \ln\sqrt{(1+x^2)}\,dx$. (AEB)

10. Given that $x > 0$ and that $y = \frac{\ln x}{x}$, find $\frac{dy}{dx}$. State the set of values of x for which $\frac{dy}{dx} > 0$ and the set of values of x for which $\frac{dy}{dx} < 0$. Hence show that y has a maximum value of $\frac{1}{e}$.

    Find the area of the finite region R bounded by the curve $y = \frac{\ln x}{x}$, the x-axis and the line $x = 5$. The region R is rotated completely about the x-axis to form a solid of revolution S. Use Simpson's rule, taking ordinates at $x = 1$, 2, 3, 4 and 5 to estimate, to 2 significant figures, the volume of S. (AEB)

11. The following approximate measurements were made of two related variables x and y.

x	2.0	2.5	3.0	3.5	4.0
y	10.9	17.5	27.0	40.3	59.1

Use the trapezium rule with five ordinates to estimate the value of $\int_2^4 y\,dx$ giving your answer to 1 decimal place. (AEB)

12. Evaluate $\int_1^5 \frac{x}{x^2+1}\,dx$ giving your answer in terms of a natural logarithm. Use Simpson's rule with four intervals to find an estimate of the integral. Hence find an approximation for ln13, giving your answer to two decimal places. (AEB)

13. Given that $I_n = \int_0^{\frac{1}{4}\pi} \tan^n x\,dx$, show that for $n \ge 0$,

$I_{n+2} + I_n = \frac{1}{n+1}$. Calculate the values of I_4 and I_5. (AEB)

14. A curve has equation $9ay^2 = 8x^3$, where a is a positive constant. The tangent at $P\left(a, \frac{2\sqrt{2}}{3}a\right)$ meets the curve again at Q. Prove that the x-coordinate of Q is $\frac{1}{4}a$ and show further that QP is the normal to the curve at Q. Show that $1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{2x}{a}$. Find, in terms of a, the length of the arc QP of the curve. (AEB)

15. Sketch the curve with equation $y = \sec 2x$ for $-\frac{1}{2}\pi \le x \le \frac{1}{2}\pi$. The region bounded by the curve, the x-axis and the lines $x = -\frac{\pi}{6}$ and $x = \frac{\pi}{6}$ is R.

Show that $\left(\frac{dy}{dx}\right)^2 + 1 = \left(2\sec^2 2x - 1\right)^2$ and hence prove that the perimeter of R is $4 + 2\sqrt{3}$. (AEB)

16. A curve has equation $y = \cosh x$ and the points P and Q on the curve have x-coordinates 0 and ln2 respectively. Find the length of the arc PQ of the curve. (AEB)

17. Given that $I_n = \int_0^1 x^n \sqrt{(1-x)}\,dx$, prove that, if $n > 0$, then $(2n+3)I_n = 2n\,I_{n-1}$.

A curve has equation $y^2 = x^4(1-x)$.

(a) Find the coordinates of the turning point of the curve and sketch its graph.

(b) Find the area of the loop of the curve. (AEB)

18. Derive a reduction formula for I_n in terms of I_{n-1} when $I_n = \int x^3 (\ln x)^n\,dx$.

Hence find $\int x^3 (\ln x)^3\,dx$. (AEB)

19. The tangent at a point P on the curve whose parametric equations are $x = a\left(t - \frac{1}{3}t^3\right)$, $y = at^2$, cuts the x-axis at T. Prove that the distance of the point T from the origin O is one half of the length of the arc OP. (AEB)

20. A curve is given by the parametric equations

$$x = 4\cos t + \cos 2t, \quad y = \sin 2t + 4\sin t + 2t.$$

Find the length of the curve between the points $t = 0$ and $t = \frac{\pi}{4}$. (AEB)

21. A curve has parametric equations

$$x = 4t^2; \quad y = t^4 - 4\ln t.$$

Find the length of the arc of the curve from $t = 1$ to $t = 2$. Find also the area of the surface formed when this arc is rotated completely about the y-axis. (AEB)

22. Given that $I_n = \int \frac{x^n}{\left(a^2+x^2\right)^{\frac{1}{2}}}\,dx$, where a is a constant, show that for $n \ge 2$

$$n\,I_n = x^{n-1}\left(a^2+x^2\right)^{\frac{1}{2}} - (n-1)a^2 I_{n-2} + \text{constant}.$$

Evaluate $\int_0^{\sqrt{3}} \frac{x^5}{\left(1+x^2\right)^{\frac{1}{2}}}\,dx$ and $\int_0^1 \frac{x^4}{\left(4+x^2\right)^{\frac{1}{2}}}\,dx$.