9 MATRICES AND TRANSFORMATIONS

Objectives

After studying this chapter you should

- be able to handle matrix (and vector) algebra with confidence, and understand the differences between this and scalar algebra;
- be able to determine inverses of 2×2 matrices, recognising the conditions under which they do, or do not, exist;
- be able to express plane transformations in algebraic and matrix form;
- be able to recognise and use the standard matrix form for less straightforward transformations;
- be able to use the properties of invariancy to help describe transformations;
- appreciate the composition of simple transformations;
- be able to derive the eigenvalues and eigenvectors of a given 2×2 matrix, and interpret their significance in relation to an associated plane transformation.

9.0 Introduction

A **matrix** is a rectangular array of numbers. Each entry in the matrix is called an **element**. Matrices are classified by the number of rows and the number of columns that they have; a matrix **A** with *m* rows and *n* columns is an $m \times n$ (said '*m* by *n*') matrix, and this is called the **order** of **A**.

Example

Given

$$\mathrm{A} = \begin{bmatrix} 1 & 4 & 2 \\ 3 & -1 & 0 \end{bmatrix},$$

then **A** has order 2×3 (rows first, columns second.) The elements of **A** can be denoted by a_{ij}, being the element in the *i*th row and *j*th column of **A**. In the above case, $a_{11} = 1, a_{23} = 0$, etc.

Addition and subtraction of matrices is defined only for matrices of equal order; the sum (difference) of matrices **A** and **B** is the matrix obtained by adding (subtracting) the elements in corresponding positions of **A** and **B**.

Thus

$$\mathrm{A} = \begin{bmatrix} 1 & 4 & 2 \\ 3 & -1 & 0 \end{bmatrix} \text{ and } \mathrm{B} = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 3 & -3 \end{bmatrix}$$

$$\Rightarrow \quad \mathrm{A} + \mathrm{B} = \begin{bmatrix} 0 & 6 & 5 \\ 7 & 2 & -3 \end{bmatrix} \text{ and } \mathrm{A} - \mathrm{B} = \begin{bmatrix} 2 & 2 & -1 \\ -1 & -4 & 3 \end{bmatrix}.$$

However, if

$$\mathrm{C} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix},$$

then **C** can neither be added to nor subtracted from either of **A** or **B**.

If you think of matrices as stores of information, then the addition (or subtraction) of corresponding elements makes sense.

Example

A milkman delivers three varieties of milk Pasteurised (PA), Semi-skimmed (SS) and Skimmed (SK)) to four houses (E, F, G and H) over a two-week period. The number of pints of each type of milk delivered to each house in week 1 is given in matrix **M**, while **N** records similar information for week 2.

$$\begin{array}{c} \\ \mathrm{E} \\ \mathrm{F} \\ \mathrm{G} \\ \mathrm{H} \end{array} \begin{array}{c} \begin{array}{ccc} \text{PA} & \text{SS} & \text{SK} \end{array} \\ \begin{bmatrix} 8 & 4 & 3 \\ 12 & 0 & 3 \\ 2 & 7 & 6 \\ 6 & 9 & 0 \end{bmatrix} \end{array} = \mathrm{M} \qquad \begin{array}{c} \\ \mathrm{E} \\ \mathrm{F} \\ \mathrm{G} \\ \mathrm{H} \end{array} \begin{array}{c} \begin{array}{ccc} \text{PA} & \text{SS} & \text{SK} \end{array} \\ \begin{bmatrix} 4 & 7 & 8 \\ 10 & 0 & 5 \\ 0 & 8 & 7 \\ 8 & 10 & 0 \end{bmatrix} \end{array} = \mathrm{N}$$

Then

$$\mathrm{M} + \mathrm{N} = \begin{bmatrix} 12 & 11 & 11 \\ 22 & 0 & 8 \\ 2 & 15 & 13 \\ 14 & 19 & 0 \end{bmatrix}$$

records the total numbers of pints of each type of milk delivered to each of the houses over the fortnight, and

$$\mathbf{N}-\mathbf{M}=\begin{bmatrix}-4 & 3 & 5\\ -2 & 0 & 2\\ -2 & 1 & 1\\ 2 & 1 & 0\end{bmatrix}$$

records the increase in delivery for each type of milk for each of the houses in the second week.

Suppose now that we consider the 3×2 matrix, **P**, giving the prices of each type of milk, in pence, as charged by two dairy companies:

$$\begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} \text{PA} \\ \text{SS} \\ \text{SK} \end{array} & \begin{bmatrix} 35 & 36 \\ 32 & 30 \\ 27 & 27 \end{bmatrix} \end{array} = \mathbf{P}$$

What are the possible weekly milk costs to each of the four households?

Define the cost matrix as

$$\begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} \text{E} \\ \text{F} \\ \text{G} \\ \text{H} \end{array} & \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \\ c_{41} & c_{42} \end{bmatrix} \end{array} = \mathbf{C}$$

Now c_{11} is the cost to household E if company 1 delivers the milk (in the week for which the matrix **M** records the deliveries) and so

$$\begin{aligned} c_{11} &= 8\times 35+4\times 32+3\times 27\\ &= 489\text{p} \end{aligned}$$

Essentially this is the first row $\begin{bmatrix}8 & 4 & 3\end{bmatrix}$ of **M** 'times' the first column $\begin{bmatrix}35\\32\\27\end{bmatrix}$ of **P**.

Similarly, for example, c_{32} can be thought of as the 'product' of the third row of **M**, $\begin{bmatrix}2 & 7 & 6\end{bmatrix}$, with the second column of **P**, $\begin{bmatrix}36\\30\\27\end{bmatrix}$, so that $$\begin{aligned} c_{32} &= 2\times 36 + 7\times 30 + 6\times 27 \\ &= 444\text{p} \end{aligned}$$

This is the cost to household G if they get company 2 to deliver their milk.

Matrix multiplication is defined in this way. You will see that multiplication of matrices **X** and **Y** is only possible if

$$\boxed{\text{the number of columns } \mathbf{X} = \text{the number of rows of } \mathbf{Y}}$$

Then, if **X** is an $(a\times b)$ matrix and **B** a $(c\times d)$ matrix, the product matrix **XY** exists if and only if $b = c$ and **XY** is then an $(a\times d)$ matrix. Thus, for $\mathsf{P} = \mathsf{XY}$,

$$\mathsf{P} = \left(p_{i\,j}\right),$$

where the entry $p_{i\,j}$ is the scalar product of the ith row of **X** (taken as a row vector) with the jth column of **Y** (taken as a column vector).

Example

Find **AB** when

$$\mathsf{A} = \begin{bmatrix} 1 & 4 & 2 \\ 3 & -1 & 0 \end{bmatrix}, \quad \mathsf{B} = \begin{bmatrix} 2 & 5 \\ 2 & 0 \\ -1 & 3 \end{bmatrix}$$

Solution

A is a 2×3 matrix, **B** is a 3×2 matrix. Since the number of columns of **A** = the number of rows of **B**, the product matrix **AB** exists, and has order 2×2.

$$\mathsf{P} = \mathsf{AB} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$p_{11} = \begin{bmatrix} 1 & 4 & 2 \end{bmatrix} . \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = 2 + 8 - 2 = 8, \text{ etc}$$

giving

$$\mathsf{P} = \begin{bmatrix} 8 & 11 \\ 4 & 15 \end{bmatrix}$$

The answers to the questions in the activity below should help you discover a number of important points relating the matrix arithmetic and algebra. Some of them are exactly as they are with ordinary real numbers, that is, scalars. More significantly, there are a few important differences.

Activity 1

(1) In the example above, suppose that $\mathbf{Q} = \mathbf{BA}$. What is the order of **Q**? Comment.

(2) (a) Take $\mathbf{C} = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} -1 & 0 & 3 \\ 2 & 5 & 7 \end{bmatrix}$. What is the order of **CD**? What is the order of **DC**? Comment.

(b) Take $\mathbf{E} = \begin{bmatrix} 0 & 3 \\ -7 & 2 \end{bmatrix}$ and $\mathbf{F} = \begin{bmatrix} 2 & 2 \\ 8 & -1 \end{bmatrix}$. What is the order of **FE**? Work out **EF** and **FE**. Comment.

(c) (i) Work out **CE** and **FC**

(ii) Work out $\mathbf{CE} \times \mathbf{F}$ and $\mathbf{C} \times \mathbf{EF}$. Comment

(iii) Work out $\mathbf{F} \times \mathbf{CE}$ and $\mathbf{FC} \times \mathbf{E}$. Comment.

(3) Using the matrices given above, work out

$$\mathbf{CF}, \quad \mathbf{C}+\mathbf{F}, \quad \mathbf{E}+\mathbf{F}.$$

Answer the following questions and comment on your answers.

(a) Is $\mathbf{CE} + \mathbf{CF} = \mathbf{C}(\mathbf{E} + \mathbf{F})$?

(b) Is $\mathbf{CE} + \mathbf{FE} = (\mathbf{C} + \mathbf{F})\mathbf{E}$?

(c) Is $\mathbf{CE} + \mathbf{EF} = \mathbf{E}(\mathbf{C} + \mathbf{F})$?

(d) Is $\mathbf{CE} + \mathbf{EF} = (\mathbf{C} + \mathbf{F})\mathbf{E}$?

Summary of observations

You should have noted that, for matrices **M** and **N**, say:

- the product matrix **MN** may exist, even if **NM** does not.
- even if **MN** and **NM** both exist, they may have different orders.
- even if **MN** and **NM** both exist and have the same order, it is generally not the case that $\mathbf{MN} = \mathbf{NM}$. (Matrix multiplication does not obey the **commutative law**. Matrix addition does: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ provided that **A** and **B** are of the same order.)
- when multiplying more than two matrices together, the order in which they appear is important, but the same result is obtained however they are multiplied within that order. (Matrix multiplication is said to obey the **associative law**.)

- A matrix can be pre-multiplied or post-multiplied by another. Multiplication of brackets and, conversely, factorisation is possible provided the left-to-right order of the matrices involved is maintained.

For a sensible matrix algebra to be developed, it is necessary to ensure that **MN** and **NM** both exist, and have the same order as **M** and **N**. That is, **M** and **N** must be square matrices. In the work that follows you will be working with 2×2 matrices, as well as with row vectors (1×2 matrices) and column vectors (2×1 matrices).

Exercise 9A

1. Work out the values of x and y in the following cases:

   (a) $\begin{bmatrix} 4 & 1 \\ -19 & -5 \end{bmatrix} \begin{bmatrix} 5 \\ -22 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

   (b) $\begin{bmatrix} 4 & 1 \\ -19 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -22 \end{bmatrix}$

   (c) $\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 18 \end{bmatrix}$

   (d) $\begin{bmatrix} 15 & -5 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 20 \\ -8 \end{bmatrix}$

   (e) $\begin{bmatrix} 1 & -7 \\ -3 & 21 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix}$

   Comment on your responses to parts (d) and (e).

2. $\mathrm{A} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$. Find $\mathbf{A}^2$ and $\mathbf{A}^3$. If we say $\mathrm{A}^1 = \mathrm{A}$, is there any meaning to A^0?

3. The transpose of $\mathrm{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the matrix A^{T} obtained by swapping the rows and columns of **A**; i.e. $\mathrm{A}^{\mathrm{T}} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Find the conditions necessary for it to be true that $\mathrm{AA}^{\mathrm{T}} = \mathrm{A}^{\mathrm{T}}\mathrm{A}$.

4. (a) Find the value of h for which

   $\begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = h \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   (b) Find the values of a and b for which

   $\begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -2 \begin{bmatrix} a \\ b \end{bmatrix}$

5. (a) Find a matrix **B** such that

   $\mathrm{B} \begin{bmatrix} 2 & 5 \\ -4 & 9 \end{bmatrix} = \begin{bmatrix} 12 & 30 \\ -24 & 54 \end{bmatrix}$

   (b) Find two 2×2 matrices **M** and **N** such that

   $\mathrm{MN} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ without any of the elements in **M** or **N** being zero.

9.1 Special matrices

The 2×2 matrix $\mathrm{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has the property that, for any 2×2 matrix **A**,

$$\boxed{\mathrm{IA} = \mathrm{AI} = \mathrm{A}}$$

In other words, multiplication by **I** (either pre-multiplication or post-multiplication) leaves the elements of **A** unchanged.

I is called the **identity matrix** and it is analogous to the real number 1 in ordinary multiplication.

The 2×2 matrix $Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is such that

$$Z + A = A + Z = A$$

and $$ZA = AZ = Z;$$

that is, **Z** leaves **A** unchanged under matrix addition, and itself remains unchanged under matrix multiplication. For obvious reasons, **Z** is called the **zero matrix**.

Next, although it is possible to define matrix multiplication meaningfully, there is no practical way of approaching division. However, in ordinary arithmetic, division can be approached as multiplication by reciprocals. For instance, the reciprocal of 2 is $\frac{1}{2}$, and 'multiplication by $\frac{1}{2}$' is the same as 'division by 2'. The equation $2x = 7$ can then be solved by multiplying both sides by $\frac{1}{2}$:

$$\tfrac{1}{2} \times 2x = \tfrac{1}{2} \times 7 \Rightarrow x = 3 \times 5$$

It is not necessary to have division defined as a process: instead, the use of the relations $\left(\frac{1}{2}\right) \times 2 = 1$ and $1 \times x = x$ suffices.

In matrix arithmetic we thus require, for a given matrix **A**, the matrix **B** for which,

$$AB = BA = I.$$

B is denoted by A^{-1} (just as $2^{-1} = \frac{1}{2}$) and is called the **inverse matrix** of **A**, giving

$$\boxed{AA^{-1} = A^{-1}A = I}$$

Activity 2 To find an inverse matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. You need to find matrix **B**, of the form $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ say, such that $AB = I$.

Calculate the product matrix **AB** and equate it, element by element, with the corresponding elements of **I**. This will give two pairs of simultaneous equations: two equations in e and g, and two more equations in f and h. Solve for e, f, g, h in terms of a, b, c, d and you will have found A^{-1} (i.e. **B**). Check that $BA = I$ also.

Summary

For $\mathsf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\mathsf{A}^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and $\mathsf{A}^{-1}\mathsf{A} = \mathsf{A}\mathsf{A}^{-1} = \mathrm{I}$.

The factor $(ad - bc)$ present in each term, is called the **determinant** of matrix **A**, and is a scalar (a real number), denoted $\det \mathsf{A}$.

If $ad = bc$, then $\frac{1}{ad - bc} = \frac{1}{0}$, which is not defined. In this case, A^{-1} does not exist and the matrix **A** is described as **singular** (non-invertible). If A^{-1} does exist the matrix **A** is described as being **non-singular** (invertible).

For $\mathsf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we write

$$\det \mathsf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

So a matrix **A** has an inverse if and only if $\det \mathsf{A} \neq 0$.

Example

Find the inverse of $\mathsf{X} = \begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix}$.

Hence solve the simultaneous equations

$$\begin{aligned} 2x + 4y &= 1 \\ 5x - y &= 8 \end{aligned}$$

Solution

$\det \mathsf{X} = 2 \times (-1) - 4 \times 5 = -22$.

So $$\mathsf{X}^{-1} = -\frac{1}{22} \begin{bmatrix} -1 & -4 \\ -5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{22} & \frac{2}{11} \\ \frac{5}{22} & -\frac{1}{11} \end{bmatrix}$$

The equations can be written in matrix form as

$$\begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix} \text{ or } \mathsf{X}\mathsf{u} = \mathsf{v}$$

where **u** and **v** are the column vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 8 \end{bmatrix}$ respectively.

(Pre-) Multiplying both sides by X^{-1} gives

$$\mathsf{X}^{-1}\mathsf{X}\mathsf{u} = \mathsf{X}^{-1}\mathsf{v}$$

$$\Rightarrow \quad \mathsf{I}\mathsf{u} = \mathsf{X}^{-1}\mathsf{v}$$

$$\Rightarrow \quad \mathsf{u} = \mathsf{X}^{-1}\mathsf{v}$$

Thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 1 & 4 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$= \frac{1}{22} \begin{bmatrix} 1+32 \\ 5-16 \end{bmatrix}$$

$$= \begin{bmatrix} 1\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

so that $x = \frac{3}{2}, \quad y = \frac{1}{2}$.

Exercise 9B

1. Evaluate the following determinants

   (a) $\begin{vmatrix} 12 & 5 \\ 27 & 11 \end{vmatrix}$ (b) $\begin{vmatrix} 36 & -9 \\ -4 & 1 \end{vmatrix}$

   (c) $\begin{vmatrix} x+2 & 4-x \\ x & 7 \end{vmatrix}$

2. Find the inverses of **A**, **B** and **C**, where

   $\mathsf{A} = \begin{bmatrix} 7 & 19 \\ 2 & 6 \end{bmatrix}$ $\mathsf{B} = \begin{bmatrix} 5 & \frac{1}{2} \\ 20 & 2 \end{bmatrix}$ $\mathsf{C} = \begin{bmatrix} a & -4 \\ 1 & 3 \end{bmatrix}$

3. Find all values of k for which the matrix

   $\mathsf{M} = \begin{bmatrix} 3 & k+2 \\ -k & k-2 \end{bmatrix}$ is singular.

4. By finding $\begin{bmatrix} 3 & 7 \\ 1 & 6 \end{bmatrix}^{-1}$ solve the equations

   $$3x + 7y = 9$$
   $$x + 6y = 5$$

5. Given $\mathsf{X} = \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}$, show that $\mathsf{X}^2 - 3\mathsf{X} + 8\mathsf{I} = \mathsf{Z}$.

   Deduce that $\mathsf{X}^{-1} = \frac{1}{8}(3\mathsf{I} - \mathsf{X})$.

   [Note: $3\mathsf{X} \equiv (3\mathsf{I})\mathsf{X}$, for instance.]

6. Take $\mathsf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathsf{B} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$. Prove that $\det(\mathsf{AB}) = \det \mathsf{A} \det \mathsf{B}$.

7. Show that all matrices of the form

   $$\begin{bmatrix} 6a+b & a \\ 3a & b \end{bmatrix}$$

   commute with

   $$\mathsf{A} = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}$$

   [A and B commute if $\mathsf{AB} = \mathsf{BA}$]

8. Given $\mathsf{A}=\begin{bmatrix}3 & 7\\ 1 & 2\end{bmatrix}$ and $\mathsf{B}=\begin{bmatrix}4 & -1\\ -3 & 1\end{bmatrix}$, find **AB** and $(\mathsf{AB})^{-1}$.

   (a) Show that $(\mathsf{AB})^{-1}\neq\mathsf{A}^{-1}\mathsf{B}^{-1}$.

   (b) Prove that $(\mathsf{AB})^{-1}=\mathsf{B}^{-1}\mathsf{A}^{-1}$ for all 2×2 non-singular matrices **A** and **B**.

9. $\mathsf{A}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, $\mathsf{B}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $\mathsf{C}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$, $\mathsf{D}=\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}$,

   $\mathsf{E}=\begin{bmatrix}k & 0\\ 0 & 1\end{bmatrix}$, $\mathsf{F}=\begin{bmatrix}1 & m\\ 0 & 1\end{bmatrix}$.

   Describe the effect on the rows of **A** of pre-multiplying **A** by (i) **B**, (ii) **C**, (iii) **D**, (iv) **E**, (v) **F**. (That is, **BA**, **CA**, etc).

10. (a) Show that the 2×2 matrix $\mathsf{M}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ is singular if and only if one row (or column) is a multiple of the other row (or column).

    (b) Prove that, if the matrix $(\mathsf{M}-\lambda\mathsf{I})$ is singular, where λ is some real (or complex) constant, then λ satisfies a certain quadratic equation, which you should find

9.2 Transformation matrices

Pre-multiplication of a 2×1 column vector by a 2×2 matrix results in a 2×1 column vector; for example,

$$\begin{bmatrix}3 & 4\\ -1 & 2\end{bmatrix}\begin{bmatrix}7\\ -1\end{bmatrix}=\begin{bmatrix}17\\ -9\end{bmatrix}$$

If the vector $\begin{bmatrix}7\\ -1\end{bmatrix}$ is thought of as a position vector (that is, representing the point with coordinates (7, –1), then the matrix has changed the point (7, –1) to the point (17, –9). Similarly, the matrix has an effect on each point of the plane. Calling the transformation **T**, this can be written

$$\mathsf{T}:\begin{bmatrix}x\\ y\end{bmatrix}\to\begin{bmatrix}x'\\ y'\end{bmatrix}.$$

T maps points (x, y) onto image points (x', y').

Using the above matrix,

$$\begin{aligned}\begin{bmatrix}x'\\ y'\end{bmatrix}&=\begin{bmatrix}3 & 4\\ -1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}\\ &=\begin{bmatrix}3x+4y\\ -x+2y\end{bmatrix}\end{aligned}$$

and the transformation can also be written in the form

$$\mathrm{T}: x' = 3x + 4y,\ y' = -x + 2y\,.$$

[The handling of either form may be required.]

Activity 3

You may need squared paper for this activity.

Express each of the following transformations in the form

$$x' = ax + by,\ y' = cx + dy$$

for some suitable values of the constants a, b, c and d (positive, zero or negative). Then re-write in matrix form as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

You may find it helpful at first to choose specific points (i.e. to choose some values for x and y).

(a) Reflection in the x-axis.

(b) Reflection in the y-axis.

(c) Reflection in the line $y = x$.

(d) Reflection in the line $y = -x$.

(e) Rotation through $90°$ (anticlockwise) about the origin.

(f) Rotation through $180°$ about the origin.

(g) Rotation through $-90°$ (i.e. $90°$ clockwise) about the origin.

(h) Enlargement with scale factor 5, centre the origin.

9.3 Invariancy and the basic transformations

An **invariant** (or fixed) **point** is one which is mapped onto itself; that is, it is its own image. An **invariant** (or fixed) **line** is a line all of whose points have image points also on this line. In the special case when all the points on a given line are invariant – in other words, they not only map onto other points on the line, but each maps onto itself – the line is called a **line of invariant points** (or a pointwise invariant line).

The invariant points and lines of a transformation are often its key features and, in most cases, help determine the nature of the transformation in question.

Example

Find the invariant points of the transformations defined by

(a) $x' = 1 - 2y, \quad y' = 2x - 3$

(b) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{21}{5} & \frac{8}{5} \\ \frac{8}{5} & \frac{9}{5} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Solution

(a) For invariant points, $x' = x$ and $y' = y$.

Thus

$x = 1 - 2y$ and $y = 2x - 3$

giving $\quad x = 1 - 2\{2x - 3\}$

$\Rightarrow \quad x = 1 - 4x + 6$

$\Rightarrow \quad 5x = 7$ and $x = \frac{7}{5}, y = -\frac{1}{5}$

The invariant point is $\left(\frac{7}{5}, -\frac{1}{5}\right)$.

(b) $x' = x, y' = y$ for invariant points

$\Rightarrow \quad x = \frac{21}{5}x + \frac{8}{5}y \quad$ and $\quad y = \frac{8}{5}x + \frac{9}{5}y$

$\Rightarrow \quad 5x = 21x + 8y \qquad 5y = 8x + 9y$

$\Rightarrow \quad y = -2x \qquad y = -2x$

and all points on the line $y = -2x$ are invariant.

Example

Find, in the form $y = mx + c$, the equations of all invariant lines of the transformation given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 7 & 24 \\ 24 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Solution

Firstly, note that if $y = mx + c$ is an invariant line, then all such points (x, y) on this line have image points (x', y') with $y' = mx' + c$ also.

Now $\quad x' = 7x + 24y = 7x + 24(mx + c) = (7 + 24m)x + 24c$

and $\quad y' = 24x - 7y = 24x - 7(mx + c) = (24 - 7m)x - 7c$

Therefore, as $y' = mx' + c,$ we can deduce that

$$(24 - 7m)x - 7c = m[(7 + 24m)x + 24c] + c$$

$$\Rightarrow \qquad 0 = \left(24m^2 + 14m - 24\right)x + (24m + 8)c$$

Since x is a variable taking any real value while m and c are constants taking specific values, this statement can **only** be true if the RHS is identically zero; whence

$$24m^2 + 14m - 24 = 0 \text{ and } (24m + 8)c = 0.$$

The first of these equations

$$\Rightarrow \qquad 2(4m - 3)(3m + 4) = 0$$

$$\Rightarrow \qquad m = \tfrac{3}{4} \text{ or } -\tfrac{4}{3}.$$

The second equation

$$\Rightarrow \qquad m = -\tfrac{1}{3} \text{ or } c = 0.$$

Clearly, then, $m \neq -\frac{1}{3}$ since in this case the coefficient of x, $24m^2 + 14m - 24$, would not then be zero.

There are, then, two cases:

(i) $m = \frac{3}{4}, c = 0$ giving the invariant line $y = \frac{3}{4}x$;

(ii) $m = -\frac{4}{3}, c = 0$ giving a second invariant line $y = -\frac{4}{3}x$.

Note that a transformation of the plane that can be represented by a 2×2 matrix must always include the origin as an invariant point, since

$$\mathsf{A}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ for any matrix } \mathbf{A}.$$

Therefore clearly not all transformations are matrix representable.

The six basic transformations, together with their respective characteristics and defining features, are given below. You will return to them again, and their possible matrix representations, in later sections.

Translation

Under a **translation** each point is moved a fixed distance in a given direction:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

The vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is called the **translation vector**, which completely defines the transformation. Distances and areas are preserved, and (provided that $\begin{bmatrix} a \\ b \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$) there are no invariant points. Thus a non-zero translation is not 2×2 matrix representable. Any line parallel to the translation vector is invariant.

Stretch

A **stretch** is defined parallel to a specified line or direction. Any line parallel to this direction is invariant, and there will be one line of invariant points perpendicular to this direction. Points of the plane are moved so that their distances from the line of invariant points are increased by a factor of k. Distances are, in general, not preserved and areas are increased by a factor of k.

Example

The transformation

$$\mathsf{T}: x' = x,\ y' = 2y$$

is a stretch of factor 2 in the direction of the y-axis. The line of invariant points is the x-axis.

Enlargement

Under an **enlargement** of factor k and centre C, each point P is moved k times further from point C, the single invariant point of the transformation, such that $\overrightarrow{\mathrm{CP}'} = k\overrightarrow{\mathrm{CP}}$. Any line through C is invariant. Distances are not preserved (unless $k = 1$); areas are increased by a factor of k^2.

Example

The transformation

$$\mathsf{T}: x' = 2x - 1,\ y' = 2y$$

is an enlargement of factor 2 with centre (1, 0). For each $(x,\ y)$,

$$(x',\ y') - (1,\ 0) = 2\big((x,\ y) - (1,\ 0)\big).$$

Projection

A trivial transformation whereby the whole plane is collapsed (projected) onto a single line or, in extreme cases, a single point.

Example

The transformation

$$\mathrm{T}: x' = 0,\ y' = y$$

projects the whole plane onto the y-axis.

Reflection

A **reflection** is defined by its axis or line of symmetry, i.e. the 'mirror' line. Each point P is mapped onto the point P' which is the mirror-image of P in the mirror line; i.e. such that PP' is perpendicular to the mirror and such that their distances from it are equal, with P and P' on opposite sides of the line. Thus all points on the axis are invariant, and any lines perpendicular to it are also invariant. A reflection preserves distances and is area-preserving.

Example

The transformation

$$\mathrm{T}: x' = 2 - x,\ y' = y$$

is a reflection in the line $x = 1$.

Rotation

A **rotation** is defined by its centre, C, the single invariant point and an angle of rotation (note that anticlockwise is taken to be the positive direction: thus a clockwise rotation of $90°$ can simply be described as a rotation of $270°$). Points P are mapped to points P' such that $\mathrm{CP'} = \mathrm{CP}$ and $\mathrm{P\hat{C}P'} = \theta$, the angle of rotation. There are no invariant lines, with the exception of the case of a $180°$ (or multiples of $180°$) rotation, when all lines through C will be invariant: note that a $180°$ rotation is identical to an enlargement, scale factor –1, and centre C. Distances and areas are preserved.

Example

The transformation

$$\mathrm{T}: x' = -y,\ y' = x$$

is a rotation of $90°$ with centre the origin.

Exercise 9C

1. Find the invariant points of the following transformations:

   (a) $x' = 2x + 1,\ y' = 3 - 2y$;

   (b) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + y - 1 \\ 1 - 2x \end{bmatrix}$;

   (c) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 8 & -15 \\ -7 & 16 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$;

   (d) $x' = x,\ y' = 2 - y$

   (e) $5x' = 9x + 8y - 12$
   $5y' = 8x + 21y - 24$

   (f) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

   (g) $x' = 5x + 6y - 1$,
   $y' = 2x + 4y - 4$

2. A transformation of the plane is given by the matrix

   $$\mathsf{A} = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$$

   Find the invariant lines of the transformation.

3. A transformation of the plane, **T**, is given by

   $$x' = 5 - 2y,\ y' = 4 - 2x.$$

   Find the invariant point and the invariant lines of **T**.

4. A transformation of the plane is given by the matrix

   $$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

   Find all the invariant lines of the transformation.

5. Determine any fixed lines of the transformation given by

   $$x' = \tfrac{1}{2}x + \tfrac{1}{4}y,\ y' = x + \tfrac{1}{2}y$$

   Describe the transformation geometrically.

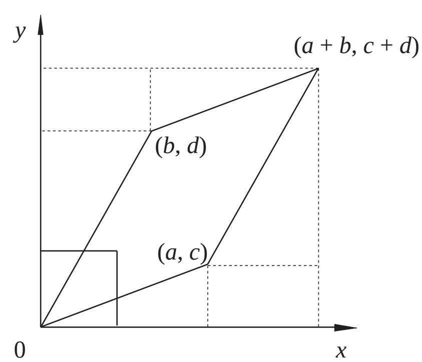
9.4 The determinant

If a plane transformation **T** is represented by a 2×2 matrix **A**, then $\det \mathsf{A}$, the determinant of **A**, represents the scale factor of the area increase produced by **T**.

To illustrate this, consider the effect of the transformation given by the matrix

$$\mathsf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

on the unit square, with vertices at (0, 0), (1, 0), (0, 1) and (1, 1), and of area 1.



The images of the square's vertices are on the diagram at (0, 0), (a, c), (b, d) and $(a + b, c + d)$ respectively. You will see that the square is transformed into a parallelogram.

By drawing in rectangles and triangles it is easily shown that the area of this parallelogram is

$$(a + b)(c + d) - 2\left[bc + \tfrac{1}{2}bd + \tfrac{1}{2}ac\right]$$

$$= ad - bc$$

$$= \det \mathsf{A}$$

Note that if the cyclic order of the vertices of a plane figure is reversed (from clockwise to anticlockwise, or vice versa) then the area factor is actually $-\det \mathbf{A}$. Strictly speaking then, you should take $|-\det \mathbf{A}|$, the absolute value of the determinant.

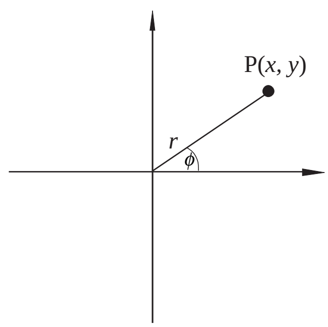
Activity 4

Write down the numerical value of some determinants of matrices which represent

(a) a stretch; (b) an enlargement; (c) a projection;
(d) a reflection; (e) a rotation.

9.5 The rotation and reflection matrices

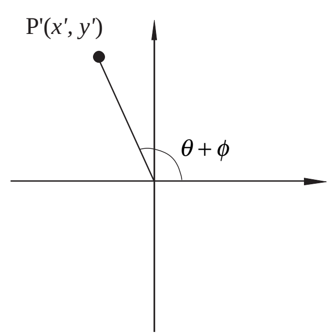
Rotation about the origin



Let O be the origin (0, 0) and consider a point $\mathrm{P}(x, y)$ in the plane, with $\mathrm{OP} = r$ and angle between OP and the x-axis equal to ϕ.

[Note that $r = \sqrt{x^2 + y^2}$ and $x = r\cos\phi,\ y = r\sin\phi$.]

Let the image of P after an anticlockwise rotation about O through an angle θ be $\mathrm{P}'(x', y')$.



Then, $$\begin{aligned} x' &= r\cos(\theta+\phi) \\ &= r(\cos\theta\cos\phi - \sin\theta\sin\phi) \\ &= r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ &= x\cos\theta - y\sin\theta \end{aligned}$$

Also, $$\begin{aligned} y' &= r\sin(\theta+\phi) \\ &= r(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ &= r\cos\phi\sin\theta + r\sin\phi\cos\theta \\ &= x\sin\theta + y\cos\theta \end{aligned}$$

Thus $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For a clockwise rotation through θ about O either

(i) replace θ by $-\theta$ and use $\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$ to get the matrix

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix};$$

or

(ii) find $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1}$

$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$ (which it clearly should be, since rotating plane figures leaves area unchanged), and the above matrix is again obtained.

Check the following results with your answers to parts (e) – (g) in Activity 3.

Putting $\theta = 90°$ gives the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\theta = 180°$ gives the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

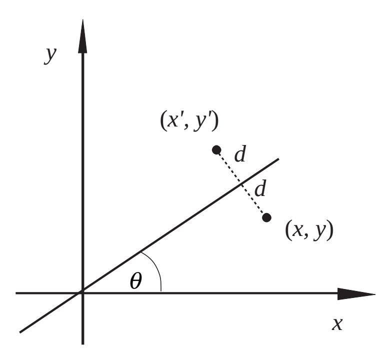
$\theta = -90°$ or $270°$ gives $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Reflection in a line through the origin

Consider the point P(x, y), a distance d from the line $y = x\tan\theta$ (where θ is the angle between the line and the positive x-axis), and its image P'(x', y') after reflection in this line.

Method 1

Using the formula for the distance of a point P(x, y) from the line $mx - y = 0$, where $m = \tan\theta$, the distance



$$d = \frac{mx - y}{\sqrt{1 + m^2}}$$

$$= \frac{x\tan\theta - y}{\sqrt{1 + \tan^2\theta}}$$

$$= \frac{x\tan\theta - y}{\sec\theta} \qquad \text{since } 1+\tan^2\theta = \sec^2\theta$$

$$= x\sin\theta - y\cos\theta .$$

Then,

$$x' = x - 2d\sin\theta$$

$$= x - 2\sin\theta\left(x\sin\theta - y\cos\theta\right)$$

$$= x\left(1-2\sin^2\theta\right) + y\times 2\sin\theta\cos\theta$$

$$= x\cos 2\theta + y\sin 2\theta .$$

Also, $\quad y' = y + 2d\cos\theta$

$$= y + 2\cos\theta\left(x\sin\theta - y\cos\theta\right)$$

$$= x\times 2\sin\theta\cos\theta + y\left(1-2\cos^2\theta\right)$$

$$= x\sin 2\theta - y\cos 2\theta .$$

Thus

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

This method is somewhat clumsy, and certainly not in the spirit of work on transformations. The following approach is both useful and powerful, requiring a few pre-requisites which cannot be quickly deduced. You should make note of it.

Method 2

Write the reflection as the composition of three simple transformations, as follows

(i) rotate the plane through θ clockwise about O, so that the line is mapped onto the x-axis. Call this T_1;

(ii) reflect in this new x-axis. Call this T_2;

(iii) rotate back through θ anticlockwise about O, so that the line is now in its original position. Call this T_3.

T_1 has matrix $\quad \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$,

T_2 has matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (see Activity 3 part (a)),and

T_3 is given by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Now, the application of these three simple transformations correspond to pre-multiplication of the position vector $\begin{bmatrix} x \\ y \end{bmatrix}$ by these three matrices in turn in the order T_1, T_2, T_3.

Applying T_1 first gives $\qquad \mathsf{T}_1 \begin{bmatrix} x \\ y \end{bmatrix}$

Applying T_2 to this gives $\qquad \mathsf{T}_2 \left\{ \mathsf{T}_1 \begin{bmatrix} x \\ y \end{bmatrix} \right\} = \left(\mathsf{T}_2 \mathsf{T}_1 \right) \begin{bmatrix} x \\ y \end{bmatrix}$

Applying T_3 then gives $\qquad \mathsf{T}_3 \left\{ \left(\mathsf{T}_2 \mathsf{T}_1 \right) \begin{bmatrix} x \\ y \end{bmatrix} \right\} = \left(\mathsf{T}_3 \mathsf{T}_2 \mathsf{T}_1 \right) \begin{bmatrix} x \\ y \end{bmatrix}$.

In this way, you will see that it is the product $\mathsf{T}_3\mathsf{T}_2\mathsf{T}_1$ that is required and not the product $\mathsf{T}_1\mathsf{T}_2\mathsf{T}_3$: it is the order of application and not the usual left-to-right writing which is important.

The transformation is then given by the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & \cos\theta\sin\theta + \sin\theta\cos\theta \\ \sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta - \cos^2\theta \end{bmatrix}$$

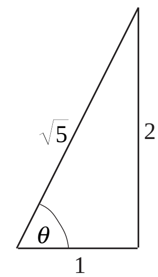
$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \text{ as before.}$$

By substituting in values of θ, you can check the matrices obtained in Activity 3 parts (a) – (d). (Parts (a) and (b) are so straightforward they hardly need checking: in fact, we used (a) in Method 2 in order to establish the more general result.) The line $y = x$ is given by $\theta = 45°$ and $y = -x$ by $\theta = 135°$ or $-45°$.

Example

Find a matrix which represents a reflection in the line $y = 2x$.

Solution



In the diagram shown opposite, $\tan\theta = 2 = \frac{2}{1}$. Drawing a right-angled triangle with angle θ and sides 2 and 1, Pythagoras' theorem gives the hypotenuse $\sqrt{5}$ whence $\cos\theta = \frac{1}{\sqrt{5}}$ and

$\sin\theta = \frac{2}{\sqrt{5}}$.

Then $\qquad \cos 2\theta = 2\cos^2\theta - 1 = 2\times\left(\frac{1}{\sqrt{5}}\right)^2 - 1 = -\frac{3}{5}$

and $\qquad \sin 2\theta = 2\sin\theta\cos\theta = 2\times\frac{2}{\sqrt{5}}\times\frac{1}{\sqrt{5}} = \frac{4}{5}$.

The required matrix is thus

$$\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

[Alternatively, the well known $t = \tan\theta$ (or $t = \tan\frac{1}{2}A$) substitution, gives $\cos 2\theta = \frac{1-t^2}{1+t^2} = -\frac{3}{5}$ and $\sin 2\theta = \frac{2t}{1+t^2} = \frac{4}{5}$ immediately.]

Example

A plane transformation has matrix $\begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$. Describe this transformation geometrically.

Solution

The matrix has the form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, where $\cos\theta = -\frac{\sqrt{3}}{2}$ and $\sin\theta = -\frac{1}{2}$, in which case $\theta = 210°$. The matrix then represents a rotation about O through $210°$ anticlockwise.

Exercise 9D

1. Describe geometrically the plane transformations with matrices

   (a) $\begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$; (b) $\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix}$;

   (c) $\begin{bmatrix} -\frac{1}{9} & \frac{4\sqrt{5}}{9} \\ \frac{4\sqrt{5}}{9} & \frac{1}{9} \end{bmatrix}$; (d) $\begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix}$;

   (e) $\begin{bmatrix} -0.28 & -0.96 \\ -0.96 & 0.28 \end{bmatrix}$.

2. Write down the matrices representing the following transformations:

   (a) Reflection in the line through the origin which makes an angle of $60°$ with the positive x-axis;

   (b) Rotation through $135°$ anticlockwise about O;

   (c) Rotation through $\cos^{-1}\left(\frac{1}{3}\right)$, clockwise about O;

   (d) Reflection in the line $y = -3x$.

9.6 Stretch and enlargement matrices

A stretch parallel to the x-axis, **scale factor** k, has matrix

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} kx \\ y \end{bmatrix}$ and points (x,y) are transformed into points with the same y-coordinate, but with x-coordinate k times further from the y-axis than they were originally.

Similarly, a stretch parallel to the y-axis, **scale factor** k is represented by the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}.$$

An enlargement, centre O and scale factor k, has matrix

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = k\mathrm{I},$$

so that $x' = kx$ and $y' = ky$.

9.7 Translations and standard forms

Having obtained matrix forms for some of the elementary plane transformations, it is now possible to extend the range of simple techniques to more complex forms of these transformations. The method employed is essentially the same as that used in the second method for deriving the general matrix for a reflection; namely, treating more complicated transformations as a succession of simpler ones for which results can be quoted without proof.

Rotation not about the origin

A rotation through θ anticlockwise about the point (a, b) can be built up in the following way:

(i) translate the plane by $\begin{bmatrix} -a \\ -b \end{bmatrix}$ so that the centre of rotation is now at the origin;

(ii) rotate about this origin through θ anticlockwise;

(iii) translate the plane back by $\begin{bmatrix} a \\ b \end{bmatrix}$ to its original position.

This leads to

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix},$$

or

$$\begin{bmatrix} x'-a \\ y'-b \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}.$$

The second form is more instructive, since it maintains the notion of a straightforward rotation with the point (a, b) as centre. This is the standard form of this (ostensibly) more complicated transformation.

Reflection in a line not through the origin

For a reflection in $y = x\tan\theta + c$,

(i) translate the plane by $\begin{bmatrix} 0 \\ -c \end{bmatrix}$ so that the crossing point of the line on the y-axis is mapped onto the origin;

(ii) reflect in this line through O;

(iii) translate back by $\begin{bmatrix} 0 \\ c \end{bmatrix}$, to get the standard form

$$\begin{bmatrix} x' \\ y'-c \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x \\ y-c \end{bmatrix}.$$

The one case that needs to be examined separately is reflection in a vertical line, $x = a$, say:

(i) translate the plane by $\begin{bmatrix} -a \\ 0 \end{bmatrix}$ so that the line becomes the *y*-axis;

(ii) reflect in the *y*-axis;

(iii) translate by $\begin{bmatrix} a \\ 0 \end{bmatrix}$, to get

$$\begin{bmatrix} x'-a \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x-a \\ y \end{bmatrix}.$$

Enlargement, centre not the origin

An enlargement, centre (a, b) and scale factor k, has standard form

$$\begin{bmatrix} x'-a \\ y'-b \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

in the same way as above.

9.8 Linear transformations of the plane

If a point P(x, y) is mapped onto point P'(x', y') by a transformation T such that

$$x' = ax + by + c, \quad y' = dx + ey + f,$$

for constants a, b, c, d, e and f, then T is said to be a **linear plane transformation**.

Such a transformation can be written algebraically, as above, or in matrix form (possibly in the translated 'standard' form discussed above),

$$\begin{bmatrix} x'-\alpha \\ y'-\beta \end{bmatrix} = \mathtt{M} \begin{bmatrix} x-\alpha \\ y-\beta \end{bmatrix},$$

where **M** is a 2×2 matrix.

Example

Express algebraically the transformation which consists of a reflection in the line $x+y=1$.

Solution

Line is $y=-x+1$ with gradient $\tan 135°$.

A reflection in this line can then be written as

$$\begin{bmatrix} x' \\ y'-1 \end{bmatrix} = \begin{bmatrix} \cos 270° & \sin 270° \\ \sin 270° & -\cos 270° \end{bmatrix} \begin{bmatrix} x \\ y-1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y-1 \end{bmatrix}$$

$$= \begin{bmatrix} -y+1 \\ -x \end{bmatrix}$$

and this is written algebraically as $x'=1-y, \quad y'=1-x$.

Example

A transformation **T** has algebraic form

$$x'=\frac{3}{5}x-\frac{4}{5}y+6, \quad y'=\frac{4}{5}x+\frac{3}{5}y-2.$$

Give a full geometrical description of **T**.

Solution

Firstly, find any invariant (fixed) points of **T**, given by $x'=x, \quad y'=y$: i.e.

$$x=\frac{3}{5}x-\frac{4}{5}y+6,$$

$$y=\frac{4}{5}x+\frac{3}{5}y-2.$$

Solving simultaneously gives $(x,y)=(5,5)$.

T can be then written in standard matrix form

$$\begin{bmatrix} x'-5 \\ y'-5 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} x-5 \\ y-5 \end{bmatrix},$$

and the matrix is clearly that of a rotation, with

$$\cos\theta = \frac{3}{5}, \quad \sin\theta = \frac{4}{5},$$

giving $\qquad \theta = \cos^{-1}\left(\frac{3}{5}\right) \quad (\approx 53.13°).$

Hence **T** is a rotation through an angle of $\cos^{-1}\left(\frac{3}{5}\right)$ anticlockwise about (5, 5).

Example

Show that the transformation $5x' = 21x + 8y, \quad 5y' = 8x + 9y$ is a stretch in a fixed direction leaving every point of a certain line invariant. Find this line and the amount of the stretch.

Solution

Firstly, $x' = x,\ y' = y$ for invariant points, giving

$$5x = 21x + 8y \text{ and } 5y = 8x + 9y.$$

Both equations give $y = -2x$ and so this is the line of invariant points.

Next, consider all possible lines perpendicular to $y = -2x$.

These will be of the form $y = \frac{1}{2}x + c$ (for constant c). Now for any point on the line $y = \frac{1}{2}x + c$,

$$x' = \frac{21}{5}x + \frac{8}{5}y = \frac{21}{5}x + \frac{8}{5}\left(\frac{1}{2}x + c\right) = 5x + \frac{8}{5}c,$$

and $\qquad y' = \frac{8}{5}x + \frac{9}{5}y = \frac{8}{5}x + \frac{9}{5}\left(\frac{1}{2}x + c\right) = \frac{5x}{2} + \frac{9}{5}c,$

whence $y' = \frac{1}{2}x' + c$ also. Hence all lines perpendicular to $y = -2x$ are invariant and the transformation is a stretch.

Finally, $\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathsf{A}\begin{bmatrix} x \\ y \end{bmatrix}$ where $\mathsf{A} = \begin{bmatrix} \frac{21}{5} & \frac{8}{5} \\ \frac{8}{5} & \frac{9}{5} \end{bmatrix}$ and

$\det \mathsf{A} = \begin{vmatrix} \frac{21}{5} & \frac{8}{5} \\ \frac{8}{5} & \frac{9}{5} \end{vmatrix} = \frac{189}{25} - \frac{64}{25} = 5$, so the stretch has scale factor 5.

Exercise 9E

1. Determine the standard matrix forms of
   (a) an enlargement, centre $(a, 0)$ and scale factor k;
   (b) a rotation of $45°$ anticlockwise about the point (1,0).

2. Give a full geometrical description of the plane transformations having matrices **A** and **B**, where

$$A = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \text{ and } B = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}.$$

   Determine the product matrices **AB** and **BA**. Give also a full geometrical description of the plane transformations having matrices **AB** and **BA**.

3. (a) The transformation **T** of the plane is defined by $x' = x,\ y' = 2 - y$. Describe this transformation geometrically.
   (b) Express algebraically the transformation **S** which is a clockwise rotation through $45°$ about the origin.

4. Describe geometrically the single transformations given algebraically by
   (a) $5x' = -3x + 4y + 12$, $5y' = -4x - 3y + 16$;
   (b) $5x' = 3x - 4y + 6$, $5y' = -4x - 3y + 12$;
   (c) $5x' = 9x + 8y - 12$, $5y' = 8x + 21y - 24$.

5. Express each of the following transformations in the form $x' = ax + by + p,\ y' = cx + dy + q$, giving the values of a, b, c, d, p and q in each case:
   (a) a reflection in the line $x + y = 0$;
   (b) a reflection in the line $x - y = 2$;
   (c) a rotation through $90°$ anticlockwise about the point (2,–1);
   (d) a rotation through $60°$ clockwise about the point (3,2).

6. Find the matrix which represents a stretch, scale factor k, parallel to the line $y = x\tan\theta$.

9.9 Composition of transformations

If a transformation of the plane T_1 is followed by a second plane transformation T_2 then the result may itself be represented by a single transformation **T** which is the composition of T_1 and T_2 taken in that order. This is written $T = T_2 T_1$.

Note, again, that the order of application is from the right: this is in order to be consistent with the pre-multiplication order of the matrices that represent these transformations.

Example (non-matrix composition)

The transformation **T** is the composition of transformations T_1 and T_2, taken in that order, where

$$T_1:\quad x' = 2x + 1,\ \ y' = 3 - 2y$$

and

$$T_2:\quad x' = x + y - 1,\ \ y' = 1 - 2x$$

Express **T** algebraically.

Solution

As T_2 is the 'second stage' transformation, write,

$$\mathsf{T}_2\text{:} \quad x'' = x' + y' - 1, \quad y'' = 1 - 2x' \ ,$$

where x' and y' represent the intermediate stage, after T_1 has been applied.

Thus $\quad \mathsf{T}_2\,\mathsf{T}_1\text{:} \quad x'' = (2x+1) + (3-2y) - 1 = 2x - 2y + 3$

$$y'' = 1 - 2(2x+1) = -4x - 1$$

and we can write

$$\mathsf{T}\text{:} \quad x' = 2x - 2y + 3, \quad y' = -4x - 1 .$$

The inverse of a transformation **T** can be thought of as that transformation **S** for which $\mathsf{TST} = \mathsf{TS} = \mathtt{I}$, the identity transformation represented by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. **S** is then denoted by T^{-1}.

Example (non-matrix inversion)

Find T^{-1}, when $\mathsf{T}\text{:} \quad x' = x + y - 1, \quad y' = 2x - y + 4$.

Interchanging x for x' and y for y' gives

$$x = x' + y' - 1, \quad y = 2x' - y' + 4 .$$

These can be treated as simultaneous equations and solved for x', y' in terms of x, y.

Adding $\quad x + y = 3x' + 3 \Rightarrow x' = \frac{1}{3}x + \frac{1}{3}y - 1;$

substituting back

$$x = \left(\frac{1}{3}x + \frac{1}{3}y - 1\right) + y' - 1 \Rightarrow y' = \frac{2}{3}x - \frac{1}{3}y + 2$$

Therefore, $\quad \mathsf{T}^{-1}\text{:} \quad x' = \frac{1}{3}x + \frac{1}{3}y - 1, \quad y' = \frac{2}{3}x - \frac{1}{3}y + 2 .$

Finding a composite transformation when its constituent parts are given in matrix form is easy, simply involving the multiplication of the respective matrices which represent those constituents. Inverses, similarly, require the finding of an inverse matrix, provided that the **T**'s matrix is non-singular: only a projection has a singular matrix.

Example

The transformation **T** is defined by $\mathsf{T} = (\mathsf{CBA})$, where **A**, **B** and **C** are the transformations:

A: a rotation about O through $30°$ anticlockwise;

B: a reflection in the line through O that makes an angle of $120°$ with the x-axis;

C: a rotation about O through $210°$ anticlockwise.

Give the complete geometrical description of **T**.

Solution

A has matrix $\begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$,

while **B** has matrix

$$\begin{bmatrix} \cos 240° & \sin 240° \\ \sin 240° & -\cos 240° \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$$

and **C** has matrix

$$\begin{bmatrix} \cos 210° & -\sin 210° \\ \sin 210° & \cos 210° \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -\sqrt{3} & 1 \\ -1 & -\sqrt{3} \end{bmatrix}.$$

Thus $\mathsf{T} = \mathsf{CBA}$ has matrix

$$\frac{1}{8}\begin{bmatrix} -\sqrt{3} & 1 \\ -1 & -\sqrt{3} \end{bmatrix} \begin{bmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \cos 60° & \sin 60° \\ \sin 60° & -\cos 60° \end{bmatrix}$$

and **T** is a reflection in $y = x \tan 30°$; i.e. $y = \dfrac{x}{\sqrt{3}}$.

Example

Describe fully the transformation **T** given algebraically by

$$x' = 8x - 15y - 37$$
$$y' = 15x + 8y - 1$$

Solution

For invariant points, set $x' = x$ and $y' = y$, giving

$$0 = 7x - 15y - 37$$

and $$0 = 15x + 7y - 1.$$

Solving simultaneously gives the single invariant point $(x, y) = (1, -2)$. **T** can then be written as

$$\begin{bmatrix} x'-1 \\ y'+2 \end{bmatrix} = \begin{bmatrix} 8 & -15 \\ 15 & 8 \end{bmatrix} \begin{bmatrix} x-1 \\ y+2 \end{bmatrix}$$

$$= 17\begin{bmatrix} \frac{8}{17} & -\frac{15}{17} \\ \frac{15}{17} & \frac{8}{17} \end{bmatrix} \begin{bmatrix} x-1 \\ y+2 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 0 \\ 0 & 17 \end{bmatrix} \begin{bmatrix} \frac{8}{17} & -\frac{15}{17} \\ \frac{15}{17} & \frac{8}{17} \end{bmatrix} \begin{bmatrix} x-1 \\ y+2 \end{bmatrix}$$

and **T** is a rotation through $\cos^{-1}\left(\frac{8}{17}\right)$ $(\approx 61.93°)$ anticlockwise about the point (1,–2) together with an enlargement, centre (1,–2) and scale factor 17.

Note that the two matrices involved here commute (remember: if $\mathbf{AB} = \mathbf{BA}$ then **A** and **B** are said to commute) so that the two components of this composite transformation may be taken in either order. This rotation-and-enlargement having the same centre is often referred to as a **spiral similarity**.

Exercise 9F

1. Write down the 2×2 matrices corresponding to:
   (a) a reflection in the line through O at $60°$ to the positive *x*-axis,
   (b) a rotation anticlockwise about O through $90°$, and
   (c) a reflection in the line through O at $120°$ to the positive *x*-axis.

   Describe geometrically the resultant
   (i) of (a) and (b);
   (ii) of (a), (b) and (c); taken in the given order.

2. Give a full geometrical description of the transformation T_1 given by

$$T_1 = x' = 5 - y, \; y' = x - 1.$$

Express in algebraic form T_2, which is a reflection in the line $y = x + 2$.

Hence express $T_3 = T_2 T_1 T_2$ algebraically and give a full geometrical description of T_3.

3. Prove that a reflection in the line $y = x \tan\theta$ followed by a reflection in the line $y = x \tan\phi$ is equivalent to a rotation. Describe this rotation completely.

4. Transformations **A**, **B** and **C** are defined algebraically by

A: $x' = -y, \quad y' = -x$,

B: $x' = y + 2, \quad y' = x - 2$,

C: $x' = -y + 1, \quad y' = x - 3$.

Transformations **U** and **V** are defined by $U = BCA$ and $V = AC^{-1}BC$. Express **U** and **V** algebraically. Show that **V** has no invariant points, and that **U** has a single invariant point. Give a simple geometrical description of **U**.

5. Given T_1: $x' = 1 - 2y, \quad y = 2x - 3$ and
T_2: $x' = 1 - y, \quad y = 1 - x$

define T_3 algebraically, where T_3 is T_1 followed by T_2. Show that T_3 may be expressed as a reflection in the line $x = \frac{4}{3}$ followed by an enlargement, and give the centre and scale factor of this enlargement.

9.10 Eigenvalues and eigenvectors

Consider the equation

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

where A is a 2×2 matrix and λ is a scalar. This is equivalent to

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \lambda I \begin{bmatrix} x \\ y \end{bmatrix} \text{ or } (A - \lambda I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Now in the case when the matrix $(A - \lambda I)$ is non-singular (i.e. its inverse exists) we can pre-multiply this equation by $(A - \lambda I)^{-1}$ to deduce that

$$\begin{bmatrix} x \\ y \end{bmatrix} = (A - \lambda I)^{-1}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In other words, if det $(A - \lambda I) \neq 0$ then the only vector $\begin{bmatrix} x \\ y \end{bmatrix}$ which satisfies the equation $A\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$ is the zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The other cases, when det $(\mathsf{A}-\lambda\,\mathsf{I})=0$, are more interesting. The equation $\mathsf{A}\begin{bmatrix}x\\y\end{bmatrix}=\lambda\begin{bmatrix}x\\y\end{bmatrix}$ has a non-trivial solution and it it easy to check that if $\begin{bmatrix}x\\y\end{bmatrix}$ is a solution so is any scalar multiple of $\begin{bmatrix}x\\y\end{bmatrix}$. In other words the solutions form a line through the origin, and indeed an invariant line of the transformation represented by **A**.

(In the special case when $\lambda=1$, $\begin{bmatrix}x'\\y'\end{bmatrix}=\mathsf{A}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$, and such vectors $\begin{bmatrix}x\\y\end{bmatrix}$ give a line of invariant points. For all other values of λ, the line will simply be an invariant line.)

Example

The matrix $\mathsf{A}=\begin{bmatrix}\frac{3}{5} & \frac{4}{5}\\ \frac{4}{5} & -\frac{3}{5}\end{bmatrix}$ represents a reflection in the line $y=x\tan\theta$, with $\cos 2\theta=\dfrac{3}{5}$, $\sin 2\theta=\dfrac{4}{5}$, giving $\tan\theta=\dfrac{1}{2}$. Then the line $y=\dfrac{1}{2}x$ is a line of invariant points under this transformation, and any line perpendicular to it (with gradient –2) is an invariant line.

To show how the equation

$$(\mathsf{A}-\lambda\,\mathsf{I})\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

can be used to find the invariate lines, first calculate those λ for which the matrix $(\mathsf{A}-\lambda\,\mathsf{I})$ is singular; i.e. det $(\mathsf{A}-\lambda\,\mathsf{I})=0$.

This gives

$$\begin{vmatrix}\frac{3}{5}-\lambda & \frac{4}{5}\\ \frac{4}{5} & -\frac{3}{5}-\lambda\end{vmatrix}=0$$

$$\Rightarrow\quad \left(\frac{3}{5}-\lambda\right)\left(-\frac{3}{5}-\lambda\right)-\frac{4}{5}\times\frac{4}{5}=0$$

$$\Rightarrow\quad \lambda^2-1=0$$

$$\Rightarrow\quad \lambda=\pm 1.$$

Remember that you are looking for solutions (x, y) to the equation

$$\begin{bmatrix} \frac{3}{5}-\lambda & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5}-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Substituting back, in turn, these two values of λ:

$$\lambda = 1 \quad \Rightarrow \quad \left.\begin{aligned} -\frac{2}{5}x + \frac{4}{5}y = 0 \\ \frac{4}{5}x - \frac{8}{5}y = 0 \end{aligned}\right\} \quad \Rightarrow \quad x = 2y,$$

and the solution vectors corresponding to $\lambda = 1$ are of the form

$\alpha\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for real α.

$$\lambda = -1 \quad \Rightarrow \quad \left.\begin{aligned} \frac{8}{5}x + \frac{4}{5}y = 0 \\ \frac{4}{5}x + \frac{2}{5}y = 0 \end{aligned}\right\} \quad \Rightarrow \quad y = -2x,$$

and the solution vectors corresponding to $\lambda = -1$ have the form

$\beta\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ for real β.

The results, then, for this reflection are that $y = \frac{1}{2}x$ is a line of invariant points (signified by $\lambda = 1$), and that $y = -2x$ is an invariant line. (Here the value of λ, namely –1, indicates that the image points of this line are the same distance from $y = \frac{1}{2}x$ as their originals, but in the opposite direction). So this method has indeed led to the invariant lines which pass through the origin.

Because the solutions to this type of matrix-vector equation provide some of the characteristics of the associated transformation, the λ's are called **characteristic values**, or **eigenvalues** (from the German word eigenschaft) of the matrix. Their associated solution vectors are called **characteristic vectors**, or **eigenvectors**. Each eigenvalue has a corresponding set of eigenvectors.

Example

Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathsf{A} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}.$$

Give a full geometrical description of the plane transformation determined by **A**.

Solution

Eigenvalues are given by det $(\mathrm{A} - \lambda \mathrm{I}) = 0$. Hence

$$\begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (3-\lambda)(3-\lambda) - 1 = 0$$

$$\Rightarrow \quad \lambda^2 - 6\lambda + 8 = 0$$

$$\Rightarrow \quad (\lambda - 2)(\lambda - 4) = 0$$

and $\lambda = 2$ or $\lambda = 4$.

Now $\quad \lambda = 2 \quad \Rightarrow \quad \left.\begin{array}{r} x - y = 0 \\ -x + y = 0 \end{array}\right\} \quad \Rightarrow \quad y = x$

and $\lambda = 2$ has eigenvectors $\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Also $\quad \lambda = 4 \quad \Rightarrow \quad \left.\begin{array}{r} -x - y = 0 \\ -x - y = 0 \end{array}\right\} \quad \Rightarrow \quad y = -x$

and $\lambda = 4$ has eigenvectors $\beta \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

The invariant lines (through the origin) of the transformation are $y = x$ and $y = -x$ (in fact there are no others).

Notice that these lines are perpendicular to each other. For $y = x$, $\lambda = 2$ means that points on this line are moved to points also on the line, twice as far away from the origin and on the same side of O. For $y = -x$, $\lambda = 4$ has a similar significance.

The transformation represented by **A** is seen to be the composition of a stretch parallel to $y = x$, scale factor 2, together with a stretch parallel to $y = -x$, factor 4, in either order.

The composition of two stretches in perpendicular directions is known as a **two-way stretch**.

In general the equation

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

is called the **characteristic equation** of the matrix $\mathsf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

In the 2×2 case the equation is the quadratic

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

The sum, $a + d$, of the entries in the leading diagonal of **A** is known as its **trace**, and so the characteristic equation is

$$\boxed{\lambda^2 - (\text{trace } \mathsf{A})\lambda + \det\ \mathsf{A} = 0}$$

Example

Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathsf{A} = \begin{bmatrix} 2 & 1 \\ -9 & 8 \end{bmatrix}.$$

Determine the coordinates of the invariant point of the transformation given algebraically by

$$x' = 2x + y - 1, \quad y' = -9x + 8y - 3\,.$$

Deduce the equations of any invariant lines of this transformation.

Solution

The characteristic equation of the matrix

$$\mathsf{A} = \begin{bmatrix} 2 & 1 \\ -9 & 8 \end{bmatrix}$$

is

$$\begin{vmatrix} 2-\lambda & 1 \\ -9 & 8-\lambda \end{vmatrix} = 0$$

or

$$\lambda^2 - \underset{\substack{\uparrow \\ \text{trace } \mathsf{A}}}{10}\lambda \;+\; \underset{\substack{\uparrow \\ \det \mathsf{A}}}{25} = 0.$$

This has root $\lambda = 5$ (twice) and

$$\lambda = 5 \quad \Rightarrow \quad \left.\begin{aligned} -3x + y &= 0 \\ -9x + 3y &= 0 \end{aligned}\right\} \quad \Rightarrow \quad y = 3x$$

Thus **A** has a single eigenvalue $\lambda = 5$, with eigenvectors $\alpha\begin{bmatrix}1\\3\end{bmatrix}$.

For invariant points $x' = x,\ y' = y$ whence $x = 2x + y - 1$, $y = -9x + 8y - 3$.

Solving simultaneously gives $(x, y) = \left(\frac{1}{4}, \frac{3}{4}\right)$.

Since the transformation can be written in the form

$$\begin{bmatrix}x' - \frac{1}{4}\\ y' - \frac{3}{4}\end{bmatrix} = \mathsf{A}\begin{bmatrix}x - \frac{1}{4}\\ y - \frac{3}{4}\end{bmatrix},$$

there is a single invariant line $\left(y - \frac{3}{4}\right) = 3\left(x - \frac{1}{4}\right)$. (This arises from the '$y = 3x$' derived from $\lambda = 5$, but the 'y' and the 'x' are translated by $\begin{bmatrix}\frac{1}{4}\\ \frac{3}{4}\end{bmatrix}$; in this instance this has given rise to the same line but this will not, in general, prove to be the case!)

Exercise 9G

1. Show that the transformation represented by the matrix $\mathsf{A} = \begin{bmatrix}2 & -2\\ -1 & 3\end{bmatrix}$ has a line of invariant points and an invariant line. Explain the distinction between the two.

2. Find the eigenvalue(s) and eigenvector(s) of the following matrices:

   (a) $\mathsf{A} = \begin{bmatrix}5 & -8\\ 2 & -3\end{bmatrix}$; (b) $\mathsf{B} = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$;

   (c) $\mathsf{C} = \begin{bmatrix}1 & 2\\ 2 & -2\end{bmatrix}$ (d) $\mathsf{D} = \begin{bmatrix}2 & 4\\ 5 & 3\end{bmatrix}$

3. Find a 2×2 matrix **M** which maps **A**(0, 2) into **A**'(1, 3) and leaves **B**(1 ,1) invariant. Show that this matrix has just one eigenvalue.

4. A linear transformation of the plane is given by

   $$\begin{bmatrix}x'\\ y'\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}.$$

   Show that the condition for the transformation to have invariant points (other than the origin) is

   $1 - a - d + ad - bc = 0$.

5. (a) Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix}0 & -2\\ -2 & 0\end{bmatrix}$.

   (b) A transformation of the plane is given by $x' = 2 - 2y,\quad y' = 7 - 2x$. Find the invariant point and give the cartesian equations of the two invariant lines. Hence give a full geometrical description of the transformation.

6. Find the eigenvalues and eigenvectors of the matrices $\mathsf{A} = \begin{bmatrix}\frac{3}{5} & \frac{4}{5}\\ \frac{4}{5} & -\frac{3}{5}\end{bmatrix}$ and $\mathsf{B} = \begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$. Show that the plane transformations represented by **A** and **B** have the same line of invariant points and state its cartesian equation.

7. The eigenvalues of the matrix

   $$\mathsf{T} = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \quad (b > 0, c \geq 0)$$

   are equal. Prove that $a = d$ and $c = 0$. If **T** maps the point (2, –1) into the (1, 2), determine the elements of **T**.

9.11 Miscellaneous Exercises

1. By writing the following in standard matrix form describe the transformations of the plane given by

   (a) $x' = 3x + 4$, $y' = 3y + 2$

   (b) $x' = \frac{3}{5}x + \frac{4}{5}y - \frac{6}{5}$, $y' = \frac{4}{5}x - \frac{3}{5}y + \frac{12}{5}$

   (c) $5x' = 3x - 4y + 8$, $5y' = 4x + 3y - 6$

   (d) $5x' = 13x - 4y - 4$, $5y' = -4x + 7y + 2$

2. Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$. Deduce the equations of the invariant lines of the transformation defined by

   $x' = 2x - y, \quad y' = -4x + 2y$.

   Explain why one of these lines has an image which is not a line at all. Describe this transformation geometrically.

3. A reflection in the line $y = x - 1$ is followed by an anticlockwise rotation of $90°$ about the point (–1, 1). Express the resultant transformation algebraically.

   Show that this resultant has an invariant line, and give the equation of this line. Describe the resultant transformation in relation to this line.

4. Find the eigenvalues and corresponding eigenvectors of the matrices

   (a) $\mathsf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$; (b) $\mathsf{B} = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$.

   In each case deduce the equations of any invariant lines of the transformations which they represent.

   A plane figure F, with area 1 square unit, is transformed by each of these transformations in turn. Write down the area of the image of F in each case.

   Describe geometrically the two transformations.

5. The linear transformation **T** leaves the line $y = x \tan \frac{\pi}{6}$ invariant and increases perpendicular distances from that line by a factor of 3.

   (a) Determine the 2×2 matrix **A** representing **T**.

   (b) Write down the eigenvalues and corresponding eigenvectors of **A**.

6. A rotation about the point (–c, 0) through an angle θ is followed by a rotation about the point (c, 0) through an angle $-\theta$.

   Show that the resultant of the two rotations is a translation and give the x and y components of this translation in terms of c and θ.

7. A transformation **T** is defined algebraically by $x' = y - \sqrt{2}$, $y' = x + \sqrt{2}$. Find the invariant points of T and hence give its full geometrical description.

8. A plane transformation **T** consists of a reflection in the line $y = 2x + 1$ followed by a rotation through $\frac{\pi}{2}$ anticlockwise about the point (2,–1).

   (a) Express **T** algebraically.

   (b) Show that **T** can also be obtained by a translation followed by a reflection in a line through the origin, giving full details of the translation and reflection.

9. A linear transformation **T** of the plane has one eigenvector $\begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$ with corresponding eigenvalue 1, and one eigenvector $\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ with eigenvalue -4.

   Give a geometric description of **T**, and find the matrix **A** representing **T**.

10. An anticlockwise rotation about (0,1) through an angle θ is followed by a clockwise rotation about the point (2, 0) through an angle θ. Show that the resultant is a translation, stating its vector in terms of θ.

11. Express each of the following transformations of the plane in the form

    $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} p \\ q \end{bmatrix},$$

    giving the values of a, b, c, d, p, q in each case:

    (a) T_1: reflection in the line $y = x$.

    (b) T_2: reflection in the line $x + y = 3$.

    (c) T_3: rotation through $90°$ anticlockwise about the point (1,4).

    (d) $\mathsf{T}_4 = \mathsf{T}_2 \mathsf{T}_3 \mathsf{T}_1$, that is T_1 followed by T_3 followed by T_2.

    Show that T_4 has a single invariant point and give a simple geometrical description of T_4.

    (Oxford)

12. (a) The transformation T_1 is represented by the matrix $A = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$.

    (i) Find the eigenvalues and eigenvectors of **A**.

    (ii) State the equation of the line of invariant points and describe the transformation T_1 geometrically.

    (b) Find the 2×2 matrix **B** which represents the transformation T_2, a rotation about the origin through $\tan^{-1}\frac{3}{4}$ anticlockwise.

    (c) The transformation $T_3 = T_1 T_2 T_1$. Find the 2×2 matrix **C** which represents T_3 and hence describe T_3 geometrically. (Oxford)

13. Explain the difference between an invariant (fixed) line and a line of invariant points.
    A transformation of the plane is given by the equations $x' = 7 - 2y, \quad y' = 5 - 2x$.

    (a) The images of A(0, 2), B(2, 2) and C(0, 4) are A', B', C' respectively. Find the ratio of area A'B'C': area ABC.

    (b) Calculate the coordinates of the invariant point.

    (c) Determine the equations of the invariant lines of the transformation.

    (d) Give a full geometrical description of the transformation. (Oxford)

14. A plane transformation **T** is defined by
    $x' = 7x - 24y + 12, \quad y' = -24x - 7y + 56$.

    Show that **T** has just one invariant point P, and find its coordinates.

    Prove that **T** is a reflection in a line through P together with an enlargement centre P.

    State the scale factor of the enlargement and determine the equation of the line of reflection. (Oxford)

15. The transformation T_1 has a line of fixed points $y = 3x$ and perpendicular distances from this line are multiplied by a factor of 4. T_1 is represented by the 2×2 matrix A. Write down the values of the constants λ_1, λ_2 where $A\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $A\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \lambda_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Hence, or otherwise, show that

    $A = \begin{bmatrix} 3.7 & -0.9 \\ -0.9 & 1.3 \end{bmatrix}$.

    The transformation T_2 is given by

    $x' = 3.7x - 0.9y + 1.8$,
    $y' = -0.9x + 1.3y - 0.6$.

    Find the line of fixed points of T_2 and describe T_2 geometrically.

    Given that T_1^{-1} is the inverse transformation of T_1, express $T_3 = T_1^{-1} T_2$ in the form

    $x' = ax + by + c$,

    $y' = dx + ey + f$,

    and describe T_3 geometrically. (Oxford)