1 Indices

1.1 Multiplication and Division

It is important to be able to multiply and divide simple numbers quickly. Learning the tables listed below will help you to do this.

$0\times0=0$					
$1\times0=0$	$1\times1=1$				
$2\times0=0$	$2\times1=2$	$2\times2=4$			
$3\times0=0$	$3\times1=3$	$3\times2=6$	$3\times3=9$		
$4\times0=0$	$4\times1=4$	$4\times2=8$	$4\times3=12$	$4\times4=16$	
$5\times0=0$	$5\times1=5$	$5\times2=10$	$5\times3=15$	$5\times4=20$	$5\times5=25$
$6\times0=0$	$6\times1=6$	$6\times2=12$	$6\times3=18$	$6\times4=24$	$6\times5=30$
$7\times0=0$	$7\times1=7$	$7\times2=14$	$7\times3=21$	$7\times4=28$	$7\times5=35$
$8\times0=0$	$8\times1=8$	$8\times2=16$	$8\times3=24$	$8\times4=32$	$8\times5=40$
$9\times0=0$	$9\times1=9$	$9\times2=18$	$9\times3=27$	$9\times4=36$	$9\times5=45$
$10\times0=0$	$10\times1=10$	$10\times2=20$	$10\times3=30$	$10\times4=40$	$10\times5=50$

$6\times6=36$				
$7\times6=42$	$7\times7=49$			
$8\times6=48$	$8\times7=56$	$8\times8=64$		
$9\times6=54$	$9\times7=63$	$9\times8=72$	$9\times9=81$	
$10\times6=60$	$10\times7=70$	$10\times8=80$	$10\times9=90$	$10\times10=100$

Not all the tables are listed because $4\times8=8\times4$, etc.

Worked Example 1

Jai's mum buys 3 packets of kit-kat each week. There are 7 bars in each packet. How many bars does she buy each week?

Solution

$$\begin{aligned}\text{Number of bars} &= 3\times7\\ &= 21.\end{aligned}$$

Worked Example 2

Claire, Lauren, Rachel and Emma have 32 sweets to share equally among themselves. How many sweets do they get each?

Solution

$$\text{Number of sweets each} = 32\div4 = 8.$$

Another way of writing this is:

$$\text{Number of sweets each} = \frac{32}{4} = 8.$$

Exercises

1. Do each calculation below.

(a)	$4 \times 5 =$	(b)	$3 \times 7 =$	(c)	$4 \times 9 =$
(d)	$6 \times 7 =$	(e)	$5 \times 9 =$	(f)	$6 \times 3 =$
(g)	$4 \times 7 =$	(h)	$7 \times 7 =$	(i)	$5 \times 8 =$
(j)	$40 \div 5 =$	(k)	$45 \div 5 =$	(l)	$24 \div 6 =$
(m)	$56 \div 8 =$	(n)	$28 \div 4 =$	(o)	$63 \div 7 =$
(p)	$8 \times 0 =$	(q)	$0 \times 7 =$	(r)	$0 \times 0 =$

2. Find all the missing numbers.

(a)	$4 \times ? = 12$	(b)	$? \times 5 = 35$	(c)	$6 \times ? = 24$
(d)	$7 \times ? = 56$	(e)	$8 \times ? = 24$	(f)	$5 \times ? = 20$
(g)	$? \times 8 = 72$	(h)	$? \times 2 = 14$	(i)	$9 \times ? = 27$
(j)	$56 \div ? = 8$	(k)	$36 \div ? = 9$	(l)	$40 \div ? = 8$
(m)	$\frac{18}{?} = 9$	(n)	$\frac{20}{?} = 5$	(o)	$\frac{63}{?} = 9$
(p)	$9 \times 0 = ?$	(q)	$4 \times ? = 0$	(r)	$0 \times ? = 0$

3, A milk crate has 6 rows of 4 bottles. How many bottles are in the crate?

4. Football teams score 3 points for every match they win. How many points would be scored for 9 wins?

5. Chocolate bars are sold in packs that contain 4 bars. How many bars are there in

(a)	4 packs	(b)	7 packs	(c)	8 packs?

6. A class of 24 children is to be divided into teams. How many children would be in each team if the number of teams was:

(a)	4	(b)	8	(c)	6?

7. A tube of smarties contains 40 sweets. Five children share the sweets equally among themselves. How many sweets do they each get?

8. How many days are there in:

(a)	5 weeks	(b)	a fortnight	(c)	6 weeks?

9. Daniel puts 5p into his money box every day for 2 weeks.
Joel puts 8p into his money box every day for a week.

   (a) How much money do they each put into their money boxes?

   (b) How much more money has Daniel put in?

10. A booklet is made using 8 sheets of paper. How many sheets of paper would be needed for:

    (a) 10 booklets? (b) 8 booklets (c) 5 booklets?

11. A five-a-side football team wins a £30 prize. How much would each player get if the money was shared equally?

12. A company buys 36 tyres. How many of its cars can be fitted with new tyres if:

    (a) spare tyres are not replaced,

    (b) spare tyres are also replaced.

13. In a tournament, points are awarded as below.

    Win : 3 points
    Draw : 1 point
    Lose : 0 points

    Find the total points scored by each team in the table below.

Team	Games Won	Games Drawn	Games Lost
A	1	4	7
B	6	3	3
C	3	5	4
D	2	8	2

14. 68 sweets are shared equally among 9 children. Each child is given a whole number of sweets.

    (a) How many sweets does each child get?

    (b) How many sweets are left over after the sharing has been completed? *(MEG)*

1.2 Squares, Cubes, Square Roots and Cube Roots

When a number is multiplied by itself, we say that the number has been *squared*.

For example, 3 squared means $3 \times 3 = 9$. This is written as $3^2 = 9$.
We could also say that 9 is the square of 3.

When a number is *cubed* it is written down 3 times and multiplied.

For example 2 cubed means $2 \times 2 \times 2 = 8$. This is written as $2^3 = 8$.
We could also say that 8 is the cube of 2.

Sometimes the reverse process is needed to answer questions such as:

What number squared gives 25?

The answer would be 5. We say that 5 is the square root of 25, or write $\sqrt{25} = 5$.

Another question might be:

What number cubed gives 8?

The answer would be 2. We would say that the cube root of 8 is 2.

We could also write $\sqrt[3]{8} = 2$.

Worked Example 1

Find

(a) 8^2 (b) 4^2 (c) 5^3.

Use your answers to find

(d) $\sqrt{64}$ (e) $\sqrt{16}$ (f) $\sqrt[3]{125}$

Solution

(a) $8^2 = 8 \times 8 = 64$

(b) $4^2 = 4 \times 4 = 16$

(c) $5^3 = 5 \times 5 \times 5 = 125$

(d) $\sqrt{64} = 8$ because $8^2 = 64$

(e) $\sqrt{16} = 4$ because $4^2 = 16$

(f) $\sqrt[3]{125} = 5$ because $5^3 = 125$

Exercises

1. Find

(a) 5^2 (b) 6^2 (c) 1^2 (d) 7^2

Use your answers to find

(e) $\sqrt{36}$ (f) $\sqrt{1}$ (g) $\sqrt{49}$ (h) $\sqrt{25}$

2. Find

(a) 3^3 (b) 4^3 (c) 6^3 (d) 10^3

Use your answers to find

(e) $\sqrt[3]{27}$ (f) $\sqrt[3]{1000}$ (g) $\sqrt[3]{216}$ (h) $\sqrt[3]{64}$

3. Find

(a) 10^2 (b) 2^2 (c) 4^2 (d) 7^2

(e) 8^2 (f) 9^2 (g) 1^3 (h) 7^3

(i) 8^3 (j) 0^2 (k) 0^3 (l) 2^3

4. Find

(a) $\sqrt{100}$ (b) $\sqrt{4}$ (c) $\sqrt{81}$ (d) $\sqrt{64}$

(e) $\sqrt{16}$ (f) $\sqrt{9}$

5. Use a calculator to find

(a) 12^2 (b) 11^2 (c) 15^3 (d) 13^3

(e) 13^2 (f) 15^2 (g) 20^2 (h) 11^3

Without a calculator, find

(i) $\sqrt{121}$ (j) $\sqrt{400}$ (k) $\sqrt{169}$ (l) $\sqrt{225}$

(m) $\sqrt[3]{3375}$ (n) $\sqrt[3]{2197}$ (o) $\sqrt{144}$ (p) $\sqrt[3]{1331}$

6. Find

(a) $6^2 + 4^2$ (b) $3^2 - 2^2$ (c) $10^2 + 4^2$ (d) $3^2 + 4^2$

(e) $5^2 - 3^2$ (f) $4^3 + 2^3$ (g) $1^3 + 10^3$ (h) $6^2 + 8^2$

1.3 Index Notation

Index notation is a useful way of writing expressions like

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

in a shorter format. The above could be written with index notation as 2^7.
The small number, 7, is called the *index* or *power*.

Worked Example 1

Find (a) 3^4 (b) 4^5 (c) 7^1

Solution

(a) $3^4 = 3 \times 3 \times 3 \times 3$
$= 81$

(b) $4^5 = 4 \times 4 \times 4 \times 4 \times 4$
$= 1024$

(c) $7^1 = 7$

Worked Example 2

Find the missing number.

(a) $3^4 \times 3^6 = 3^{?}$ (b) $4^2 \times 4^3 = 4^{?}$ (c) $\dfrac{5^7}{5^4} = 5^{?}$

Solution

(a) $3^4 \times 3^6 = (3 \times 3 \times 3 \times 3) \times (3 \times 3 \times 3 \times 3 \times 3 \times 3)$
$= 3^{10}$

(b) $4^2 \times 4^3 = (4 \times 4) \times (4 \times 4 \times 4)$

$= 4^5$

(c) $\dfrac{5^7}{5^4} = \dfrac{5 \times 5 \times 5 \times \cancel{5}^{1} \times \cancel{5}^{1} \times \cancel{5}^{1} \times \cancel{5}^{1}}{\cancel{5}_{1} \times \cancel{5}_{1} \times \cancel{5}_{1} \times \cancel{5}_{1}}$

$= 5 \times 5 \times 5$

$= 5^3$

Note

$a^m \times a^n = a^{m+n}$ and $\dfrac{a^n}{a^m} = a^{n-m}$

These rules apply whenever index notation is used.

Using these rules,

$\dfrac{a^3}{a^3} = a^{3-3} = a^0$ or $\dfrac{a^3}{a^3} = \dfrac{\cancel{a}^{1} \times \cancel{a}^{1} \times \cancel{a}^{1}}{\cancel{a}_{1} \times \cancel{a}_{1} \times \cancel{a}_{1}} = 1$

So $a^0 = 1$

Worked Example 3

Find

(a) $(2^3)^4$ (b) $(3^2)^3$

Solution

(a) $(2^3)^4 = (2 \times 2 \times 2) \times (2 \times 2 \times 2) \times (2 \times 2 \times 2) \times (2 \times 2 \times 2)$

$= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

$= 2^{12}$

(b) $(3^2)^3 = (3 \times 3) \times (3 \times 3) \times (3 \times 3)$

$= 3 \times 3 \times 3 \times 3 \times 3 \times 3$

$= 3^6$

Note

$(a^m)^n = a^{m \times n}$

Exercises

1. Write each of the following using index notation.

(a) $4 \times 4 \times 4 \times 4 \times 4$ (b) $3 \times 3 \times 3$

(c) $6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6$ (d) $7 \times 7 \times 7 \times 7$

(e) $18 \times 18 \times 18$ (f) 19×19

(g) $4 \times 4 \times 4 \times 4 \times 4 \times 4$ (h) $7 \times 7 \times 7 \times 7 \times 7$

(i) $10 \times 10 \times 10 \times 10 \times 10 \times 10$ (j) $100 \times 100 \times 100 \times 100 \times 100$

2. Find the value of each of the following.

(a) 3^4 (b) 5^4 (c) 7^4 (d) 10^4

(e) 5^0 (f) 3^6 (g) 2^7 (h) 2^1

(i) 8^4 (j) 4^1 (k) 3^0 (l) 5^2

3. Fill in the missing numbers.

(a) $2^7 \times 2^4 = 2^?$ (b) $3^4 \times 3^5 = 3^?$ (c) $3^6 \times 3^7 = 3^?$

(d) $4^? \times 4^2 = 4^7$ (e) $5^? \times 5^2 = 5^6$ (f) $5^4 \times 5^? = 5^9$

(g) $?^2 \times 4^4 = 4^6$ (h) $5^7 \div 5^4 = 5^?$ (i) $3^4 \div 3^2 = 3^?$

(j) $7^{14} \div 7^{10} = 7^?$ (k) $17^5 \div 17^? = 17^3$ (l) $9^7 \div 9^? = 9^3$

(m) $4^6 \times 4^? = 4^{11}$ (n) $4^? \div 4^6 = 4^{10}$ (o) $3^? \times 3^2 = 3^8$

(p) $3^6 \div 3^6 = ?$ (q) $3^7 \div 3^6 = ?$ (r) $3^0 \times 3^? = 3^5$

(s) $3^0 \times 3^7 = 3^?$ (t) $4^1 \times 4^? = 4^8$ (u) $5^2 \times 5^? = 5^2$

4. Fill in the missing numbers.

(a) $4 = 2^?$ (b) $8 = 2^?$ (c) $16 = 2^?$

(d) $64 = 2^?$ (e) $27 = 3^?$ (f) $25 = 5^?$

(g) $64 = 4^?$ (h) $81 = 3^?$ (i) $125 = ?^3$

5. Simplify the following expressions, giving your answer in index notation.

(a) $3^7 \times 3^6 =$ (b) $2 \times 2^7 =$ (c) $4^5 \times 4^6 =$

(d) $3^6 \times 3^4 =$ (e) $2^4 \times 2^5 =$ (f) $2^6 \times 2^4 =$

(g) $3^7 \div 3^2 =$ (h) $3 \times 3^6 =$ (i) $3^6 \div 3 =$

(j) $\dfrac{8^{12}}{8^2} =$ (k) $\dfrac{7^6}{7^3} =$ (l) $\dfrac{9^2}{9^0} =$

(m) $4 \times 2^2 =$ (n) $\dfrac{2^5}{4} =$ (o) $\dfrac{2^6}{8} =$

6. Fill in the missing powers.

(a) $8 = 2^{?}$ (b) $1000 = 10^{?}$ (c) $16 = 2^{?}$

(d) $27 = 3^{?}$ (e) $81 = 3^{?}$ (f) $10\,000 = 10^{?}$

(g) $625 = 5^{?}$ (h) $64 = 4^{?}$ (i) $1296 = 6^{?}$

(j) $1 = 2^{?}$ (k) $36 = 6^{?}$ (l) $1 = 5^{?}$

7. Simplify the following, giving your answers in index form.

(a) $\left(2^3\right)^2 =$ (b) $\left(3^2\right)^2 =$ (c) $\left(6^2\right)^3 =$

(d) $\left(5^3\right)^2 =$ (e) $\left(2^2\right)^4 =$ (f) $\left(4^2\right)^3 =$

(g) $\left(3^2\right)^4 =$ (h) $\left(5^2\right)^4 =$ (i) $\left(3^3\right)^2 =$

8. Fill in the missing numbers.

(a) $\left(2^2\right)^4 = 2^{?}$ (b) $\left(2^{?}\right)^3 = 2^{12}$ (c) $\left(3^2\right)^5 = ?^{10}$

(d) $\left(5^{?}\right)^4 = 5^{12}$ (e) $\left(10^5\right)^{?} = 10^{15}$ (f) $\left(7^5\right)^{?} = 7^{20}$

9. Simplify each of the following, giving your answer in index notation.

(a) $3^2 \times 3^0 \times 3^4 =$ (b) $2^6 \times 2^7 \times 2 =$ (c) $5^2 \times 5^7 \times 5^3 =$

(d) $\dfrac{7^2 \times 7^4}{7^3} =$ (e) $\dfrac{7^4 \times 7^5}{7^2 \times 7^3} =$ (f) $\dfrac{2^3 \times 2^8}{2^3 \times 2} =$

(g) $\dfrac{3^2 \times 3^3}{3^5} =$ (h) $\dfrac{4^7 \times 4^8}{4^5 \times 4^9} =$ (i) $\dfrac{2^3 \times 2^0}{2^2} =$

10. Simplify each of the following expressions.

(a) $a^3 \times a^2 =$ (b) $a^4 \times a^6 =$ (c) $x^2 \times x^7 =$

(d) $x^4 \div x^2 =$ (e) $y^3 \times y^0 =$ (f) $p^7 \div p^4 =$

(g) $q^6 \div q^3 =$ (h) $x^7 \times x =$ (i) $b^4 \div b =$

(j) $\dfrac{b^6}{b^0} =$ (k) $\dfrac{c^7}{c^4} =$ (l) $\dfrac{x^8}{x^3} =$

(m) $\dfrac{y^3}{y} =$ (n) $\dfrac{x^4}{x^4} =$ (o) $x^2 \times x^3 \times x^3 =$

(p) $\dfrac{p^2 \times p^7}{p^5} =$ (q) $\dfrac{x^{10}}{x^2 \times x^5} =$ (r) $\dfrac{y^3 \times y^7}{y^2 \times y^4} =$

(s) $\dfrac{x^2 \times x^3}{x^5} =$ (t) $\dfrac{x^7 \times x}{x^3 \times x^4} =$ (u) $\dfrac{x^8 \times x^4}{x^0} =$

(v) $\left(x^2\right)^4 =$ (w) $\left(x^3\right)^5 =$ (x) $\left(x^2 \times x^7\right)^6 =$

11. 243 can be written as 3^5.

Find the values of p and q in the following:

(a) $64 = 4^p$ (b) $5^q = 1$

(SEG)

12. Express as simply as possible:

$$\frac{4x^2 \times 6x^5}{12x^3}$$

(MEG)

1.4 Factors

A factor of a number will divide exactly into it.

Worked Example 1

List all the factors of 20.

Solution

The factors of 20 are:

1, 2, 4, 5, 10, 20

These are all numbers that divide exactly into 20.

Worked Example 2

Write the number 12 as the product of two factors in as many ways as possible.

Solution

$12 = 1 \times 12$ $12 = 4 \times 3$

$12 = 2 \times 6$ $12 = 6 \times 2$

$12 = 3 \times 4$ $12 = 12 \times 1$

Exercises

1. List the factors of these numbers.

(a)	14	(b)	27	(c)	6	(d)	15
(e)	18	(f)	25	(g)	40	(h)	100
(i)	45	(j)	50	(k)	36	(l)	28

2. Write each number below as the product of two factors in as many ways as possible.

(a)	10	(b)	8	(c)	7	(d)	9
(e)	16	(f)	22	(g)	11	(h)	24

3. Fill in the missing numbers.

(a) $32 = 4 \times 2 \times ?$ (b) $45 = ? \times 3 \times 5$

(c) $27 = 3 \times 3 \times ?$ (d) $40 = 5 \times ? \times 2$

(e) $50 = 5 \times 2 \times ?$ (f) $88 = 11 \times 2 \times ?$

(g) $66 = 2 \times 3 \times ?$ (h) $21 = ? \times 3 \times 7$

4. Here is a Bingo card.

6		10		20		9	
	3		8		17		15
2		24		55		4	

(a) Circle those numbers that 2 will divide into exactly.

(b) Cross out those numbers that 5 will divide into exactly.

(LON)

5. 20 21 22 23 24 25 26 27 29

(a) In the row of numbers above:

(i) *circle* all numbers divisible by 2, e.g. (20)

(ii) *cross out* all numbers divisible by 3, e.g. ~~24~~

(iii) *underline* all numbers divisible by 5. e.g. 25

(b) Describe the numbers which are not circled, crossed out or underlined.

(MEG)

6. A pattern of counting numbers is shown.

14, 15, 16, 17, 18, 19, 20, ...

(a) (i) Which of these numbers is a square number?

(ii) Which of these numbers is a multiple of nine?

The pattern is continued.

(b) (i) What is the next square number?

(ii) What is the next number that is a multiple of nine?

(SEG)

Investigation

Han Sin, a Chinese general, devised a method to count the number of soldiers that he had. First, he ordered his soldiers to form groups of 3, followed by groups of 5 and then groups of 7. In each case he noted down the remainder. Using the three remainders, he was able to calculate the exact number of soldiers he had without doing the actual counting.
Do you know how he did it?

1.5 Prime Factors

Any number can be written as the product of a number of prime factors. For example,

$$20 = 2^2 \times 5$$

or

$$180 = 2^2 \times 3^2 \times 5.$$

Note

A *prime number* is a number that can be divided exactly by only 1 and itself.

The first few prime numbers are 2, 3, 5, 7, 11, ...

Worked Example 1

Write the number 276 as a product of prime numbers.

Solution

Write 276 as a product of two factors:

$$276 = 2 \times 138$$

But $138 = 2 \times 69$ so $276 = 2 \times 2 \times 69$

But $69 = 3 \times 23$ so $276 = 2 \times 2 \times 3 \times 23$

This expression contains only prime numbers, so

$$276 = 2^2 \times 3 \times 23 .$$

This is called the *product of prime factors*.

Worked Example 2

(a) Write the numbers 660 and 470 as the product of prime factors.

(b) Find the largest common factor that will divide into both 660 and 470.

Solution

(a)
$$\begin{aligned} 660 &= 2 \times 330 \\ &= 2 \times 2 \times 165 \\ &= 2 \times 2 \times 3 \times 55 \\ &= 2 \times 2 \times 3 \times 5 \times 11 \end{aligned}$$

So as a product of prime factors,

$$660 = 2^2 \times 3 \times 5 \times 11.$$

$$\begin{aligned} 470 &= 2 \times 235 \\ &= 2 \times 5 \times 47 \end{aligned}$$

So as a product of prime factors,

$$470 = 2 \times 5 \times 47.$$

(b) To find the largest common factor that will divide into both 660 and 470, look at the factors common to each of the products of primes.

The numbers that appear in both are 2 and 5, so the largest number that will divide into both 660 and 470 is $2 \times 5 = 10$.

This number is called the *highest common factor* or *HCF*.

Exercises

1. Which of the following are prime numbers?

   1, 2, 3, 5, 7, 9, 13, 15, 18, 19, 21, 23, 25

2. Which numbers between 50 and 60 are prime numbers?

3. Write each number below as a product of prime factors.

(a)	10	(b)	42	(c)	68
(d)	168	(e)	250	(f)	270
(g)	429	(h)	825	(i)	1001

4. (a) Express 32 and 56 as the product of prime factors.

   (b) By comparing the answers to (a) find the HCF of 32 and 56.

5. Find the highest common factors of each pair of numbers below.

(a)	36, 42	(b)	30, 42	(c)	45, 105
(d)	42, 50	(e)	50, 80	(f)	70, 315
(g)	216, 240	(h)	156, 234	(i)	735, 1617

6. (a) Express each of the following numbers as the product of prime factors:

   45, 99, 135.

   (b) By considering the products of the prime factors, find the highest common factor of

   (i) 45 and 99 (ii) 99 and 135 (iii) 45 and 135

   (c) What is the highest common factor of all three numbers?

7. Find the highest common factor (HCF) for each set of three numbers given below.

(a)	20, 35, 105	(b)	90, 225, 405	(c)	16, 24, 56
(d)	200, 210, 220	(e)	72, 168, 312	(f)	330, 450, 630
(g)	216, 324, 432	(h)	660, 572, 528	(i)	1008, 1260, 1764

Just for Fun

You open a book. Two pages face you. If the product of the two page numbers is 3 192, what are the two page numbers?

1.6 Further Index Notation

Indices can be negative or fractions. The rules below explain how to use these types of indices.

$$a^{-1} = \frac{1}{a}$$ This is called the *reciprocal* of a.

$$a^{-n} = \frac{1}{a^n}$$

$$a^{\frac{1}{2}} = \sqrt{a}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Worked Example 1

Find:

(a) 2^{-4} (b) 3^{-2} (c) 5^{-1}

(d) $4^{\frac{1}{2}}$ (e) $8^{\frac{1}{3}}$ (d) $9^{\frac{3}{2}}$

Solution

(a) $2^{-4} = \frac{1}{2^4}$

$= \frac{1}{2 \times 2 \times 2 \times 2}$

$= \frac{1}{16}$

(b) $3^{-2} = \frac{1}{3^2}$

$= \frac{1}{3 \times 3}$

$= \frac{1}{9}$

(c) $5^{-1} = \frac{1}{5}$

(d) $4^{\frac{1}{2}} = \sqrt{4}$

$= 2$

(e) $8^{\frac{1}{3}} = \sqrt[3]{8}$

$= 2$

(f) $9^{\frac{3}{2}} = \left(9^{\frac{1}{2}}\right)^3$

$= 3^3$

$= 3 \times 3 \times 3$

$= 27$

Worked Example 2

Find

(a) $2^{-5} \times 2^6$ (b) $m^2 \times m^{-4}$ (c) $\frac{3^{-7}}{3^2}$

(d) $\left(2^8 \times 2^6\right)^{\frac{1}{2}}$ (e) $\left(a^2 \times b^{-2}\right)^{-1}$ (f) $\left(\frac{m^2}{a}\right)^{-2}$

Solution

(a) $2^{-5} \times 2^{6} = 2^{-5+6}$
$= 2^{1}$
$= 2$

(b) $m^{2} \times m^{-4} = m^{2-4}$
$= m^{-2}$
$= \dfrac{1}{m^{2}}$

(c) $\dfrac{3^{-7}}{3^{2}} = 3^{-7-2}$
$= 3^{-9}$
$= \dfrac{1}{3^{9}}$

(d) $\left(2^{8} \times 2^{6}\right)^{\frac{1}{2}} = \left(2^{8+6}\right)^{\frac{1}{2}}$
$= \left(2^{14}\right)^{\frac{1}{2}}$
$= 2^{14 \times \frac{1}{2}}$
$= 2^{7}$

(e) $\left(a^{2} \times b^{-2}\right)^{-1} = a^{-2} \times b^{2}$
$= \dfrac{b^{2}}{a^{2}}$

(f) $\left(\dfrac{m^{2}}{a}\right)^{-2} = \left(m^{2}a^{-1}\right)^{-2}$
$= m^{-4}a^{2}$
$= \dfrac{a^{2}}{m^{4}}$

Exercises

1. Find as fractions that do not involve indices, *without* using a calculator:

(a) $4^{-2} =$ (b) $2^{-3} =$ (c) $6^{-1} =$

(d) $7^{-1} =$ (e) $9^{\frac{1}{2}} =$ (f) $64^{\frac{1}{2}} =$

(g) $16^{\frac{1}{4}} =$ (h) $27^{\frac{1}{3}} =$ (i) $1^{\frac{1}{3}} =$

(j) $5^{-2} =$ (k) $16^{\frac{3}{4}} =$ (l) $4^{\frac{5}{2}} =$

(m) $9^{\frac{7}{2}} =$ (n) $25^{\frac{3}{2}} =$ (o) $8^{-\frac{1}{3}} =$

2. Complete the missing numbers, *without* using a calculator.

(a) $3^{?} = \dfrac{1}{81}$ (b) $2^{?} = \dfrac{1}{2}$ (c) $5^{?} = \dfrac{1}{125}$

(d) $36^{?} = 6$ (e) $36^{?} = \dfrac{1}{6}$ (f) $7^{?} = 49$

(g) $7^{?} = 343$ (h) $17^{?} = \dfrac{1}{17}$ (i) $125^{?} = 5$

(j) $\dfrac{1}{2} = 2^{?}$ (k) $\dfrac{1}{4} = 2^{?}$ (l) $\dfrac{1}{100} = 10^{?}$

(m) $\dfrac{1}{a^{3}} = a^{?}$ (n) $\sqrt{m} = m^{?}$ (o) $\dfrac{1}{p^{2}} = p^{?}$

(p) $\sqrt[3]{q} = q^{?}$ (q) $\sqrt[3]{q^{2}} = q^{?}$ (r) $\sqrt[5]{q^{2}} = q^{?}$

3. Use a calculator to find:

(a) 8^{-1} (b) 20^{-1} (c) $\left(\frac{1}{2}\right)^{-1}$ (d) $\left(\frac{1}{4}\right)^{-1}$

(e) 15^{-2} (f) 20^{-3} (g) $81^{\frac{3}{2}}$ (h) $243^{\frac{3}{5}}$

(i) $16^{-\frac{1}{4}}$ (j) $144^{\frac{3}{2}}$ (k) $169^{\frac{7}{2}}$ (l) $121^{\frac{3}{2}}$

4. Simplify the following expressions, so that they contain no negative indices.

(a) $a^6 \times a^{-7} =$ (b) $\frac{a^7}{a^{-3}} =$ (c) $\frac{a^{-5}}{a^{-9}} =$

(d) $a^{-4} \times a^{-2} =$ (e) $\left(a^2\right)^{-1}$ (f) $\left(a^2\right)^{-3} =$

(g) $\left(a^{-2}\right)^{-4} =$ (h) $\left(a^{\frac{1}{2}}\right)^5 =$ (i) $\left(a^3\right)^{-\frac{1}{2}} =$

(j) $\left(a^6\right)^{\frac{1}{3}} =$ (k) $\left(a^9\right)^{-\frac{1}{3}} =$ (l) $\left(a^{-12}\right)^{-\frac{1}{4}} =$

(m) $\left(\frac{a}{b}\right)^2 =$ (n) $\left(a^2 \times b^{-4}\right)^3 =$ (o) $\left(a^3 b^{\frac{1}{2}}\right)^4 =$

(p) $\left(a^2 b^{-2}\right)^{-2} =$ (q) $\left(\frac{a^2}{b^3}\right)^4 =$ (r) $\left(m^{-1} n^3\right)^{-2} =$

(s) $\left(\frac{a^6}{b^{10}}\right)^{\frac{1}{2}} =$ (t) $\left(\frac{a^2}{m^4}\right)^{-\frac{1}{2}} =$ (u) $\left(\frac{a^8 b^2}{c^6}\right)^{-\frac{1}{2}} =$

(v) $\left(\frac{m^2}{x}\right)^{-1} =$ (w) $\left(\frac{x^2 y}{z^3}\right)^{-4} =$ (x) $\left[\left(a^3 b^{-8}\right)^{-\frac{1}{3}}\right]^2 =$

5. (a) Express $81^{-\frac{1}{2}}$ as a fraction in the form $\frac{a}{b}$, where a and b are integers.

(b) Simplify $a^6 \div a^2$.

(c) Find the value of y for which $2 \times 4^y = 64$.

(LON)

1.7 Standard Form

Standard form is a way to write very large or very small numbers. It is particularly useful when working with a scientific calculator.

In standard form a number is written as

$$a \times 10^n$$

where a is a number greater or equal to 1 and less than 10, and n is a positive or negative whole number.

Note

Because of the way that the powers of 10 are used in standard form, it is important to remember that:

$$10^4 = 10000 \qquad \text{and} \qquad 10^{-4} = 0.0001$$
$$10^3 = 1000 \qquad\qquad 10^{-3} = 0.001$$
$$10^2 = 100 \qquad\qquad 10^{-2} = 0.01$$
$$10^1 = 10 \qquad\qquad 10^{-1} = 0.1$$
$$10^0 = 1$$

Worked Example 1

Find

(a) $4.21 \times 10^2 =$ (b) $3.1 \times 10^5 =$

(c) $3.6 \times 10^{-2} =$ (d) $4.7 \times 10^{-3} =$

Solution

(a) $4.21 \times 10^2 = 4.21 \times 100$
$= 421$

(b) $3.1 \times 10^5 = 3.1 \times 100\,000$
$= 310\,000$

(c) $3.6 \times 10^{-2} = 3.6 \times 0.01$
$= 0.036$

(d) $4.7 \times 10^{-3} = 4.7 \times 0.001$
$= 0.0047$

Worked Example 2

Write each of the following numbers in standard form.

(a) $346\,000\,000$ (b) 2710 (c) $0.000\,543$

Solution

(a) $346\,000\,000 = 3.46 \times 100\,000\,000$
$= 3.46 \times 10^8$

(b) $2710 = 2.71 \times 1000$
$= 2.71 \times 10^3$

(c) $0.000\,543 = 5.43 \times 0.0001$
$= 5.43 \times 10^{-4}$

Information

On average, a human heart beats 75 times a minute, 4 500 times an hour, 108 000 times a day, 39 420 000 times a year and 3 153 600 000 times for someone who lives 80 years.

Worked Example 3

Write each of the following numbers in normal decimal form.

(a) 3.217×10^3 (b) 3.68×10^5 (c) 4.7×10^{-4}

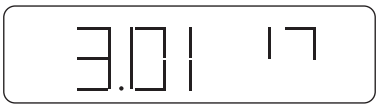
Solution

(a) $3.217 \times 10^3 = 3.217 \times 1000$
$= 3217$

(b) $3.68 \times 10^5 = 3.68 \times 100\,000$
$= 368\,000$

(c) $4.7 \times 10^{-4} = 4.7 \times 0.0\,001$
$= 0.00\,047$

Most calculators can work with numbers in standard form. On a calculator display 3.01×10^{17} would look like



Exercises

1. Write each of the following numbers in standard form.

(a)	47 000	(b)	52 100	(c)	32 000 000
(d)	324 100	(e)	420	(f)	81 000
(g)	5 000	(h)	47 000 000 000	(i)	3 200 000 000
(j)	0.000 62	(k)	0.0571	(l)	0.000 000 2
(m)	0.124	(n)	0.0371	(o)	0.000 21
(p)	0.000 07	(q)	0.471	(r)	0.0003

2. Write each of these numbers in standard form.

(a)	1 million	(b)	15 thousand	(c)	6.4 million
(d)	30.4 million	(e)	4 million	(f)	0.4 million

3. Write each of the following numbers using normal decimal notation.

(a)	6×10^5	(b)	4.31×10^2	(c)	5.86×10^7
(d)	8.3×10^{-4}	(e)	4.172×10^3	(f)	6.42×10^{-5}
(g)	4.7×10^1	(h)	3.2×10^{-1}	(i)	8.47×10^{-4}
(j)	3.34×10^8	(k)	3.471×10^{-4}	(l)	8.421×10^2
(m)	1.675×10^1	(n)	8.4×10^{-6}	(o)	7.12×10^{-4}

4. For each of the numbers below state whether or not it is in standard form. If it is not in standard form, write it in standard form.

(a) 3.2×10^8 (b) 43.2×10^2 (c) 15.6×10^{-8}

(d) 0.4×10^3 (e) 1.3×10^{-8} (f) 0.7×10^{-4}

(g) 5.471×10^2 (h) 54.71×10^3 (i) 8.21×10^6

5. Give the answers to the following calculations in standard form.

(a) $2000 \times 30 =$ (b) $4000^2 =$ (c) $50^3 =$

(d) $\dfrac{4}{1000} =$ (e) $\dfrac{6}{3000} =$ (f) $0.04^2 =$

(g) $0.004 \times 0.7 =$ (h) $22 \times 400 =$ (i) $\dfrac{18}{20\,000} =$

(j) $30^2 =$ (k) $0.02^2 =$ (l) $100^2 =$

6. There are 1000 m in 1 km. Convert the following distances to metres, giving your answers in standard form.

(a) 50 km (b) 620 km (c) 1456 km

7. Find the number of:

(a) hours in a year (b) minutes in a week

(c) seconds in a day,

giving your answers in standard form.

8. The radius of the earth is 6370 km.

(a) Write this in a normal decimal form.

(b) Find the radius of the earth in metres and express it in both decimal form and standard form.

(c) Find the circumference of the earth in metres, giving the answer in standard form.

9. The mass of the earth is 5.9×10^{24} kg. Write this as a decimal number.

10. The width of a thin strip of metal is $\dfrac{3}{100}$ mm. Write this in standard form.

11. Scientists estimate the mass of a newly discovered planet as 482 000 000 kg. Write this in standard form.

12. The distance of the earth from the sun varies between 1.53×10^8 km and 1.47×10^8 km.

(a) Write these numbers in a decimal format.

(b) Convert both distances to metres and write them in standard form.

13. Carry out each of the following calculations on a calculator and write the answers in standard form.

(a) $66\,666^2$ (b) $54\,321 \times 6\,789 =$

(c) $2\,000^3 =$ (d) $640\,000 \times 240\,000 =$

(e) $88\,000 \times 188\,000 =$ (f) $56\,000 \div 0.000\,025 =$

1.8 Calculations with Standard Form

When using standard form it is possible to multiply and divide numbers, taking advantage of the form in which they are written.

Worked Example 1

Find

$$4 \times 10^{18} \times 3 \times 10^4$$

Solution

To do this calculation, you multiply together the 4 and the 3 and then multiply together the 10^{18} and the 10^4.

$$\begin{aligned} 4 \times 10^{18} \times 3 \times 10^4 &= 4 \times 3 \times \left(10^{18} \times 10^4\right) \\ &= 12 \times 10^{22} \end{aligned}$$

This result is not in standard form so the final stage is to convert the result to standard form.

$$\begin{aligned} 12 \times 10^{22} &= 1.2 \times 10 \times 10^{22} \\ &= 1.2 \times 10^{23} \end{aligned}$$

$$4 \times 10^{18} \times 3 \times 10^4 = 1.2 \times 10^{23}$$

Worked Example 2

Find

(a) $3.2 \times 10^4 \times 5 \times 10^{-3}$ (b) $\left(6 \times 10^8\right) \div \left(3 \times 10^4\right)$

(c) $\left(7.2 \times 10^3\right) \div \left(6 \times 10^4\right)$

Solution

(a) Multiply together the 3.2 and the 5 and then multiply together the 10^4 and the 10^{-3}.

$$\begin{aligned} 3.2 \times 10^4 \times 5 \times 10^{-3} &= 3.2 \times 5 \times 10^4 \times 10^{-3} \\ &= 16.0 \times 10^1 \end{aligned}$$

This number is not in standard form so converting gives

$$16.0 \times 10^1 = 1.6 \times 10 \times 10^1$$
$$= 1.6 \times 10^2.$$

(b) Division follows a similar approach to multiplication. First divide 6 by 3 and then divide 10^8 by 10^4.

$$\left(6 \times 10^8\right) \div \left(3 \times 10^4\right) = \left(6 \div 3\right) \times \left(10^8 \div 10^4\right)$$
$$= 2 \times 10^4$$

This result is in standard form so no further work is required.

(c) First divide 7.2 by 6 and then divide 10^3 by 10^4.

$$\left(7.2 \times 10^3\right) \div \left(6 \times 10^4\right) = \left(7.2 \div 6\right) \times \left(10^3 \div 10^4\right)$$
$$= 1.2 \times 10^{-1}$$

This result is in standard form.

Problems can be done directly on a calculator, or by entering numbers using the EE or EXP keys.

Exercises

1. Do the following calculations, making sure that your answer is in standard form. *Do not use a calculator.*

(a)	$3 \times 10^8 \times 2 \times 10^4 =$	(b)	$2 \times 10^5 \times 4 \times 10^3 =$
(c)	$9 \times 10^6 \times 1 \times 10^{10} =$	(d)	$5 \times 10^3 \times 4 \times 10^8 =$
(e)	$6 \times 10^3 \times 4 \times 10^{11} =$	(f)	$3 \times 10^{-2} \times 4 \times 10^8 =$
(g)	$1.2 \times 10^6 \times 2.4 \times 10^5 =$	(h)	$1.1 \times 10^6 \times 2 \times 10^{-4} =$
(i)	$8.1 \times 10^8 \times 7.2 \times 10^{-2} =$	(j)	$5.2 \times 10^3 \times 1.3 \times 10^{-7} =$
(k)	$6.2 \times 10^{-3} \times 2.1 \times 10^{-6} =$	(l)	$1.8 \times 10^{-4} \times 2.5 \times 10^{-9} =$

2. Give the answers to the following calculations in standard form. *Do not use a calculator.*

(a)	$\left(8 \times 10^6\right) \div \left(2 \times 10^2\right) =$	(b)	$\left(9 \times 10^5\right) \div \left(3 \times 10^2\right) =$
(c)	$\left(8 \times 10^4\right) \div \left(4 \times 10^2\right) =$	(d)	$\left(1.6 \times 10^5\right) \div \left(2 \times 10^2\right) =$
(e)	$\left(3.6 \times 10^8\right) \div \left(3 \times 10^2\right) =$	(f)	$\left(4.8 \times 10^{12}\right) \div \left(4 \times 10^3\right) =$
(g)	$\left(8.1 \times 10^4\right) \div \left(3 \times 10^5\right) =$	(h)	$\left(4.5 \times 10^3\right) \div \left(9 \times 10^{-5}\right) =$
(i)	$\left(1.64 \times 10^8\right) \div \left(4 \times 10^{-12}\right) =$	(j)	$\left(1.32 \times 10^5\right) \div \left(1.2 \times 10^{-3}\right) =$
(k)	$\left(9.6 \times 10^4\right) \div \left(3.2 \times 10^{-5}\right) =$	(l)	$\left(1.21 \times 10^{-4}\right) \div \left(1.1 \times 10^6\right) =$

3. Do the following using a calculator, giving your answers in standard form.

(a) $\left(4.2 \times 10^{6}\right)^{2} =$ (b) $\left(3.7 \times 10^{-2}\right)^{2} =$

(c) $\left(1.2 \times 10^{-5}\right)^{3} =$ (d) $\sqrt{8.1 \times 10^{8}} =$

(e) $6.2 \times 10^{8} \times 1.2 \times 10^{14} =$ (f) $3.8 \times 10^{4} \times 4.1 \times 10^{-12} =$

(g) $\left(1.84 \times 10^{6}\right) \div \left(1.92 \times 10^{7}\right) =$ (h) $\dfrac{4.7 \times 10^{8}}{3.2 \times 10^{5}} =$

(i) $\dfrac{1.62 \times 10^{-5}}{3.2 \times 10^{4}} =$ (j) $\sqrt{\dfrac{3 \times 10^{8}}{5 \times 10^{3}}} =$

(k) $4.8 \times 10^{11} + 3.2 \times 10^{10} =$ (l) $6.8 \times 10^{12} - 4.7 \times 10^{10} =$

4. There are 8.64×10^{4} seconds in one day. How many seconds are there in:

(a) 10 days (b) 1 week (c) 1 year?

5. The mass of an electron is 9.1×10^{-31} kg. Find the mass of:

(a) 3×10^{18} electrons (b) 4×10^{32} electrons (c) 7×10^{8} electrons.

6. A rectangle has sides of length 3×10^{5} mm and 4.2×10^{6} mm.
Find the area of the rectangle.

7. The speed of sound is $3.32 \times 10^{2}\ \text{m s}^{-1}$.

(a) How far would the noise from a 'bang' travel in:

(i) 10 seconds (ii) 3×10^{3} seconds (iii) 4×10^{-2} seconds?

(b) How long would it take the noise from a 'bang' to travel:

(i) 10 metres (ii) 2×10^{3} metres (iii) 2×10^{-2} metres?

8. The speed of light is $3 \times 10^{8}\ \text{m s}^{-1}$.

(a) How far would light travel in 100 seconds?

(b) The mean distance of the earth from the sun is 1.5×10^{11} m.
How long does it take for light to travel from the sun to the earth?

9. The distance from the earth to the moon is 3.84×10^{5} km.

(a) Find this distance in metres.

(b) How long would it take a spaceship to travel to the moon from earth if its average speed was $400\ \text{m s}^{-1}$?

Investigation

Find four integers, a, b, c and d such that $a^3 + b^3 + c^3 = d^3$.

10. The density of air is 1.3×10^{-6} kg/cm^3.

    (a) How many cubic centimetres are in one cubic metre?

    (b) Find the mass of $1\ \text{m}^3$ of air.

    (c) Find the volume of air that will have a mass of 3 grams.

    (d) The density of hydrogen is 9×10^{-8} kg/cm^3.
    Repeat (b) and (c) for hydrogen.

11. The population of the world was estimated to be 4.5×10^9 at the beginning of 1990. If the population increases by 3% each year, find the population at the beginning of the year 2000.

12. (a) The approximate population of the United Kingdom is given in standard form as 5.2×10^7. Write this as an ordinary number.

    (b) The thickness of grade A paper is 6.0×10^{-2} cm. Grade B paper is twice as thick as grade A.

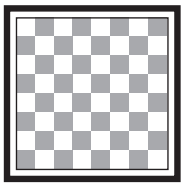
    Calculate, in centimetres, the thickness of grade B paper. Write your answer in standard form.

(SEG)

13. Between 1950 and 1985 the number of people living in towns and cities in developing countries increased from 2.86×10^8 to 1.14×10^9.

    Calculate the increase in the number of people, giving your answer in standard form.

(MEG)

14. 

    Centuries ago, a man promised to give his wife some grains of rice. He took a chess board and placed one grain on the first square, two grains on the second square, four grains on the third square, eight grains on the fourth square, and so on.

    If he had completed all 64 squares on the chess board he would have used approximately 1.845×10^{19} grains of rice.

    One grain of rice weighs about 0.01 grams. Calculate an estimate of the weight of rice used. Give your answer in tonnes, correct to one significant figure.

    [1 tonne = 1000 kg]

(MEG)

15. (a) Evaluate 1.2^8.

    (b) Evaluate $(0.0009)^{\frac{1}{2}}$. Give your answer in standard form.

(SEG)

Investigation

Find the positive integers, x, y and z such that $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$.

16. The mean distance of the earth from the sun is 149.6 million kilometres.

(a) Write the number 149.6 million in standard index form.

The earth travels a distance, D km, in one day. The value of D is given by the formula

$$D = \frac{2\pi \times \text{distance of earth from sun}}{365}$$

(b) Calculate the value of D, giving your answer in standard index form.

(LON)

17. The number 10^{100} is called a googol.

(a) Write the number, 50 googols, in standard index form.

A nanometre is 10^{-9} metres.

(b) Write 50 nanometres in metres.
Give your answer in standard index form.

(LON)

Just for Fun

In astronomy, the distance between stars is measured in light years, which is the distance travelled by light in a year.

One light year $= 3 \times 10^5 \times 60 \times 60 \times 24 \times 365$ *km.*

This is approximately 9 460 800 000 000 km.

How long would it take for light to travel from the Sun to the Earth if their distance apart is 1.5×10^8 *km?*