10 Equations

10.1 Negative Numbers

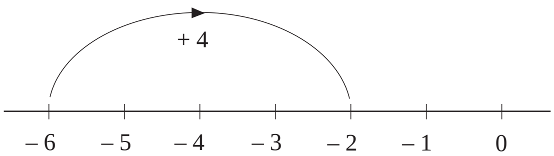
A number line like this is very helpful when dealing with negative numbers.

Worked Example 1

An ice cube taken from a deep freeze is at a temperature of $-6°C$. Its temperature rises by 4 degrees. What is its new temperature?

Solution

The number line can be used as shown.



$$-6 + 4 = -2$$

Worked Example 2

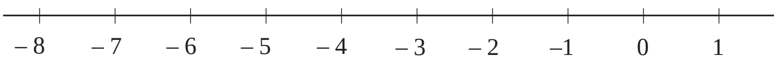
Write down numbers which are:

(a) less than -4

(b) greater than -5 but less than -2

(c) less than -3 but greater than -8.

Solution



(a) Any number less than -4 must be to the left of -4 on the number line, for example -5, -6, -7, etc.

(b) A number greater than -5 must be to the right of -5, and to be less than -2 must also be to the left of -2, for example -4 and -3.

(c) To be less than -3 the number must be to the left of -3 but to be greater than -8 the number must be to the right of -8, for example -7, -6, -5, -4.

Investigation

Within 4 consecutive years, Mrs. White gave birth to four lovely children. Today, x years later, Mr and Mrs. White find out that the product of their four children's ages is 3 024. How old is each child now, assuming that all the children are of different ages?

Exercises

1. Copy and complete the following table of temperature changes.

Original Temperature	*Temperature Change*	*Final Temperature*
6° C	-10° C	
-4° C	$+2^\circ$ C	
-5° C	-3° C	
-10° C	-4° C	
2° C	-10° C	
-6° C	$+1^\circ$ C	
-5° C	-2° C	

2. Place each set of numbers in increasing order.

(a) $-10, -4, -6, 2, -5, 3$ (b) $3, -1, 0, -4, 7$

(c) $7, -2, -4, -6, 3, 0$ (d) $-4, 0, -7, 7, 5, -2$

(e) $-1, -4, 0, -5, 6, -3$

3. Write down all the integers which lie between:

(a) -6 and -10 (b) -4 and 2 (c) -3 and 0

(d) -1 and -3 (e) -9 and 0.

4. Write down a whole number which is:

(a) greater than -10 but less than -6 (b) less than -6

(c) greater than -5 but less than 0 (d) greater than -10

(e) less than -6 but greater than -9 (f) less than -4.

5. Insert a $<$ sign or a $>$ sign between each pair of numbers.

(a) $3 \quad 2$ (b) $-5 \quad -6$ (c) $0 \quad -1$

(d) $-7 \quad -10$ (e) $-2 \quad -4$ (f) $-1 \quad -6$

(g) $-5 \quad 0$ (h) $-9 \quad -6$ (i) $-8 \quad -2$

Just For Fun

If $ABCDE \times 4 = EDCBA$, *find A, B, C, D and E if none of them is zero.*

10.2 Arithmetic with Negative Numbers

The following rules for working with negative numbers are important to remember.

Multiplication	*Division*
$+ \times + \Rightarrow +$	$+ \div + \Rightarrow +$
$+ \times - \Rightarrow -$	$+ \div - \Rightarrow -$
$- \times + \Rightarrow -$	$- \div + \Rightarrow -$
$- \times - \Rightarrow +$	$- \div - \Rightarrow +$

Worked Example 1

(a) $6 \times (-4) = ?$ (b) $-6 \times (-7) = ?$

(c) $42 \div (-2) = ?$ (d) $-33 \div (-11) = ?$

Solution

(a) $6 \times (-4) = -24$ (b) $-6 \times (-7) = 42$

(c) $42 \div (-2) = -21$ (d) $-33 \div (-11) = 3$

Worked Example 2

(a) $-6 + 7 = ?$ (b) $-6 - 8 = ?$

(c) $-6 - (-4) = ?$ (d) $-7 + (-3) = ?$

Solution

(a) $-6 + 7 = 1$ (b) $-6 - 8 = -14$

(c) $-6 - (-4) = -6 + 4$
$= -2$

(d) $-7 + (-3) = -7 - 3$
$= -10$

Exercises

1. (a) $-6 \times -2 = ?$ (b) $-7 \times 2 = ?$ (c) $-8 + -4 = ?$
   (d) $-8 - 12 = ?$ (e) $-9 + 2 = ?$ (f) $-4 + (-2) = ?$
   (g) $-6 - (-5) = ?$ (h) $-7 \times (-4) = ?$ (i) $-3 - (-4) = ?$
   (j) $-64 \div (-2) = ?$ (k) $-18 \div 9 = ?$ (l) $-21 \div (-3) = ?$
   (m) $-4 + (-6) = ?$ (n) $-2 - (-4) = ?$ (o) $-6 - 7 = ?$
   (p) $-24 \div 2 = ?$ (q) $-7 \times (-8) = ?$ (r) $-7 - (-8) = ?$
   (s) $-14 + 8 = ?$ (t) $-12 - 3 = ?$ (u) $-16 - (-8) = ?$
   (v) $4 - 12 = ?$ (w) $3 - (-2) = ?$ (x) $8 \div (-4) = ?$

10.3 Simplifying Expressions

When simplifying expressions you should bring together terms which contain the same letter.

Note

x and x^2 must be treated as if they were different letters. You cannot add an x term to an x^2 term. The + and – signs go with the term which follows.

Worked Example 1

Simplify each expression below.

(a) $4a + 3a + 6 + 2$ (b) $4a + 8b - 2a + 3b$

(c) $x^2 + 5x - 8x + x^2 - 4$ (d) $8x + y - 4x - 6y$

Solution

(a) The terms which involve a can be brought together. Also the 6 and 2 can be added.

$$4a + 3a + 6 + 2 = 7a + 8$$

(b) The terms involving a are considered together, and then the terms involving b.

$$\begin{aligned} 4a + 8b - 2a + 3b &= 4a - 2a + 8b + 3b \\ &= 2a + 11b \end{aligned}$$

(c) Here the x and x^2 must be treated as if they are different letters.

$$\begin{aligned} x^2 + 5x - 8x + x^2 - 4 &= x^2 + x^2 + 5x - 8x - 4 \\ &= 2x^2 - 3x - 4 \end{aligned}$$

(d) The different letters, x and y, must be considered in turn.

$$\begin{aligned} 8x + y - 4x - 6y &= 8x - 4x + y - 6y \\ &= 4x - 5y \end{aligned}$$

When a bracket is to be multiplied by a number or a letter, every term inside the bracket must be multiplied.

Worked Example 2

Remove the brackets from each expression below.

(a) $6(x + 5)$ (b) $3(2x + 7)$ (c) $4(x - 3)$ (d) $x(x - 4)$

Solution

(a) $$\begin{aligned} 6(x + 5) &= 6 \times x + 6 \times 5 \\ &= 6x + 30 \end{aligned}$$

(b) $$\begin{aligned} 3(2x + 7) &= 3 \times 2x + 3 \times 7 \\ &= 6x + 21 \end{aligned}$$

(c) $$\begin{aligned} 4(x - 3) &= 4 \times x - 4 \times 3 \\ &= 4x - 12 \end{aligned}$$

(d) $$\begin{aligned} x(x - 4) &= x \times x - x \times 4 \\ &= x^2 - 4x \end{aligned}$$

Exercises

1. Simplify each of these expressions.

(a) $a + 2a + 3a$ (b) $3a + 2 + 4 + 6$

(c) $3a + 2b + 8a + 4b$ (d) $4x + 2y + 8y + y$

(e) $5x + 2y + 8x - 3y$ (f) $6a + 7b + 3b - 4a$

(g) $4 + 6a - 3a + 2 + b$ (h) $p + q + 2p - 8q + 3p$

(i) $x + y - 8x + 2y$ (j) $4x - 3p + 2p - 2x$

(k) $7x - 4z + 8x - 5z$ (l) $3z - 4x + 2z - 10x$

(m) $3q - 4x + 8a - 2x + q$ (n) $x + y + z - p - q - y$

(o) $x + 6 + y + 4 + 2x - 3y$ (p) $4x - 8q + 17x - 24q$

(q) $-x + y + x + y$ (r) $4x + 7y - 3x - 8y + x + y$

(s) $-8x + 7y - 11x + 4y$ (t) $6x - 18y + 17x - 4$

(u) $x + y - 8x - 11y$ (v) $4p + 8q - 8p - 4q$

2. Simplify each of the following expressions.

(a) $2x^2 + 3x + 4x^2 + 5x$ (b) $x^2 + 8x + 5x + 10$

(c) $x^2 + 6x + 4x + x^2$ (d) $x^2 + x + 10 + x + 4x^2$

(e) $5x^2 - x - 6x^2 + 8x$ (f) $4x^2 - 3y^2 - x^2 + y^2$

(g) $x^2 + y^2 - x - y + x^2$ (h) $4x^2 - 7x + 1 + x^2 + 4x - 11$

(i) $x^2 - y^2 - x - y + 2x^2 - 2y^2$ (j) $y^2 + y - 4 + y + 4y^2$

(k) $ab + cd + 4ab$ (l) $xy + xz + xy + 4xz$

(m) $4ab + 7ab - 3ad$ (n) $4pq - 3qr + 5pq$

3. Remove the brackets from each expression below.

(a) $3(x + 5)$ (b) $4(6 + x)$ (c) $7(x + 2)$

(d) $2(x + 6)$ (e) $5(x + 2)$ (f) $4(2x + 3)$

(g) $5(3x + 2)$ (h) $8(5x + 3)$ (i) $7(x - 6)$

(j) $8(5 - x)$ (k) $4(2x - 7)$ (l) $7(5x - 3)$

(m) $6(3x - 5)$ (n) $4(x - 2y)$ (o) $5(x + 2y + 3z)$

(p) $x(5 + x)$ (q) $a(2 - a)$ (r) $4(b - 3)$

(s) $2x(x - 6)$ (t) $4x(2x + 3)$ (u) $3x(7 - 2x)$

(v) $8x(x - 5)$

4. Simplify each of the following expressions, by first removing all the brackets.

(a) $3(a+2)+4(a+5)$ (b) $2(2x+4)+3(x+5)$

(c) $5(x+2)+3(x+4)$ (d) $x(x+1)+x(x+6)$

(e) $4(x+1)+x(x+5)$ (f) $2(a+b)+5(2a+3b)$

10.4 Simple Equations

To solve simple equations you must carry out the *same* operation (addition, subtraction, multiplication or division) on *both* sides of the equation so that the new equation is still balanced.

Worked Example 1

Solve each of the following equations.

(a) $x+3=8$ (b) $x-8=11$ (c) $4x=32$ (d) $\frac{x}{6}=7$

Solution

(a) To solve this equation, subtract 3 from both sides.

$$x+3=8$$
$$x+3-3=8-3$$
$$x=5.$$

(b) To solve this equation, add 8 to both sides.

$$x-8=11$$
$$x-8+8=11+8$$
$$x=19.$$

(c) To solve this equation divide both sides by 4.

$$4x=32$$
$$\frac{{}^{1}\cancel{4}x}{\cancel{4}_{1}}=\frac{\cancel{32}^{8}}{\cancel{4}_{1}}$$
$$x=8.$$

(d) To solve this equation multiply both sides by 6.

$$\frac{x}{6}=7$$
$$\frac{x}{\cancel{6}_{1}}\times\cancel{6}^{1}=7\times 6$$
$$x=42.$$

Worked Example 2

A packet of sweets is divided equally among 5 children and each child is given 4 sweets. Write down an equation to describe this situation and solve it to find the number of sweets in the packet.

Solution

Let x be the number of sweets in the packet.

Then
$$\frac{x}{5} = 4,$$

since the 5 children have 4 sweets each. Now the equation can be solved by multiplying both sides by 5.

$$\frac{x}{\cancel{5}_{1}} \times \cancel{5}^{1} = 4 \times 5$$
$$x = 20.$$

Exercises

1. Solve each of these equations.

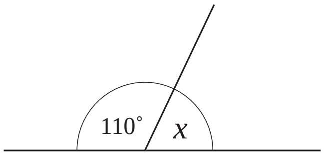
(a)	$x + 6 = 10$	(b)	$x - 7 = 3$	(c)	$x + 4 = 7$
(d)	$x - 1 = 11$	(e)	$x - 6 = 8$	(f)	$x + 5 = 3$
(g)	$5x = 45$	(h)	$6x = 24$	(i)	$6x = 108$
(j)	$7x = 56$	(k)	$3x = 102$	(l)	$6x = 42$
(m)	$\frac{x}{2} = 5$	(n)	$\frac{x}{2} = 12$	(o)	$\frac{x}{5} = 10$
(p)	$\frac{x}{3} = 4$	(q)	$\frac{x}{7} = 3$	(r)	$\frac{x}{6} = 11$
(s)	$x - 5 = 3$	(t)	$x + 8 = 14$	(u)	$4x = 104$
(v)	$\frac{x}{2} = 18$	(w)	$x - 3 = 12$	(x)	$x + 7 = 11$

2. Solve each of these equations.

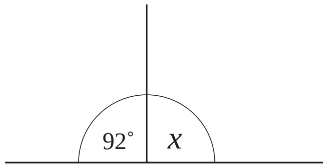
(a)	$x + 6 = 2$	(b)	$x + 8 = 3$	(c)	$x - 5 = -2$
(d)	$x + 2 = -4$	(e)	$x - 2 = -6$	(f)	$x + 4 = -10$
(g)	$2x = -12$	(h)	$3x = -24$	(i)	$5x = -60$
(j)	$\frac{x}{2} = -8$	(k)	$x - 2 = -5$	(l)	$x + 6 = -14$
(m)	$x - 10 = -2$	(n)	$x - 12 = -4$	(o)	$x - 7 = -1$
(p)	$x + 4 = -1$	(q)	$x + 12 = 2$	(r)	$x + 10 = -16$

3. The angles on a straight line add up to $180°$. Write down and solve an equation for each diagram shown below.

(a)



(b)



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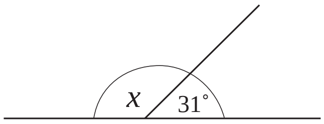
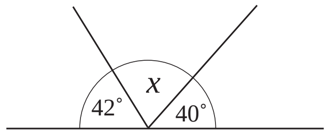


(d)



4. Sanjit is two years older than his brother. His brother is 16. Write down an equation which uses x to represent Sanjit's age. Solve the equation for x..

5. To pay for a school trip, 12 children take the same amount of money to school. If the total money collected is £54 and the amount each child takes is x, write down an equation to describe this situation. Solve your equation for x.

6. The train fare for a long journey increases by £3 to £41. If x is the old fare, write down an equation to describe this situation. Solve your equation for x.

7. Majid knows that when a certain number is doubled, the answer is 52. Explain in words, starting with 52, how he can work out the number.

(SEG)

8. Tim thought of a number. He doubled his number. His answer was 24.
What number did Tim think of?

(LON)

9. Ali is twice as old as Sue. Sue is 2 years younger than Philip. Philip is 11 years old.

(a) How old is Sue? (b) How old is Ali?

(NEAB)

10.5 Solving Equations

Most equations require a number of steps to solve them. These steps must be logical so that the new equation still balances. Whatever you do to one side of an equation you *must* do the same to the other side. The following examples illustrate these steps.

Worked Example 1

Solve the following equations.

(a) $3x + 7 = 13$ (b) $5x - 8 = 13$ (c) $\dfrac{x}{5} - 2 = 3$

(d) $4(x - 3) = 8$

Solution

(a) First subtract 7 from both sides of the equation.

$$\begin{aligned} 3x + 7 &= 13 \\ 3x + 7 - 7 &= 13 - 7 \\ 3x &= 6 \end{aligned}$$

Next divide both sides of the equation by 3.

$$\begin{aligned} \frac{\cancel{3}^{1}x}{\cancel{3}_{1}} &= \frac{\cancel{6}^{2}}{\cancel{3}_{1}} \\ x &= 2. \end{aligned}$$

(b) First add 8 to both sides of the equation.

$$\begin{aligned} 5x - 8 &= 13 \\ 5x - 8 + 8 &= 13 + 8 \\ 5x &= 21. \end{aligned}$$

Then divide both sides of the equation by 5.

$$\begin{aligned} \frac{\cancel{5}^{1}x}{\cancel{5}_{1}} &= \frac{21}{5} \\ x &= \frac{21}{5} \\ &= 4\tfrac{1}{5}. \end{aligned}$$

(c) First add 2 to both sides of the equation.

$$\begin{aligned} \frac{x}{5} - 2 &= 3 \\ \frac{x}{5} - 2 + 2 &= 3 + 2 \\ \frac{x}{5} &= 5 \end{aligned}$$

Then multiply both sides of the equation by 5.

$$\begin{aligned} \frac{x}{\cancel{5}_{1}} \times \cancel{5}^{1} &= 5 \times 5 \\ x &= 25. \end{aligned}$$

(d) First remove the brackets, multiplying each term inside the bracket by 4.

$$\begin{aligned} 4(x - 3) &= 8 \\ 4x - 12 &= 8. \end{aligned}$$

Then add 12 to both sides of the equation.

$$\begin{aligned} 4x - 12 + 12 &= 8 + 12 \\ 4x &= 20. \end{aligned}$$

Finally divide both sides by 4.

$$\frac{{}^{1}\cancel{4}x}{\cancel{4}_{1}} = \frac{\cancel{20}^{5}}{\cancel{4}_{1}}$$

$$x = 5$$

Sometimes equations may contain the letter x on both sides of the equation or a $-x$ term. The next examples show how to deal with these cases.

Worked Example 2

Solve these equations.

(a) $4x + 6 = 3x + 10$ (b) $6 - 2x = 8$ (c) $4x - 2 = 8 - 6x$

Solution

(a) As x appears on both sides of the equation, first subtract $3x$ from both sides.

$$\begin{aligned} 4x + 6 &= 3x + 10 \\ 4x + 6 - 3x &= 3x + 10 - 3x \\ x + 6 &= 10. \end{aligned}$$

Then subtract 6 from both sides.

$$\begin{aligned} x + 6 - 6 &= 10 - 6 \\ x &= 4. \end{aligned}$$

(b) As the left-hand side contains $-2x$, add $2x$ to both sides.

$$\begin{aligned} 6 - 2x &= 8 \\ 6 - 2x + 2x &= 8 + 2x \\ 6 &= 8 + 2x \end{aligned}$$

Then subtract 8 from both sides.

$$\begin{aligned} 6 - 8 &= 8 + 2x - 8 \\ -2 &= 2x. \end{aligned}$$

Finally divide both sides by 2.

$$\frac{\cancel{-2}^{-1}}{\cancel{2}_{1}} = \frac{{}^{1}\cancel{2}x}{\cancel{2}_{1}}$$

$$-1 = x \quad \text{or} \quad x = -1.$$

(c) As one side contains $-6x$, add $6x$ to both sides.

$$\begin{aligned} 4x - 2 &= 8 - 6x \\ 4x - 2 + 6x &= 8 - 6x + 6x \\ 10x - 2 &= 8 \end{aligned}$$

Then add 2 to both sides of the equation.

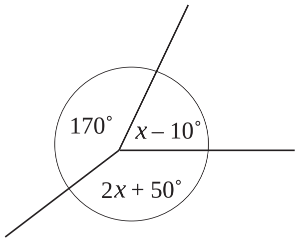
$$10x - 2 + 2 = 8 + 2$$
$$10x = 10$$

Finally divide both sides by 10.

$$\frac{{}^{1}\cancel{10}x}{{}_{1}\cancel{10}} = \frac{\cancel{10}^{1}}{\cancel{10}_{1}}$$
$$x = 1.$$

Worked Example 3

Use the information in the diagram to write down an equation and then find the value of x.



Solution

The three angles shown must add up to $360°$, so

$$170 + 2x + 50 + x - 10 = 360$$
$$210 + 3x = 360.$$

Subtracting 210 from both sides gives

$$210 + 3x - 210 = 360 - 210$$
$$3x = 150.$$

Then dividing both sides by 3 gives

$$\frac{{}^{1}\cancel{3}x}{{}_{1}\cancel{3}} = \frac{\cancel{150}^{50}}{\cancel{3}_{1}}$$
$$x = 50.$$

Exercises

1. Solve each of these equations.

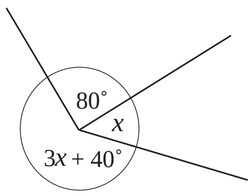
(a)	$3x + 6 = 48$	(b)	$5x - 6 = 39$	(c)	$2x - 6 = 22$
(d)	$6x - 7 = 41$	(e)	$8x - 3 = 29$	(f)	$6x + 12 = 20$
(g)	$4x + 18 = 2$	(h)	$5x + 10 = 5$	(i)	$3x + 6 = 1$
(j)	$5(x + 2) = 45$	(k)	$3(x - 2) = 12$	(l)	$2(x + 7) = 10$
(m)	$3(2x - 1) = 57$	(n)	$3(2x + 7) = 27$	(o)	$5(5x + 1) = 20$
(p)	$4(2x + 3) = -8$	(q)	$5(3x - 1) = -2$	(r)	$2(8x + 5) = -2$
(s)	$6x - 8 = -26$	(t)	$4(x + 15) = 60$	(u)	$5x - 8 = -10$
(v)	$\frac{x}{4} - 1 = 8$	(w)	$\frac{x}{3} + 2 = 7$	(x)	$\frac{2x}{5} + 1 = 3$

2. Solve these equations.

(a)	$2x + 6 = x + 3$	(b)	$4x - 8 = 5x - 2$
(c)	$6x + 7 = 2x + 20$	(d)	$x + 6 = 2x - 8$
(e)	$3x + 7 = 2x + 11$	(f)	$10x + 2 = 8x + 22$
(g)	$6 - x = 5$	(h)	$2 - x = 5$
(i)	$3 - x = -10$	(j)	$14 - 3x = 5$
(k)	$10 - 2x = 2$	(l)	$4 - 3x = 2$
(m)	$x + 2 = 8 - x$	(n)	$x + 4 = 10 - 2x$
(o)	$x + 4 = 9 - 2x$	(p)	$8 - x = 12 - 2x$
(q)	$22 - 4x = 18 - 2x$	(r)	$3 - 6x = 2 - 4x$
(s)	$3(x + 2) = 5(x - 2)$	(t)	$4 = 8 - \frac{x}{3}$
(u)	$3 - \frac{x}{4} = -5$	(v)	$4(x - 2) = 3(x + 2)$
(w)	$5 = 18 - \frac{x}{3}$	(x)	$2 - \frac{x}{4} = 1 - \frac{x}{6}$

3. For each diagram below, write down an equation involving x and solve it.

(a)



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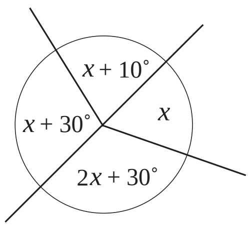

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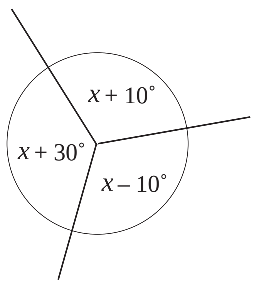

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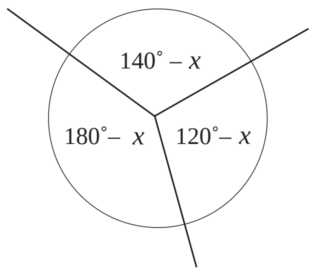


4. A rope of length 10 m is used to mark out a rectangle, so that the two long sides are 1 m longer than the short sides. If x is the length of the short sides, write down an equation to describe this situation and hence find x.

5. You ask a friend to think of a number, double it and add 10. His answer is 42. If x is the number your friend thought of, write down the relevant equation and find x.

6. Six teams enter a competition. There are x members in each team. If 8 people drop out and 34 complete the competition, write down an equation and solve it to find the number in each team at the start of the competition.

7. Three people drive a car on a long journey. John drives for 2 hours more than Mary. Philip drives for twice as long as Mary. The whole journey takes 6 hours. Use an equation to find out for how long each person drives.

8. A driver travels 80 miles to a motorway and then travels at a steady 60 mph for x hours. Write down and solve equations involving x if the driver travels a total of

   (a) 290 miles (b) 220 miles.

   Give your answers in hours and minutes.

9. A child was asked to think of a number and follow these instructions. In each case, let x be the number the child thinks of, write down an equation, and find the value of x.

   (a) Think of a number, add 6 and double it. *Answer* 18

   (b) Think of a number, divide by 2 and add 10. *Answer* 16

   (c) Think of a number, divide by 2, add 2 and multiply by 2. *Answer* 9

   (d) Think of a number, subtract 7, divide by 2 and multiply by 10. *Answer* 115

10. Four consecutive numbers, when added together, give a total of 114. If x is the lowest number, write down an equation and solve it.

11. Adrian thinks of a number. He doubles it and then adds 5. The answer is 17. What was his number?

(SEG)

12. (a) Write, in symbols, the rule,

To find y, double x and add 1.

   (b) Use your rule from part (a) to calculate the value of x when $y = 9$.

(LON)

13. Jacob uses this rule,

Start with a number, divide it by 2 and then add 3. Write down the result.

   (a) What is the result when Jacob starts with 8?

   (b) What number did Jacob start with when the result is 5?

(SEG)

14. Solve the equation

$$11x + 5 = x + 25.$$

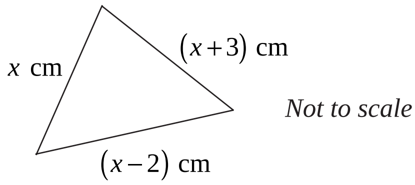
(LON)

15. Solve the equations

   (a) $4x - 1 = 17$ (b) $11y + 3 = 5y + 27$

(MEG)

16. The lengths of the sides of a triangle are x cm, $(x + 3)$ cm and $(x - 2)$ cm.



(a) What is the perimeter of the triangle in terms of x?

(b) The triangle has a perimeter of 22 cm.

(i) Write down an equation in x.

(ii) Use your equation to find the length of each side of the triangle

(SEG)

17. A pint of milk costs x pence.

(a) Write, in terms of x, the cost of two pints of milk.

The cost of a carton of fruit juice is 10 p more than the cost of a pint of milk.

(b) Write in terms of x, the cost of a carton of fruit juice.

Sam pays £1.70 for three pints of milk and two cartons of fruit juice. He writes down the correct equation

$$3x + 2(x + 10) = 170$$

(c) Solve this equation to find the cost of a pint of milk.

(SEG)

10.6 Trial and Improvement Method

The equations which have been solved so far are called *linear* equations, as they contain only x or number terms. However, some equations are *not* linear, that is they contain, for example, x^2 or x^3 terms.

Equations such as these can be solved by a *trial and improvement* method.

Worked Example 1

Solve the equation

$$x^3 + x = 5$$

by using a trial and improvement method.

Solution

First give a value of x to start the process, for example 1.

Start with 1 and substitute 1 into $x^3 + x$, giving

$$\begin{aligned} 1^3 + 1 &= 1 + 1 \\ &= 2. \end{aligned}$$

As this is less than 5, x must be greater than 1. So choose a larger value, say 2.

Then substitute 2 into $x^3 + x$ to get

$$2^3 + 2 = 8 + 2$$
$$= 10.$$

As this is greater than 5, x must be less than 2. But x is also greater than 1, so you know that x must lie in the interval

$$1 < x < 2.$$

The next stage is to choose a value between 1 and 2, say 1.5. This is substituted into $x^3 + x$ and the process is repeated.

The results of each trial can be shown in a table, as below.

Trial Value of x	$x^3 + x$ (to 4 d.p.)	*Comment*	*Conclusion*	
1	2	less than 5	$x > 1$	
2	10	greater than 5	$1 < x < 2$	
1.5	5.0625	greater than 5	$1 < x < 1.5$	
1.49	4.9288	less than 5	$1.49 < x < 1.50$	← so $x = 1.5$ to 1 d.p.
1.495	4.9953	less than 5	$1.495 < x < 1.500$	← so $x = 1.50$ to 2 d.p.
1.497	5.0221	greater than 5	$1.495 < x < 1.497$	
1.496	5.0087	greater than 5	$1.495 < x < 1.496$	
1.4955	5.0020	greater than 5	$1.4950 < x < 1.4955$	← so $x = 1.495$ to 3 d.p.

Note

This process could be continued to give the answer correct to any number of decimal places. Usually questions will specify the accuracy required and you must always state the accuracy of your solution. You should ensure that all working is correct to at least two decimal places (or significant figures) more than is required in the final answer; it is a very common mistake to approximate answers too soon in the calculation.

So in this worked example, the answer is 1.495 correct to 3 d.p.

Worked Example 2

Solve the equation

$$\sqrt{x} + 1 - x = 0$$

using the trial and improvement method, giving your answer to 2 decimal places.

Solution

The working is displayed in a table, as in the previous example.

Trial Value of x	$\sqrt{x} + 1 - x$ (to 4 d.p.)	*Comment*	*Conclusion*
1	1	greater than 0	$x > 1$
2	0.4142	greater than 0	$x > 2$
3	-0.2679	less than 0	$2 < x < 3$
2.5	0.0811	greater than 0	$2.5 < x < 3$
2.6	0.0125	greater than 0	$2.6 < x < 3$
2.7	-0.0568	less than 0	$2.6 < x < 2.7$
2.65	-0.0221	less than 0	$2.60 < x < 2.65$
2.62	-0.0014	less than 0	$2.60 < x < 2.62$
2.61	0.0055	greater than 0	$2.61 < x < 2.62$
2.615	0.0021	greater than 0	$2.615 < x < 2.62$

So $x = 2.62$ to 2 decimal places.

Exercises

1. Each of the following equations have solutions which are whole numbers.
   Find the solution of each equation by using a trial and improvement method.

   (a) $x^2 + x = 30$ (b) $x^2 - x = 702$

   (c) $x^3 + 2x = 135$ (d) $x^3 + 2x = 1752$

   (e) $x^5 - x = 240$ (f) $x + \sqrt{x} = 72$

   (g) $x - \sqrt{x} = 90$ (h) $x - x^3 = -1320$.

2. Solve each of the following equations, giving your answers to 1 decimal place.

   (a) $x^3 - 1 = 10$ (b) $x + \sqrt{x} = 7$ (c) $x^3 - x = 4$

   (d) $\sqrt{x} + x = 8$ (e) $x^2 - \sqrt{x} = 5$ (f) $x^3 + x = 16$

3. Trial and improvement can be used to find cube roots of numbers. To find the cube root of 6, you need to solve $x^3 = 6$.

   (a) Solve $x^3 = 6$ to 2 decimal places, starting with a first trial value of 2.

   (b) What would be a good first trial value when solving $x^3 = 1020$? Find x.

   (c) In the same way find the cube root of 350.

Just For Fun

Find a number, the double of which is 99 greater than its half.

4. Some equations have more than one solution. The equations below all have two solutions, some of which are negative. Find the solutions by using the trial and improvement method.

(a) $x^2 + x = 56$ (b) $x^2 - x = 72$ (c) $x^4 - 1 = 0$

(d) $x^2 - 2x = 80$ (e) $x^2 - 6x = 7$ (f) $x^2 + 5x - 24 = 0$

5. Find both solutions to the following equations, giving your answers to 2 decimal places.

(a) $x^2 + 5x - 7 = 0$ (b) $x^2 - 6x + 4 = 0$

(c) $x^2 + x - 10 = 0$ (d) $x^2 + 6x = 8$

6. (a) Explain why the equation $\sqrt{x} + x = 8$ cannot have a solution which is negative.

(b) Find the solution correct to 3 decimal places.

7. Solve the following equations, giving your answers to 2 decimal places.

(a) $\frac{1}{x} + x = 8$ (b) $x - \frac{1}{x} = 5$ (c) $\frac{2}{x} + x = 4$

(d) $\frac{4}{x} + \sqrt{x} = 5$ (e) $5x - \frac{2}{x} = 8$ (f) $\frac{5}{x} + x = 7$

8. Jimmy tries to solve this problem using trial and improvement.

$$number \times 3.1 = 5.6$$

Try 2.0	$2.0 \times 3.1 = 6.2$	Too large
Try 1.5	$1.5 \times 3.1 = 4.65$	Too small
. . .	. . .	. . .

Continue Jimmy's working until you know the number correct to one decimal place. Write down this number.

(SEG)

9. Judy is using 'trial and improvement' to solve the equation

$$x^2 + x = 11.$$

Copy and complete her working and find a solution correct to one decimal place.

Try $x = 3.5$	$3.5^2 + 3.5 = 15.75$	Too large
Try $x = 2.5$	$2.5^2 + 2.5 = 8.75$	Too small
Try $x = 3.0$	. . .	. . .
. . .	. . .	. . .

(SEG)

10. (a) Solve the equation $3x + 2 = 17$.

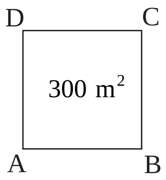
(b) (i) Work out $4^3 = ?$, $5^3 = ?$

(ii) Use a trial and improvement method to solve, correct to 1 decimal place, the equation

$$x^3 = 100.$$

Show all your trials.

(MEG)

11. 

The length, AB, of a side of a square with area 300 m^2 is $\sqrt{300}$ m.

Sabeeha wants to find this length but her calculator does not have a $\boxed{\sqrt{\ }}$ key, so she has to use a trial and improvement method.

Her first 2 trials are as follows:

$$18 \times 18 = 324 \qquad \text{(too big)}$$
$$17 \times 17 = 289 \qquad \text{(too small)}$$

(a) Showing your working clearly, continue for at least 2 more trials until you have found the length of AB correct to 1 decimal place.

(b) Change your answer to part (a) into centimetres.

(MEG)

12. The number x satisfies the equation

$$x^3 = 20.$$

(a) Between which two consecutive whole numbers does x lie?

(b) Use a trial and improvement method to find this value of x correct to one decimal place. *Show all your working clearly.*

(NEAB)

10.7 Expanding Brackets

Equations or formulae may contain brackets, for example

$$P = 8(x - 7) \quad \text{or} \quad y = (2x - 3)(x - 8)$$

Removing brackets from these types of expressions is known as *expanding*.

When evaluating or manipulating expressions it is sometimes important to be able to use the original equation, whereas at other times the expanded version might be helpful.

The reverse of expanding, the process of factorisation, is even more important and becomes easier once you have gained confidence in expanding brackets.

Worked Example 1

Expand $2x(5x-8)$.

Solution

$$\begin{aligned} 2x(5x-8) &= 2x\times 5x - 2x\times 8 \\ &= 10x^2 - 16x. \end{aligned}$$

Worked Example 2

Expand $-4(x-6)$.

Solution

$$\begin{aligned} -4(x-6) &= -4\times x + (-4)\times(-6) \\ &= -4x + 24. \end{aligned}$$

Sometimes situations will arise where a bracket has to be multiplied by another bracket, as in the next example.

Worked Example 3

Expand $(x+2)(x+5)$.

Solution

The first bracket $(x+2)$ is split so that each if its terms, (x) and $(+2)$, can multiply the other bracket.

$$\begin{aligned} (x+2)(x+5) &= x(x+5) + 2(x+5) \\ &= x^2 + 5x + 2x + 10 \\ &= x^2 + 7x + 10. \qquad \text{(collecting like terms)} \end{aligned}$$

Worked Example 4

Expand $(4x-3)(2x-7)$

Solution

Splitting the first bracket and multiplying the other bracket by each of its terms gives,

$$\begin{aligned} (4x-3)(2x-7) &= 4x(2x-7) - 3(2x-7) \\ &= 8x^2 - 28x - 6x + 21 \\ &= 8x^2 - 34x + 21. \end{aligned}$$

Just For Fun

Which area is larger and by how much – a half km square or a half square km?

Worked Example 5

Expand $(x+6)^2$.

Solution

First note that

$$(x+6)^2 = (x+6)(x+6).$$

Then the brackets can be expanded.

$$\begin{aligned}(x+6)(x+6) &= x(x+6)+6(x+6)\\ &= x^2+6x+6x+36\\ &= x^2+12x+36.\end{aligned}$$

Worked Example 6

Expand $(3x-2n)(4x+5n)$.

Solution

$$\begin{aligned}(3x-2n)(4x+5n) &= 3x(4x+5n)-2n(4x+5n)\\ &= 12x^2+15xn-8nx-10n^2\\ &= 12x^2+7xn-10n^2.\end{aligned}$$

Note that xn is the same as nx.

Exercises

1. Simplify each expression below, expanding the brackets as necessary.

(a)	$x(x+1)$	(b)	$-x(3x-2)$	(c)	$3n(2a-5)$
(d)	$x^2(1-x)$	(e)	$4x(x-7)$	(f)	$5+3(x+2)$
(g)	$4x+x(x-6)$	(h)	$x(1+x^2)$	(i)	$5-4(x^2+2)$
(j)	$x^2-x(x+5)$	(k)	$x^2-4x(5-x)$	(l)	$x^2+4(x^2-2)$
(m)	$x^3+x^2(5-2x)$	(n)	$4(x+4)-x$	(o)	$5x-x(3x-2)$

2. Expand and simplify each of the following.

(a)	$(x+1)(x+6)$	(b)	$(x+4)(x+7)$	(c)	$(x-2)(x+8)$
(d)	$(x-1)(x+4)$	(e)	$(x-7)(x-1)$	(f)	$(2x+1)(3x-1)$
(g)	$(4x+3)(2x+7)$	(h)	$(5x-2)(2x+3)$	(i)	$(6x-1)(4x+3)$
(j)	$(3x-1)(4x-3)$	(k)	$(5x-6)(3x-8)$	(l)	$(3a-7)(5a+2)$
(m)	$(4a-5)(3a-5)$	(n)	$(6n+1)(4n-2)$	(o)	$(4x+3)(5x-8)$

3. Expand and simplify each of the following.

(a) $(x+1)^2$ (b) $(x-1)^2$ (c) $(x+8)^2$

(d) $(2x+1)^2$ (e) $(3x+4)^2$ (f) $(6x-1)^2$

(g) $(2x-5)^2$ (h) $(6x-7)^2$ (i) $(5x+9)^2$

4. Expand and simplify, whenever possible, the following.

(a) $(a+b)(c+d)$ (b) $(a+c)(2a+c)$ (c) $(3a+d)(a-5d)$

(d) $(4x-y)(3x+y)$ (e) $(4a+d)(a+3d)$ (f) $(2a+b)(a+4c)$

(g) $(6x-y)(y-3x)$ (h) $(p+q)(q-2p)$ (i) $(5x+y)(9x-2y)$

(j) $(4x-2)(x+y)$ (k) $(2a+b)(2a-c)$ (l) $(4x-5y)(x-6y)$

(m) $(p-2q)(p+2q)$ (n) $(5a+2b)(a-3b)$ (o) $(5x-6y)(2x-3y)$

5. (a) Expand and simplify

(i) $(x+1)(x-1)$ (ii) $(x+4)(x-4)$ (iii) $(x+6)(x-6)$

(b) Without expanding the brackets, write down what you would expect to get if you expanded

$$(x+5)(x-5)$$

(c) Expand and simplify

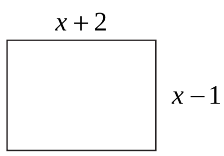
$$(a+b)(a-b)$$

(d) What would you expect to obtain if you expanded

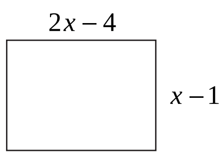
$$(5x+2)(5x-2)?$$

6. Find the area of each rectangle.

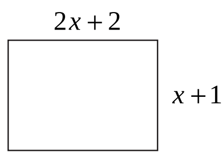
(a)



(b)



(c)



7. Expand and simplify each expression.

(a) $x(x+2)(x-5)$ (b) $(x+2)(x^2+2x-8)$

(c) $(x+1)(x-2)(x+3)$ (d) $(2x+1)(3x-4)(x+7)$

(e) $(x+1)^3$ (f) $(x-4)^3$

10.8 Simultaneous Linear Equations

A pair of equations which use both terms at the same time, such as

$$x + 2y = 8$$
$$2x + y = 7,$$

are known as a pair of *simultaneous* equations.

Worked Example 1

Solve the pair of simultaneous equations

$$x + 2y = 8$$
$$2x + y = 7.$$

Solution

First it is helpful to label the equations (1) and (2).

$$\begin{aligned} x + 2y &= 8 \qquad (1) \\ 2x + y &= 7 \qquad (2) \end{aligned}$$

Equation (1) is multiplied by 2, so that it contains the same number of x's as equation (2).

Let the new equation be labelled (3).

$$\begin{aligned} 2x + 4y &= 16 \qquad (3) \qquad [2 \times (1)] \\ 2x + y &= 7 \qquad (2) \end{aligned}$$

Equation (2) is now subtracted from equation (3).

$$\begin{aligned} 2x + 4y &= 16 \qquad (3) \\ 2x + y &= 7 \qquad (2) \\ \hline 3y &= 9 \qquad (3) - (2) \end{aligned}$$

Solving $3y = 9$ gives $y = 3$.

This value of y can now be substituted into equation (1) to give:

$$\begin{aligned} x + 2 \times 3 &= 8 \\ x + 6 &= 8 \end{aligned}$$

Solving this gives $x = 2$. So the solution to the equation is $x = 2$, $y = 3$.

Worked Example 2

Solve the simultaneous equations

$$\begin{aligned} 3x + 5y &= 2 \\ -4x + 7y &= -30. \end{aligned}$$

Solution

First label the equations (1) and (2) as shown below.

$$\begin{aligned} 3x + 5y &= 2 \qquad &(1) \\ -4x + 7y &= -30 \qquad &(2) \end{aligned}$$

Then multiply equation (1) by 4 and equation (2) by 3 to make the number of x's in both equations the same.

$$\begin{aligned} 12x + 20y &= 8 \qquad &(3) \qquad &[4 \times (1)] \\ -12x + 21y &= -90 \qquad &(4) \qquad &[3 \times (2)] \end{aligned}$$

Now add together equations (3) and (4) to give

$$\begin{aligned} 12x + 20y &= 8 \qquad &(3) \\ -12x + 21y &= -90 \qquad &(4) \\ \hline 41y &= -82 \qquad &(3) + (4) \end{aligned}$$

Solving the equation $41y = -82$ gives $y = -2$.

This value for y can be substituted into equation (1) to give

$$\begin{aligned} 3x + 5 \times (-2) &= 2 \\ \text{or} \qquad 3x - 10 &= 2. \end{aligned}$$

Solving this equation gives:

$$\begin{aligned} 3x - 10 &= 2 \\ 3x &= 12 \\ x &= \frac{12}{3} \\ &= 4. \end{aligned}$$

So the solution is $x = 4$ and $y = -2$.

Note

It is a good idea to check that solutions are correct by substituting these values back into the original equations. Here,

$$3 \times 4 + 5 \times (-2) = 2$$

and

$$-4 \times 4 + 7 \times (-2) = -30$$

You must check *both* equations to make sure that you have the correct answer.

Worked Example 3

Denise sells 300 tickets for a concert. Some tickets are sold to adults at £5 each and some are sold to children at £4 each. If she collects in £1444 in ticket sales, how many tickets have been sold to adults and how many to children?

Solution

Let x = number of adults' tickets
and y = number of children's tickets.

She has sold 300 tickets, so

$$x + y = 300.$$

The value of the adult tickets sold is £5x, and the value of the children's tickets is £4y.

As the value of all the tickets sold is £1444, then

$$5x + 4y = 1444.$$

The two simultaneous equations

$$x + y = 300 \qquad (1)$$
$$5x + 4y = 1444 \qquad (2)$$

can now be solved. First multiply equation (1) by 5 and subtract equation (2) to give

$$5x + 5y = 1500 \qquad (3) \qquad [5 \times (1)]$$
$$\underline{5x + 4y = 1444} \qquad (2)$$
$$y = 56 \qquad (3) - (2)$$

This value can then be substituted into equation (1) to give

$$x + 56 = 300$$
or
$$x = 244.$$

So the solution is $x = 244$ and $y = 56$. That is, 244 adults' tickets and 56 children's tickets have been sold.

Investigation

Consider the following simultaneous equations.

$$2x + y = 6 \qquad (1)$$
$$x = 1 - \tfrac{1}{2}y \qquad (2)$$

If (2) is substituted for x into (1), then

$$2\left(1 - \tfrac{1}{2}y\right) + y = 6$$
$$2 - y + y = 6$$
$$2 = 6!$$

Find out where the problem lies.

Exercises

1. Solve each pair of simultaneous equations.

(a) $\begin{aligned} x + 2y &= 5 \\ 3x + y &= 5 \end{aligned}$ (b) $\begin{aligned} 3x + 2y &= 19 \\ x + 5y &= 15 \end{aligned}$ (c) $\begin{aligned} x - 2y &= 4 \\ 4x + 3y &= 49 \end{aligned}$

(d) $\begin{aligned} 2x + 3y &= 14 \\ 5x + 2y &= 24 \end{aligned}$ (e) $\begin{aligned} 3x + 4y &= 2 \\ 7x - 5y &= 9 \end{aligned}$ (f) $\begin{aligned} 4x + 2y &= 16 \\ -3x + 2y &= -19 \end{aligned}$

(g) $\begin{aligned} 5x + y &= 2 \\ -4x + 3y &= 44 \end{aligned}$ (h) $\begin{aligned} 6x - 4y &= 12 \\ -9x + 2y &= -66 \end{aligned}$ (i) $\begin{aligned} 7x - 2y &= 23 \\ 3x + 4y &= 39 \end{aligned}$

(j) $\begin{aligned} 8x + 4y &= 7 \\ -12x + 8y &= -6 \end{aligned}$ (k) $\begin{aligned} 4x - 2y &= -0.1 \\ 5x + 2y &= 1.5 \end{aligned}$ (l) $\begin{aligned} 6x - 5y &= 41 \\ 4x + 15y &= 31 \end{aligned}$

(m) $\begin{aligned} -2x + 5y &= 14 \\ 10x + 7y &= 26 \end{aligned}$ (n) $\begin{aligned} 8x + 5y &= -29 \\ 3x - 7y &= -2 \end{aligned}$ (o) $\begin{aligned} 6x - 5y &= -14 \\ 18x - 4y &= 6 \end{aligned}$

(p) $\begin{aligned} 6x - 8y &= -2 \\ 5x + 2y &= 1.8 \end{aligned}$ (q) $\begin{aligned} \frac{1}{2}x - \frac{1}{4}y &= 0 \\ \frac{1}{3}x + \frac{2}{3}y &= 10 \end{aligned}$ (r) $\begin{aligned} \frac{1}{5}x - \frac{1}{10}y &= -1 \\ \frac{1}{4}x + \frac{1}{2}y &= 10 \end{aligned}$

2. Find the coordinates of the point of intersection of the lines:

(a) $x + y = 8$ and $y = 2x - 1$

(b) $x + y = 10$ and $y = 2x + 1$

(c) $x + y = 4$ and $y = 2 - \dfrac{x}{10}$

3. Describe the problems you encounter when you try to solve the simultaneous equations:

$$\begin{aligned} 3x - 2y &= 8 \\ 9x - 6y &= 2. \end{aligned}$$

4. (a) Check that $x = 2$ and $y = 5$ is a solution of both the equations below.

$$\begin{aligned} x + 2y &= 12 \\ 3x + 6y &= 36 \end{aligned}$$

(b) Try to solve the equations. What happens?

(c) Write both equations in the form $y = \ldots$ and comment on the equations you obtain.

5. Ian rows a boat in a river where there is a steady current. He travels at 1.3 mph upstream and 3.7 mph downstream.

   Use a pair of simultaneous equations to find v, the speed of the boat in still water, and c, the speed of the current.

6. A machine sells tickets for travel on a tram system. A single ticket costs £1 and a return ticket costs £2. In one day, the machine sells 100 tickets and takes £172. How many of each type of ticket were sold?

7. A shopkeeper takes 200 notes to a bank. They are a mixture of £5 and £10 notes. The shopkeeper claims that there is £1395.

   (a) Find out how many of each type of note there should be.

   (b) In fact the value of the notes turns out to be £1390. How many of each type of note does this mean there should be? What error do you think the shopkeeper made?

8. A bus stops at two places. The fare to the first stop is £3 and the fare to the second stop is £5. When the bus sets out, there are 38 people on the bus and the driver has collected £158 in fares. How many people get off the bus at the first stop?

9. When Isa travels to the USA for a holiday he leaves the UK at 1.00 pm local time and lands at 5.00 pm local time. On the return journey he leaves at 8.00 pm local time and lands at 10.00 am local time the next day.

   Find the length of the flight in hours and the time difference between the UK and the part of the USA that Isa visited.

10. Solve the simultaneous equations,

$$\begin{aligned} 2x + 3y &= 23 \\ x + y &= 4 \end{aligned}$$

(LON)

11. Solve the simultaneous equations,

$$\begin{aligned} 2a + 4c &= 13 \\ a + 3c &= 8 \end{aligned}$$

(NEAB)

12. The height, h metres, of a sky rocket t seconds after being launched is given by the formula

$$h = at^2 + bt + 2$$

   where a and b are constants. The heights of the rocket above the ground at two different times are given in the table below.

t (seconds)	1	2
h (metres)	37	62

(a) At what height above the ground is the rocket launched?

(b) (i) Use the table of values to show that

$$a + b = 35$$

and $$4a + 2b = 60.$$

(ii) Solve these simultaneous equations to find the value of a and the value of b.

(c) What was the height of the sky rocket $7\frac{1}{2}$ seconds after it was launched?

(MEG)

10.9 Factorisation 1

The process of *removing* brackets is known as *expanding*. The reverse process is known as *factorisation*, where an expression is re-written as a product of terms.

To factorise an expression it is necessary to identify numbers or variables which are factors common to all the terms.

Worked Example 1

Factorise $6x + 8$.

Solution

Both terms ($6x$) and (8) can be divided by 2, so the expression is factorised as

$$\begin{aligned} 6x + 8 &= (2 \times 3x) + (2 \times 4) \\ &= 2(3x + 4). \end{aligned}$$

Worked Example 2

Factorise $12a - 16$.

Solution

Here the largest number by which both terms, (12a) and (16), can be divided is 4.

$$\begin{aligned} 12a - 16 &= (4 \times 3a) - (4 \times 4) \\ &= 4(3a - 4). \end{aligned}$$

Worked Example 3

Factorise $4x^2 - 8x$.

Solution

Here 4 is the largest number that will divide both terms, but each term can also be divided by x, so $4x$ is the factor common to both terms.

$$4x^2 - 8x = (4x \times x) - (4x \times 2)$$
$$= 4x(x - 2).$$

Exercises

1. Copy and complete each of the following.

(a) $5x + 10 = ?(x + 2)$ (b) $6x - 8 = ?(3x - 4)$

(c) $15x + 25 = ?(3x + 5)$ (d) $12x + 8 = 4(? + ?)$

(e) $18 - 6n = 6(? - ?)$ (f) $6x - 21 = 3(? - ?)$

(g) $16a + 24 = 8(? + ?)$ (h) $33x - 9 = 3(? - ?)$

2. Factorise each of the following expressions.

(a) $6x + 24$ (b) $5x - 20$ (c) $16 - 8x$

(d) $8n + 12$ (e) $12x - 14$ (f) $3a - 24$

(g) $11x - 66$ (h) $10 + 25x$ (i) $100x - 40$

(j) $50 - 40x$ (k) $6x - 30$ (l) $5y - 45$

(m) $12 + 36x$ (n) $16x + 32$ (o) $27x - 33$

3. Complete a copy of each of the following.

(a) $x^2 + x = ?(x + 1)$ (b) $x^2 + 2x = ?(x + 2)$

(c) $2a^2 - 5a = ?(2a - 5)$ (d) $4x^2 + x = x(? + ?)$

(e) $x^2 + 4x = x(? + ?)$ (f) $xa + xb = x(? + ?)$

(g) $6x^2 + 3x = 3x(? + ?)$ (h) $4x^2 - 2ax = 2x(? - ?)$

4. Factorise each of the following expressions.

(a) $5x^2 + x$ (b) $a^2 + 3a$ (c) $5n^2 + 2n$

(d) $6n^2 + 3n$ (e) $5n^2 - 10n$ (f) $3x^2 + 6x$

(g) $15x^2 + 30x$ (h) $14x^2 + 21x$ (i) $16x^2 + 24x$

(j) $30x^2 - 18x$ (k) $5 + 5n^2$ (l) $10n^2 - 15$

(m) $3n^3 + 9n$ (n) $9x^2 - 27x$ (o) $10x^3 - 5x^2$

5. Factorise each of the following expressions.

(a) $ax + ax^2$ (b) $bx + cx^2$ (c) $2pq - 4rq$

(d) $15xy - 5y^2$ (e) $16pq + 24p^2$ (f) $6x^2 + 18xy$

(g) $3p^2 - 9px$ (h) $24px + 56x^2$ (i) $16x^2y - 18xy^2$

6. For each factorisation shown below, state whether it can be factorised further. If the answer is yes, give the complete factorisation.

(a) $6x^2 + 4x = 2\left(3x^2 + 2x\right)$ (b) $16x^3 + 8x^2 = 8x\left(2x^2 + x\right)$

(c) $5x^2 - 60x = 5x(x - 12)$ (d) $3x^2y - 18xy^2 = 3x\left(xy - 6y^2\right)$

10.10 Factorisation 2

It is also possible to factorise expressions such as

$$x^2 + 5x + 6$$

to obtain

$$(x + 2)(x + 3).$$

First consider what happens when two brackets are multiplied together. For example,

$$(x + 2)(x + 3) = x^2 + 5x + 6.$$

Note that the 5 is given by $2 + 3$ and the 6 is given by 2×3.

When factorising a quadratic like this we need to find two numbers which, when added together, give one number and when multiplied together give the other number.

For example, when factorising

$$x^2 + 8x + 12$$

we need two numbers which give 12 when multiplied and 8 when added. These are, of course, 2 and 6. Hence $x^2 + 8x + 12 = (x + 2)(x + 6)$.

Worked Example 1

Factorise $x^2 + 9x + 20$.

Solution

The solution will be of the form

$$(x + a)(x + b)$$

where $a \times b = 20$ and $a + b = 9$.

You may immediately see that the two numbers are 4 and 5. However, it will not always be obvious. A helpful approach is to write the possible pairs of numbers which multiply to give 20.

$$x^2 + 9x + 20 = (x + \quad)(x + \quad)$$

1	20
2	10
4	5

It is then easy to see that only the third pair of numbers add up to 9. So

$$x^2 + 9x + 20 = (x + 4)(x + 5).$$

Worked Example 2

Factorise $x^2 - 3x - 10$.

Solution

The solution will be of the form $(x \quad)(x \quad)$. Considering the ways of obtaining -10 by multiplication gives

$$x^2 - 3x - 10 = (x \quad)(x \quad)$$

-1	$+10$
-2	$+5$
$+1$	-10
$+2$	-5

Only the fourth possibility gives a total of -3 when the two terms are added, so

$$x^2 - 3x - 10 = (x + 2)(x - 5).$$

Worked Example 3

Factorise $x^2 - 5x + 6$.

Solution

The solution will be of the form $(x \quad)(x \quad)$. Considering ways of obtaining $+6$ (including negative factors since the x component has a negative coefficient) gives:

$$x^2 - 5x + 6 = (x \quad)(x \quad)$$

$+6$	$+1$
$+3$	$+2$
-6	-1
-3	-2

The last of these gives a total of -5 when the two terms are added, so

$$x^2 - 5x + 6 = (x - 3)(x - 2).$$

Worked Example 4

Factorise $2x^2 - x - 3$.

Solution

The solution will be of the form $(2x \quad)(x \quad)$ to give the $2x^2$ term. Considering the ways of obtaining -3 gives:

$$2x^2 - x - 3 = (2x \quad)(x \quad)$$

$$\begin{array}{r|r|} -1 & +3 \\ +1 & -3 \\ +3 & -1 \\ -3 & +1 \end{array}$$

Note that these will be multiplied by the 2 in the $2x$ term

From the last of these we can obtain the middle middle term,

$$(-3 \times x) + (1 \times 2x) = -3x + 2x$$
$$= -x$$

so

$$2x^2 - x - 3 = (2x - 3)(x + 1).$$

You can check that these brackets multiply out to give the original expression.

Note

It is often a good idea to check the answers you obtain by expanding the brackets.

You may remember that

$$\boxed{(a - b)(a + b) = a^2 - b^2}$$

This result is known as the *difference between two squares* and can be used to factorise some expressions.

Worked Example 5

Factorise the following using the difference between two squares result.

(a) $x^2 - 9$ (b) $4x^2 - 25$.

Solution

(a) $x^2 - 9 = x^2 - 3^2$

$= (x + 3)(x - 3)$

(b) $4x^2 - 25 = (2x)^2 - 5^2$

$= (2x - 5)(2x + 5)$

Exercises

1. Factorise each of the following expressions.

(a) $x^2 + 4x + 4$ (b) $x^2 + 7x + 12$ (c) $x^2 + 6x + 8$

(d) $x^2 + 7x + 6$ (e) $x^2 + 10x + 16$ (f) $x^2 + 4x + 3$

(g) $x^2 + 8x + 15$ (h) $x^2 + 3x + 2$ (i) $x^2 + 5x + 4$

(j) $x^2 + 11x + 24$ (k) $x^2 + 12x + 11$ (l) $x^2 + 15x + 56$

(m) $x^2 + 6x + 9$ (n) $x^2 + 7x + 10$ (o) $x^2 + 9x + 14$

(p) $x^2 + 11x + 30$ (q) $x^2 + 9x + 8$ (r) $x^2 + 12x + 32$

2. Factorise the following expressions.

(a) $x^2 + x - 2$ (b) $x^2 - x - 12$ (c) $x^2 - 3x - 10$

(d) $x^2 + 4x - 5$ (e) $x^2 - 5x - 14$ (f) $x^2 - 2x - 8$

(g) $x^2 + 2x - 15$ (h) $x^2 - 3x + 2$ (i) $x^2 - 9x + 20$

(j) $x^2 - 10x + 21$ (k) $x^2 - 9x + 14$ (l) $x^2 - 7x + 10$

(m) $x^2 - 6x - 16$ (n) $x^2 - 17x + 72$ (o) $x^2 - 5x - 24$

3. Factorise each of the following using the difference of two squares result.

(a) $x^2 - 1$ (b) $x^2 - 16$ (c) $x^2 - 81$

(d) $9x^2 - 4$ (e) $16x^2 - 36$ (f) $4x^2 - 100$

(g) $x^4 - 100$ (h) $x^4 - 4$ (i) $4x^4 - 9$

4. Complete each of the following factorisations.

(a) $2x^2 + 3x + 1 = (2x + 1)(x + ?)$

(b) $3x^2 + 7x + 2 = (3x + 1)(x + ?)$

(c) $2x^2 + 5x + 3 = (2x + ?)(x + 1)$

(d) $3x^2 + 14x + 8 = (3x + ?)(x + 4)$

(e) $2x^2 + 9x - 5 = (2x - 1)(x + ?)$

(f) $4x^2 - 5x - 6 = (4x + ?)(x - ?)$

(g) $3x^2 + x - 10 = (3x - ?)(x + ?)$

(h) $3x^2 - 23x + 14 = (3x - ?)(x - ?)$

(i) $6x^2 + 17x + 5 = (3x + ?)(2x + ?)$

(j) $8x^2 - 6x + 1 = (4x - ?)(2x - ?)$

5. Factorise each of the following expressions.

(a)	$3x^2 - 2x - 1$	(b)	$3x^2 + 4x + 1$	(c)	$2x^2 + 5x + 2$
(d)	$3x^2 + 8x + 4$	(e)	$3x^2 + 8x - 3$	(f)	$4x^2 - 11x - 3$
(g)	$5x^2 + 3x - 2$	(h)	$3x^2 - 8x + 4$	(i)	$5x^2 + 13x - 6$
(j)	$6x^2 + 5x + 1$	(k)	$6x^2 - 7x + 2$	(l)	$10x^2 - 3x - 1$
(m)	$8x^2 + 10x - 3$	(n)	$6x^2 + 19x - 7$	(o)	$6x^2 - 17x + 12$

6. (a) Expand $(2x + 1)(x + 4)$.

   (b) Factorise completely $4x^2 - 6x$.

(LON)

7. (a) Factorise completely $12p^2q - 15pq^2$.

   (b) Expand and simplify $(2x - 3)(x + 5)$.

   (c) The cost, C pence, of printing n party invitations is given by

$$C = 120 + 40n.$$

   Find a formula for n in terms of C.

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10.11 Solving Quadratic Equations by Factorisation

Equations of the form

$$\boxed{ax^2 + bx + c = 0}$$

are called *quadratic* equations. Many can be solved using factorisation. If a quadratic equation can be written as

$$(x - a)(x - b) = 0,$$

then the equation will be satisfied if *either* bracket is equal to zero. That is,

$$(x - a) = 0 \quad \text{or} \quad (x - b) = 0.$$

So there would be two possible solutions, $x = a$ and $x = b$.

Worked Example 1

Solve $x^2 + 6x + 5 = 0$.

Solution

Factorising gives

$$(x + 5)(x + 1) = 0.$$

So

$$x + 5 = 0 \quad \text{or} \quad x + 1 = 0,$$

therefore

$$x = -5 \quad \text{or} \quad x = -1.$$

Worked Example 2

Solve $x^2 + 5x - 14 = 0$.

Solution

Factorising gives

$$(x - 2)(x + 7) = 0.$$

So

$$x - 2 = 0 \quad \text{or} \quad x + 7 = 0,$$

therefore

$$x = 2 \quad \text{or} \quad x = -7.$$

Worked Example 3

Solve $x^2 - 12x = 0$.

Solution

Factorising gives

$$x(x - 12) = 0.$$

So

$$x = 0 \quad \text{or} \quad x - 12 = 0,$$

therefore

$$x = 0 \quad \text{or} \quad x = 12.$$

Worked Example 4

Solve

$$4x^2 - 81 = 0.$$

Solution

Factorising gives

$$(2x - 9)(2x + 9) = 0.$$

So

$$2x - 9 = 0 \quad \text{or} \quad 2x + 9 = 0,$$

therefore

$$x = \frac{9}{2} \quad \text{or} \quad x = -\frac{9}{2}$$

$$= 4\frac{1}{2} \qquad\qquad = -4\frac{1}{2}.$$

Worked Example 5

Solve $x^2 - 4x + 4 = 0$.

Solution

Factorising gives

$$(x-2)(x-2) = 0.$$

So

$$x - 2 = 0 \quad \text{or} \quad x - 2 = 0,$$

therefore

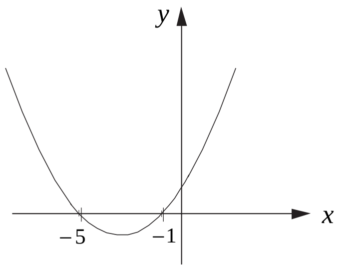
$$x = 2 \quad \text{or} \quad x = 2.$$

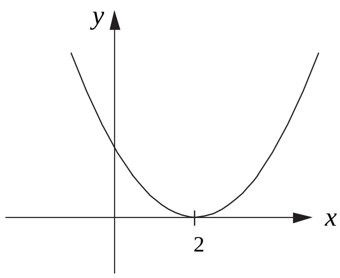
This type of solution is often called a *repeated* solution.

Most of these examples have had two solutions, but the last example had only one solution.

The graphs below show

$$y = x^2 + 6x + 5 \qquad \text{and} \qquad y = x^2 - 4x + 4.$$





The curve crosses the x-axis at $x = -5$ and $x = -1$.
These are the solutions of $x^2 + 6x + 5 = 0$.

The curve touches the x-axis at $x = 2$.
This is the solution of $x^2 - 4x + 4 = 0$.

Exercises

1. Solve the following quadratic equations.

(a) $x^2 + x - 12 = 0$ (b) $x^2 - 2x - 15 = 0$ (c) $x^2 + 4x - 12 = 0$

(d) $x^2 + 6x = 0$ (e) $3x^2 - 4x = 0$ (f) $4x^2 - 9x = 0$

(g) $x^2 - 9 = 0$ (h) $x^2 - 49 = 0$ (i) $9x^2 - 64 = 0$

(j) $x^2 - 8x + 16 = 0$ (k) $x^2 + 10x + 25 = 0$ (l) $x^2 - 3x - 18 = 0$

(m) $x^2 - 11x + 28 = 0$ (n) $x^2 + x - 30 = 0$ (o) $x^2 - 14x + 40 = 0$

(p) $2x^2 + 7x + 3 = 0$ (q) $2x^2 + 5x - 12 = 0$ (r) $3x^2 - 7x + 4 = 0$

(s) $4x^2 + x - 3 = 0$ (t) $2x^2 + 5x - 3 = 0$ (u) $2x^2 - 19x + 35 = 0$

2. The equations of a number of curves are given below. Find where each curve crosses the x-axis and use this to draw a sketch of the curve.

   (a) $y = x^2 + 6x + 9$ (b) $y = x^2 - 4$

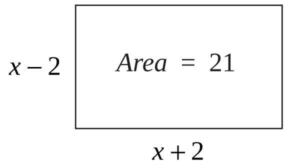
   (c) $y = 2x^2 - 3x$ (d) $y = x^2 + x - 12$

3. Use the difference of two squares result to solve the following equations.

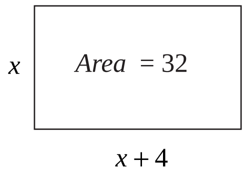
   (a) $x^4 - 16 = 0$ (b) $x^4 - 625 = 0$ (c) $x^6 - 1 = 0$

4. Find the lengths of each side of the rectangles given below.

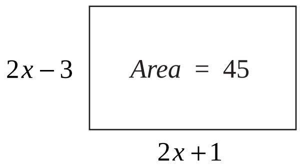
   (a)



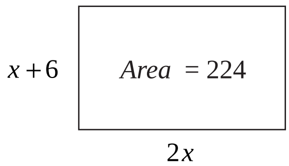
   (b)



   (c)



   (d)

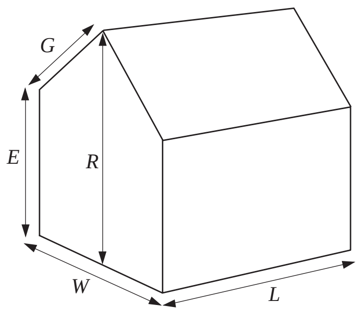


5. The height of a ball thrown straight up from the ground into the air at time, t, is given by

$$h = 8t - 10t^2 .$$

   Find the time it takes for the ball to go up and fall back to ground level.

6. The diagram represents a greenhouse.



   The volume of the greenhouse is given by the formula

$$V = \frac{1}{2} LW(E + R).$$

   (a) Make L the subject of the formula, giving your answer as simply as possible.

   The surface area, A, of the greenhouse, is given by the formula

$$A = 2GL + 2EL + W(E + R),$$

   where $V = 500$, $A = 300$, $E = 6$ and $G = 4$.

   (b) By substituting these values into the equations for V and A show that L satisfies the equation

$$L^2 - 15L + 50 = 0.$$

   Make the steps in your working clear.

   (c) Solve the equation $L^2 - 15L + 50 = 0$.

(LON)

10.12 Solving Quadratic Equations Using the Formula

Quadratic equations of the form

$$ax^2 + bx + c = 0$$

can be solved using the formula

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

This formula is particularly useful for quadratics which cannot be factorised. To prove this important result requires some quite complex analysis, using a technique called *completing the square*, which will be discussed again in *Section 10.14*.

Proof

The equation $ax^2 + bx + c = 0$ is first divided by the non-zero constant, a, giving

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Note that

$$\begin{aligned}
\left(x + \frac{b}{2a}\right)^2 &= \left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right) \\
&= x^2 + \frac{bx}{2a} + \frac{bx}{2a} + \left(\frac{b}{2a}\right)^2 \quad \text{(expanding)} \\
&= x^2 + \frac{2bx}{2a} + \left(\frac{b}{2a}\right)^2 \quad \text{(adding like terms)} \\
&= x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2. \quad \text{(simplifying)}
\end{aligned}$$

The first two terms are identical to the first two terms in our equation, so you can re-write the equation as

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\begin{aligned}
\left(x + \frac{b}{2a}\right)^2 &= \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \\
&= \frac{b^2}{4a^2} - \frac{c}{a}
\end{aligned}$$

i.e. $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Taking the square root of both sides of the equation gives

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$= \frac{\pm\sqrt{b^2 - 4ac}}{2a}.$$

Hence

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

or $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

as required.

Worked Example 1

Solve

$$x^2 + 6x - 8 = 0$$

giving the solution correct to 2 decimal places.

Solution

Here $a = 1$, $b = 6$ and $c = -8$. These values can be substituted into

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to give

$$x = \frac{-6 \pm \sqrt{6^2 - (4 \times 1 \times -8)}}{2 \times 1}$$

$$= \frac{-6 \pm \sqrt{68}}{2}$$

$$= \frac{-6 + \sqrt{68}}{2} \quad \text{or} \quad \frac{-6 - \sqrt{68}}{2}$$

$$= 1.12 \quad \text{or} \quad -7.12 \qquad \text{(to 2 d.p.)}$$

Worked Example 2

Solve the quadratic equation

$$4x^2 - 12x + 9 = 0.$$

Solution

Here $a = 4$, $b = -12$ and $c = 9$. Substituting the values into

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

gives

$$x = \frac{12 \pm \sqrt{(-12)^2 - (4 \times 4 \times 9)}}{2 \times 4}$$

$$= \frac{12 \pm \sqrt{144 - 144}}{8}$$

$$= \frac{12 \pm \sqrt{0}}{8}$$

$$= \frac{12}{8}$$

$$= 1.5.$$

Worked Example 3

Solve the quadratic equation

$$x^2 + x + 5 = 0.$$

Solution

Here $a = 1$, $b = 1$ and $c = 5$. Substituting the values into the formula gives

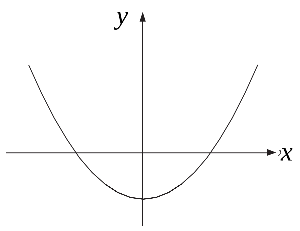
$$x = \frac{-1 \pm \sqrt{1^2 - (4 \times 1 \times 5)}}{2 \times 1}$$

$$= \frac{-1 \pm \sqrt{1 - 20}}{2}$$

$$= \frac{-1 \pm \sqrt{-19}}{2}.$$

As it is not possible to find $\sqrt{-19}$, this equation has *no* solutions.

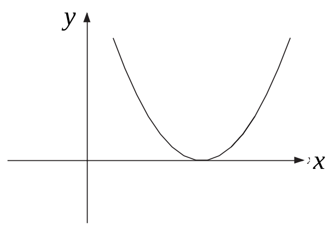
These three examples illustrate that a quadratic equation can have 2, 1 or 0 solutions. The graphs below illustrate these graphically and show how the number of solutions depends on the sign of $(b^2 - 4ac)$ which is part of the quadratic formula.



Two solutions

$b^2 - 4ac > 0$

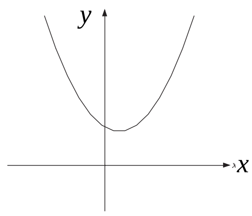
(Worked Example 1)



One solution

$b^2 - 4ac = 0$

(Worked Example 2)



No solutions

$b^2 - 4ac < 0$

(Worked Example 3)

Exercises

1. Use the quadratic equation formula to find the solutions, where they exist, of each of the following equations. Give answers to 2 decimal places.

(a)	$4x^2 - 7x + 3 = 0$	(b)	$2x^2 + x - 10 = 0$	(c)	$9x^2 - 6x - 11 = 0$
(d)	$3x^2 - 5x - 7 = 0$	(e)	$x^2 + x - 8 = 0$	(f)	$4x^2 - 6x - 9 = 0$
(g)	$2x^2 + 17x - 9 = 0$	(h)	$x^2 - 14x = 0$	(i)	$x^2 + 2x - 10 = 0$
(j)	$3x^2 + 8x - 1 = 0$	(k)	$x^2 + 6 = 0$	(l)	$2x^2 - 8x + 3 = 0$
(m)	$4x^2 - 5x - 3 = 0$	(n)	$5x^2 - 4x + 12 = 0$	(o)	$x^2 - 6x - 5 = 0$

2. A ticket printing and cutting machine cuts rectangular cards which are 2 cm longer than they are wide.

   (a) If x is the width of a ticket, find an expression for the area of the ticket.

   (b) Find the size of a ticket with an area of 10 cm^2.

3. A window manufacturer makes a range of windows for which the height is 0.5 m greater than the width.

   Find the width and height of a window with an area of 2 m^2.

4. The height of a stone launched from a catapult is given by

$$h = 20t - 9.8t^2$$

   where t is the time after the moment of launching.

   (a) Find when the stone hits the ground.

   (b) For how long is the stone more than 5 m above the ground?

   (c) Is the stone ever more than 12 m above ground level?

   (d) If m is the maximum height of the stone, write down a quadratic equation which involves m. Explain why this equation has only one solution and use this fact to find the value of m, to 2 decimal places.

5. The equation below is used to find the maximum amount, x, which a bungee cord stretches during a bungee jump:

$$mgx + mgl - \frac{1}{2}kx^2 = 0,$$

where	m	=	mass of bungee jumper	
	l	=	length of rope when not stretched	(10 m)
	k	=	stiffness constant	(120 Nm^{-1})
	g	=	acceleration due to gravity	(10 ms^{-2})

   (a) Find the maximum amount that the cord stretches for a bungee jumper of mass 60 kg.

(b) How much more would the cord stretch for a person of mass 70 kg?

6. Solve the equation $x^2 = 5x + 7$, giving your answers correct to 3 significant figures.

(NEAB)

10.13 Algebraic Fractions

Some algebraic fractions can be simplified if the numerator and denominator are factorised.

Worked Example 1

Simplify

$$\frac{x^2 + 2x - 15}{x^2 + x - 12}.$$

Solution

Both the top and bottom of the fraction can be factorised to give

$$\frac{x^2 + 2x - 15}{x^2 + x - 12} = \frac{(x+5)(x-3)}{(x+4)(x-3)}.$$

The term $(x-3)$ appears as a factor on both the top and bottom of the fraction so it can be cancelled to give

$$\frac{(x+5)\cancel{(x-3)}}{(x+4)\cancel{(x-3)}} = \frac{x+5}{x+4}$$

No further simplification is possible.

Worked Example 2

Simplify

$$\frac{x}{x-1} \times \frac{x^2 - 1}{x^3}.$$

Solution

First multiply together the fractions to give

$$\frac{x}{x-1} \times \frac{x^2 - 1}{x^3} = \frac{x\left(x^2 - 1\right)}{x^3 (x-1)}.$$

The bracket $\left(x^2 - 1\right)$ can be factorised using the difference of two squares to give

$$\frac{x\left(x^2 - 1\right)}{x^3 (x-1)} = \frac{x(x+1)(x-1)}{x^3 (x-1)}.$$

Cancellation can then take place to give

$$\frac{\overset{1}{\cancel{x}}(x+1)\overset{1}{\cancel{(x-1)}}}{{}_{x^2}\cancel{x^3}\underset{1}{\cancel{(x-1)}}} = \frac{x+1}{x^2}$$

No further simplification is possible.

Worked Example 3

Simplify

$$\frac{x+1}{x^2} \div \frac{x^2-1}{x}.$$

Solution

Recall from working with fractions that

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$

So

$$\begin{aligned}
\frac{x+1}{x^2} \div \frac{x^2-1}{x} &= \frac{x+1}{{}_{x}\cancel{x^2}} \times \frac{\overset{1}{\cancel{x}}}{x^2-1} \\
&= \frac{x+1}{x\left(x^2-1\right)} \\
&= \frac{\overset{1}{\cancel{x+1}}}{x\underset{1}{\cancel{(x+1)}}(x-1)} \\
&= \frac{1}{x(x-1)}.
\end{aligned}$$

Worked Example 4

Simplify

$$\frac{x}{x+2} + \frac{2x}{x+1}.$$

Solution

Algebraic fractions are added in the same way as ordinary fractions, by using the lowest common denominator. In this case it will be $(x+2)(x+1)$.

$$\begin{aligned}
\frac{x}{x+2} + \frac{2x}{x+1} &= \frac{x(x+1)}{(x+2)(x+1)} + \frac{2x(x+2)}{(x+2)(x+1)} \\
&= \frac{x^2 + x + 2x^2 + 4x}{(x+2)(x+1)}
\end{aligned}$$

$$= \frac{3x^2 + 5x}{(x+2)(x+1)}$$

$$= \frac{x(3x+5)}{(x+2)(x+1)}$$

No further simplification is possible.

Exercises

1. Simplify each of the following expressions.

(a) $\dfrac{x^6}{x^4}$ (b) $\dfrac{x(x+2)}{x^2(x-2)}$ (c) $\dfrac{(x+2)(x-3)}{(x-2)(x+2)}$

(d) $\dfrac{(x-4)(2x-1)}{x(x-4)(x+1)}$ (e) $\dfrac{x^3(x-6)}{x(x+6)}$ (f) $\dfrac{x(x-2)(x+3)}{x^2(x-2)(x+3)}$

(g) $\dfrac{(2x+1)(5x-3)}{x(2x+1)}$ (h) $\dfrac{(2x+1)(x-2)(x+7)}{x(x+7)(x-2)}$

(i) $\dfrac{(x+1)(x-3)(x+4)}{(x+4)(x+2)(x+1)}$

2. Simplify each of the following expressions.

(a) $\dfrac{x^2+x}{x^3+x^2}$ (b) $\dfrac{x^2+2x}{x+2}$ (c) $\dfrac{4x^2+6x}{2x+3}$

(d) $\dfrac{x^4+5x^2}{x(x+5)}$ (e) $\dfrac{x^3-x^2}{x}$ (f) $\dfrac{x^6-x^3}{x(x-1)}$

(g) $\dfrac{x^2+x}{x^2-1}$ (h) $\dfrac{x^3-4x}{x^2-3x-4}$ (i) $\dfrac{x^2-9}{x^2+4x+3}$

(j) $\dfrac{x^3-16x}{x^2-6x+8}$ (k) $\dfrac{x^2+7x+10}{x^2-x-6}$ (l) $\dfrac{x^3-6x^2+8x}{x^2-3x-4}$

(m) $\dfrac{x^3+x^2-2x}{x^4-x^3-6x^2}$ (n) $\dfrac{x^3-x}{x^2+9x+8}$ (o) $\dfrac{x^2-7x+10}{x^3+4x^2-12x}$

(p) $\dfrac{10x^2-27x+18}{2x^2-x-3}$ (q) $\dfrac{5x^2+23x-10}{x^2-2x-35}$ (r) $\dfrac{3x^2-5x-2}{5x^2-9x-2}$

(s) $\dfrac{2x^2+3x-35}{3x^2+17x+10}$ (t) $\dfrac{12x^2-5x-3}{15x^2+2x-1}$ (u) $\dfrac{9x^3-4x}{12x^2+x-6}$

3. Simplify each of the following expressions.

(a) $\dfrac{x+1}{x+4} \times \dfrac{x}{x+1}$

(b) $\dfrac{1}{(x+1)^2} \times \dfrac{x^2-1}{x}$

(c) $\dfrac{x^2+5x+6}{x} \times \dfrac{x+2}{x+3}$

(d) $\dfrac{3x}{x^2-4} \times \dfrac{x-2}{x^3}$

(e) $\dfrac{x+3}{x} \div \dfrac{x^2}{x+3}$

(f) $\dfrac{(x+3)^2}{x} \div \dfrac{x+3}{x^2}$

(g) $\dfrac{x^3}{x+6} \div \dfrac{x^2+x}{x^2+5x-6}$

(h) $\dfrac{x^3-x}{x+4} \div \dfrac{x^2+3x-4}{x^2}$

(i) $\dfrac{x^2+5x+6}{x-4} \div \dfrac{x^2-2x-8}{x+3}$

4. Express each of the following as a single fraction, simplifying if possible.

(a) $\dfrac{1}{x+2} + \dfrac{2}{x-5}$

(b) $\dfrac{4}{x+2} - \dfrac{5}{x-1}$

(c) $\dfrac{1}{x^2-1} + \dfrac{1}{x+1}$

(d) $\dfrac{3}{x+2} + \dfrac{5}{x+2}$

(e) $\dfrac{1}{x^2-9} + \dfrac{1}{x-3}$

(f) $\dfrac{4}{x-3} + \dfrac{5}{x-6}$

(g) $\dfrac{x}{x-4} + \dfrac{x}{x+2}$

(h) $\dfrac{x^2}{x+1} + \dfrac{x}{x+1}$

(i) $\dfrac{x}{x-6} + \dfrac{4}{2x-1}$

(j) $\dfrac{3x}{x+2} + \dfrac{2x}{x-6}$

(k) $\dfrac{x^2}{x^2-1} + \dfrac{x}{x+1}$

(l) $\dfrac{x+1}{x^2} + \dfrac{x+7}{x}$

(m) $\dfrac{3x}{x+6} + \dfrac{2x}{x+7}$

(n) $\dfrac{x-1}{x+2} + \dfrac{x+7}{x+3}$

(o) $\dfrac{x-3}{x+2} + \dfrac{x-7}{x+1}$

10.14 Completing the Square

Completing the square is a technique which can be used to solve quadratic equations that do not factorise. It can also be useful when finding the minimum or maximum value of a quadratic.

A general quadratic ax^2+bx+c is written in the form $a(x+p)^2+q$ when completing the square. You need to find the constants p and q so that the two expressions are identical.

Worked Example 1

Complete the square for $x^2+10x+2$.

Solution

First consider the $x^2 + 10x$. These terms can be obtained by expanding $(x+5)^2$.

But $$(x+5)^2 = x^2 + 10x + 25,$$

so $$x^2 + 10x = (x+5)^2 - 25.$$

Therefore
$$\begin{aligned} x^2 + 10x + 2 &= (x+5)^2 - 25 + 2 \\ &= (x+5)^2 - 23. \end{aligned}$$

Worked Example 2

Complete the square for $x^2 + 6x - 8$.

Solution

To obtain $x^2 + 6x$ requires expanding $(x+3)^2$.

But $$(x+3)^2 = x^2 + 6x + 9,$$

so $$x^2 + 6x = (x+3)^2 - 9.$$

Therefore
$$\begin{aligned} x^2 + 6x - 8 &= (x+3)^2 - 9 - 8 \\ &= (x+3)^2 - 17. \end{aligned}$$

Note

When completing the square for $x^2 + bx + c$, the result is

$$\boxed{x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c}$$

Worked Example 3

Complete the square for $3x^2 + 6x + 7$.

Solution

As a first step, the quadratic can be rearranged as shown below.

$$3x^2 + 6x + 7 = 3\left(x^2 + 2x\right) + 7.$$

Then note that $$x^2 + 2x = (x+1)^2 - 1,$$

so
$$\begin{aligned} 3\left(x^2 + 2x\right) + 7 &= 3\left[(x+1)^2 - 1\right] + 7 \\ &= 3(x+1)^2 - 3 + 7 \\ &= 3(x+1)^2 + 4. \end{aligned}$$

Worked Example 4

(a) Complete the square for $y = 2x^2 - 8x + 2$.

(b) Find the minimum value of *y*.

(c) Sketch the graph of $y = 2x^2 - 8x + 2$.

Solution

(a) First rearrange the quadratic as shown.

$$2x^2 - 8x + 2 = 2\left(x^2 - 4x\right) + 2.$$

Then $x^2 - 4x$ can be written as $(x-2)^2 - 4$ to give

$$\begin{aligned} 2\left(x^2 - 4x\right) + 2 &= 2\left[(x-2)^2 - 4\right] + 2 \\ &= 2(x-2)^2 - 8 + 2 \\ &= 2(x-2)^2 - 6. \end{aligned}$$

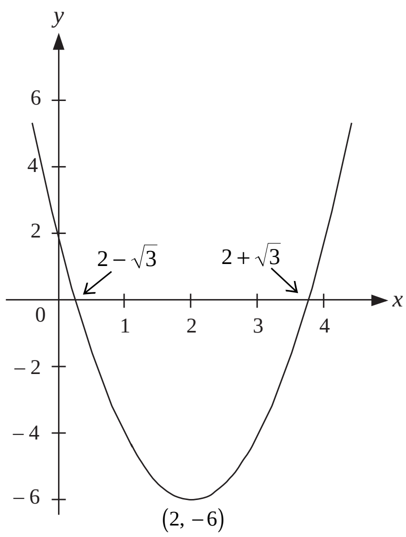
(b) As $y = 2(x-2)^2 - 6$, the minimum possible value of *y* is -6, which is obtained when $x - 2 = 0$ or $x = 2$.

(c) Before sketching the graph, it is also useful to find where the curve crosses the *x*-axis, that is when $y = 0$. To do this, solve

$$\begin{aligned} 0 &= 2(x-2)^2 - 6 \\ 2(x-2)^2 &= 6 \\ (x-2)^2 &= 3 \\ x - 2 &= \pm\sqrt{3} \\ x &= 2 \pm \sqrt{3} \end{aligned}$$

So the curve crosses the *x*-axis at $2 + \sqrt{3}$ and $2 - \sqrt{3}$, and has a minimum at $(2, -6)$.

This is shown in the graph opposite.



Exercises

1. Complete the square for each of the expressions below.

(a) $x^2 + 4x - 5$ (b) $x^2 + 6x - 1$ (c) $x^2 + 10x - 2$

(d) $x^2 - 8x + 2$ (e) $x^2 + 12x + 3$ (f) $x^2 - 20x + 10$

(g) $x^2 + 3x - 1$ (h) $x^2 - 5x + 2$ (i) $x^2 - x + 4$

2. Use the completing the square method to solve each of the following equations.

(a) $x^2 - 4x + 3 = 0$ (b) $x^2 - 6x - 4 = 0$ (c) $x^2 + 10x - 8 = 0$

(d) $x^2 + 5x + 1 = 0$ (e) $x^2 + x - 1 = 0$ (f) $x^2 + 2x - 4 = 0$

(g) $x^2 + 4x - 8 = 0$ (h) $x^2 + 5x - 2 = 0$ (i) $x^2 + 7x + 1 = 0$

3. Complete the square for each of the following expressions.

(a) $2x^2 + 8x - 1$ (b) $2x^2 + 10x - 3$ (c) $2x^2 + 2x + 1$

(d) $3x^2 + 6x - 2$ (e) $5x^2 + 15x - 4$ (f) $7x^2 - 14x + 2$

(g) $3x^2 + 12x - 4$ (h) $4x^2 + 20x - 3$ (i) $2x^2 - 12x + 3$

4. Solve each of the following equations by completing the square.

(a) $2x^2 + 4x - 5 = 0$ (b) $2x^2 + 16x - 3 = 0$

(c) $3x^2 + 12x - 8 = 0$ (d) $4x^2 + 2x - 1 = 0$

(e) $2x^2 + x - 6 = 0$ (f) $5x^2 - 20x + 1 = 0$

5. Sketch the graph of each equation below, showing its minimum or maximum point and where it crosses the x-axis.

(a) $y = x^2 - 2x - 1$ (b) $y = x^2 + 6x + 8$

(c) $y = x^2 - 10x + 24$ (d) $y = x^2 + 5x - 14$

(e) $y = 4 + 3x - x^2$ (f) $y = 3x - 2 - x^2$

6. The height of a ball thrown into the air is given by

$$h = 1 + 20t - 10t^2 .$$

Find the maximum height reached by the ball.

Information

The word 'quadratic' comes from the Latin word 'quadratum', which means 'a squared figure'.

7. (a) By writing the quadratic expression

$$x^2 - 4x + 2$$

in the form $(x + a)^2 + b$, find a and b and hence find the minimum value of the expression.

(b) Solve the equation

$$x^2 - 4x + 2 = 0$$

giving your answers correct to 2 decimal places.

(SEG)

8. (a) Factorise the expression $x^2 + 2x - 3$.

(b) Express $x^2 + 2x - 3$ in the form $(x + a)^2 - b$, where a and b are whole numbers.

(c) Sketch the curve with equation $y = x^2 + 2x - 3$.

(MEG)

10.15 Algebraic Fractions and Quadratic Equations

This final section deals with finding the solutions of equations such as

$$\frac{x}{x-6} + \frac{x}{x+2} = \frac{2}{15}$$

and uses techniques from the earlier sections.

Worked Example 1

Find the solution of

$$\frac{x}{x-6} + \frac{x}{x+2} = \frac{2}{15}.$$

Solution

The first stage is to combine the two fractions so that they have a common denominator.

$$\frac{x(x+2) + x(x-6)}{(x-6)(x+2)} = \frac{2}{15}$$

$$\frac{x^2 + 2x + x^2 - 6x}{(x-6)(x+2)} = \frac{2}{15}$$

$$\frac{2x^2 - 4x}{(x-6)(x+2)} = \frac{2}{15}$$

Then dividing both sides by 2 gives

$$\frac{x^2 - 2x}{(x-6)(x+2)} = \frac{1}{15}$$

Then multiplying both sides by 15 and $(x-6)(x+2)$ gives

$$15\left(x^2 - 2x\right) = (x-6)(x+2).$$

Therefore

$$15x^2 - 30x = x^2 - 4x - 12$$

and

$$14x^2 - 26x + 12 = 0.$$

This factorises as

$$(x-1)(14x-12) = 0$$

so

$$x - 1 = 0 \quad \text{or} \quad 14x - 12 = 0$$

and

$$x = 1 \quad \text{or} \quad x = \frac{12}{14} = \frac{6}{7}.$$

Worked Example 2

Solve the equation

$$\frac{1}{x} + \frac{x}{x+2} = \frac{5}{3}.$$

Solution

The first step is to combine the two fractions.

$$\frac{x^2 + x + 2}{x(x+2)} = \frac{5}{3}$$

Then multiplying by 3 and $x(x+2)$ gives

$$3\left(x^2 + x + 2\right) = 5x(x+2).$$

$$3x^2 + 3x + 6 = 5x^2 + 10x$$

$$0 = 2x^2 + 7x - 6.$$

This quadratic can be solved by completing the square (or using the formula) to give

$$x^2 + \frac{7}{2}x - 3 = 0 \qquad \text{(dividing both sides by 2)}$$

$$\left(x + \frac{7}{4}\right)^2 - \frac{49}{16} - 3 = 0 \qquad \left[\left(x + \frac{7}{4}\right)\left(x + \frac{7}{4}\right) = x^2 + \frac{7}{2}x + \frac{49}{16}\right]$$

$$\left(x + \frac{7}{4}\right)^2 = \frac{49}{16} + \frac{48}{16}$$

$$= \frac{97}{16}$$

$$x + \frac{7}{4} = \pm\frac{\sqrt{97}}{4}$$

$$x = -\frac{7}{4} \pm \frac{\sqrt{97}}{4}$$

$$x = \frac{-7 + \sqrt{97}}{4} \quad \text{or} \quad \frac{-7 - \sqrt{97}}{4}.$$

Exercises

Solve each of the following equations.

1. $\dfrac{x}{x+4} + \dfrac{1}{x} = \dfrac{3}{4}$

2. $\dfrac{x}{x-2} - \dfrac{x}{x+2} = \dfrac{4}{3}$

3. $\dfrac{2x}{x+1} - \dfrac{x}{x+3} = \dfrac{25}{24}$

4. $\dfrac{x}{2x+1} + \dfrac{x}{x-1} = \dfrac{12}{5}$

5. $\dfrac{x}{x+3} - \dfrac{x}{x-5} = 2$

6. $\dfrac{x}{2x+1} - \dfrac{x}{2x+7} = \dfrac{2}{15}$

7. $\dfrac{1}{x+2} + \dfrac{1}{x+3} = \dfrac{9}{20}$

8. $\dfrac{x}{x-5} - \dfrac{x}{x+5} = \dfrac{4}{3}$

9. $\dfrac{8x}{x+3} - \dfrac{x}{x-2} = 1$

10. $\dfrac{2}{x} - \dfrac{1}{x+6} = \dfrac{7}{8}$

11. $\dfrac{4}{x} + \dfrac{3}{10-x} = 2$

12. $\dfrac{1}{2x+3} + \dfrac{1}{6-x} = \dfrac{2}{5}$

13. $\dfrac{x}{x+2} - \dfrac{x-1}{x} = 4$

14. $\dfrac{1}{10-x} + \dfrac{x}{x+4} = \dfrac{4}{7}$

15. $\dfrac{3}{x+1} + \dfrac{7}{x-6} = 5$

Information

Both linear and quadratic equations have been used for over four thousand years. The early Chinese and Babylonians made use of equations to solve daily problems such as the sharing of an inheritance.