UNIT 10 *Equations* Activities

Activities

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ACTIVITY 10.1 *Solving Equations*

Solving equations is a fundamental part of algebra. At first, it might seem as if there are different rules for different types of equations and you might be confused as to which rule applies. In fact, all algebraic manipulations are based on the concept of

balancing equations.

Whichever process is applied to one side of the equation *must* also be applied to the other side. To solve a *general* linear equation of the form

$$\boxed{ax + b = c}$$

where x is the unknown and a, b and c are given constants, we must make x the subject.

The procedure is shown below for both the *general* case and, as an example, $2x + 5 = 9$.

		General formula	*Example*
		$\boxed{ax + b = c}$	$\boxed{2x + 5 = 9}$
Step 1	Subtract b from each side:	$ax + b - b = c - b$	$2x + 5 - 5 = 9 - 5$
		$ax = c - b.$	$2x = 4$
Step 2	Divide each side by a:	$\frac{ax}{a} = \frac{c-b}{a}$	$\frac{2x}{2} = \frac{4}{2}$
		$x = \frac{c-b}{a}$	$x = 2.$

Using this method, find the value of x in the following equations.

1. (a) $3x + 7 = 10$ (b) $5x - 4 = 26$ (c) $6x + 7 = 22$

   (d) $4 - x = -3$ (e) $10 - \frac{3x}{2} = 16$ (f) $11x - 52 = 3$

The special cases where either $a = 1$ or $b = 0$ can be solved easily.

2. (a) $x + 3 = 7$ (b) $x - 5 = -4$ (c) $7 - x = -4$

3. (a) $2x = 6$ (b) $3x = -15$ (d) $\frac{x}{2} = 10$ (d) $\frac{5x}{3} = 15$

Extension

1. Find the *general* solution of $\boxed{ax + b = cx + d}$ where a, b, c and d are constants.

2. Use this method to solve:

   (a) $2x + 1 = x + 6$ (b) $3x + 2 = 5 - x$ (c) $4 - 7x = 3x + 14$

ACTIVITY 10.2 *Magic Squares*

In *magic squares* the sum of the numbers in each row, column and diagonal are all equal to the *magic number*.

1. Here is an example.

   Check that the sum of the numbers in each row, column and diagonal is equal to the magic number, 12.

3	2	7
8	4	0
1	6	5

Solving magic squares

	11	7
9		
	5	10

This magic square is more challenging! The answer may be found by trial and error but really a more systematic method is required.

Let x be the unknown number in *Column 1*, *Row 1*,
y be the unknown number in *Column 1*, *Row 3*,
n be the *magic* number.

x	11	7
9		
y	5	10

Then for *Row 1*, $n = x + 11 + 7 = x + 18$,

and for *Column 1*, $n = x + 9 + y$.

So, $x + 18 = x + 9 + y$,

$18 = 9 + y$

or $y = 9$.

From *Row 3*, $n = y + 5 + 10$, so $n = 24$. From *Row 1*, $x + 18 = n = 24$, so $x = 6$.

The other two missing numbers can then be found to be 8 (*Column 2*) and 7 (*Column 3*).

2. Use an algebraic approach to solve the following magic squares.

(a)

9	2	
12	8	

(b)

10	3	
5		9
	11	4

(c)

14		12
10		8

Extension

a	b	
c	d	

For the general magic square opposite:

1. Find an expression for the missing entries in terms of a, b, c, d and n.
2. Form equations for the sums in the two diagonals.
3. Hence solve for the unknowns, c and d, in terms of a, b and n.
4. Use your general magic square to solve the magic squares in Problem 2.

ACTIVITY 10.3 Sheet 1 *Solving Simultaneous Equations*

Simultaneous equations are of the *general* form

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

where the coefficients, a, b, c and d, and e and f are *constants*.

Before solving this *general* case, we will demonstrate three 'short-cuts' which can be used in the following *special* cases. Bear them in mind when solving the problems on *Sheet 2*.

1 *Coefficient of x (or y) is 1*

You can substitute for x directly in the other equation, for example:

$$\left.\begin{aligned} x + 2y &= 4 \\ 3x + y &= 7 \end{aligned}\right\} \Rightarrow \quad x = 4 - 2y \quad \Rightarrow \quad 3(4-2y) + y = 7$$

$$\Rightarrow \quad 12 - 6y + y = 7 \quad \Rightarrow \quad 5 = 5y$$

So $y = 1$ and $x = 4 - 2 \times 1 = 2$. [*Check* this in the second equation: $(3 \times 2) + 1 = 7$]

2 *Coefficients of x (or y) are multiples*

You can multiply one equation by the multiple, for example:

$$\left.\begin{aligned} 2x + 3y &= 3 \\ 6x - 2y &= 20 \end{aligned}\right\}$$ Multiply the first equation by 3, since $6 = 3 \times 2$, to give

$$\left.\begin{aligned} 6x + 9y &= 9 \\ 6x - 2y &= 20 \end{aligned}\right\}$$ Subtract the second equation from the first equation to give

$$9y - (-2y) = 9 - 20 \quad \Rightarrow \quad 11y = -11 \quad \Rightarrow \quad y = -1.$$

Substituting for y in the very first equation gives $2x - 3 = 3 \Rightarrow 2x = 6 \Rightarrow x = 3$.
[*Check* this in the second equation: $(6 \times 3) - (2 \times -1) = 20$]

3 *Coefficients of x (or y) have a factor in common*

Find the *lowest common multiple* (L.C.M.) for the two coefficients and multiply each equation by the factor needed to make the coefficient equal to the L.C.M, for example:

$$\left.\begin{aligned} 6x + 2y &= 10 \\ 15x + 7y &= 23 \end{aligned}\right\}$$ L.C.M. of x-coefficients is 30.
Multiply the first equation by 5 and the second by 2, giving

$$\left.\begin{aligned} 30x + 10y &= 50 \\ 30x + 14y &= 46 \end{aligned}\right\}$$ Subtract the second equation from thefirst equation to give

$$10y - 14y = 50 - 46 \quad \Rightarrow \quad -4y = 4 \quad \Rightarrow \quad y = -1.$$

Substituting for y in the very first equation gives $6x - 2 = 10 \Rightarrow 6x = 12 \Rightarrow x = 2$.
[*Check* this in the second equation: $(15 \times 2) + (7 \times -1) = 23$]

ACTIVITY 10.3 Sheet 2 *Solving Simultaneous Equations*

Use the information on *Sheet 1*, where possible, to solve the following simultaneous equations.

1. Use *Method 1* to solve:

(a) $x + 3y = 9$, $4x + y = 14$

(b) $2x + y = 7$, $5x - 3y = 23$

(c) $3x - 5y = 4$, $x + 2y = 5$

2. Use *Method 2* to solve:

(a) $3x + 2y = 10$, $12x - 5y = 14$

(b) $2x + 5y = 4$, $7x + 15y = 9$

(c) $7x + 3y = 13$, $-21x + 5y = -11$

3. Use *Method 3* to solve:

(a) $10x - 3y = 16$, $15x + 7y = 1$

(b) $8x + 5y = 2$, $12x - 3y = -18$

(c) $7x + 6y = 45$, $5x - 11y = -29$

Extension

1. (a) For the most general case,

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

where a and c are *coprimes* (i.e. have no factors in common) and b and d are coprimes, show that

$$y = \frac{ce - af}{bc - ad}$$

and hence deduce the value of x.

[*Hint*: You can use *Method 3* on *Sheet 1*, noting that the L.C.M. of a and c is $a \times c$.]

(b) When does this method *not* work?

2. Use the method you think is best to solve:

(a) $4x + 3y = 5$, $5x + 2y = 8$

(b) $3x + 7y = 8$, $15x + 2y = -26$

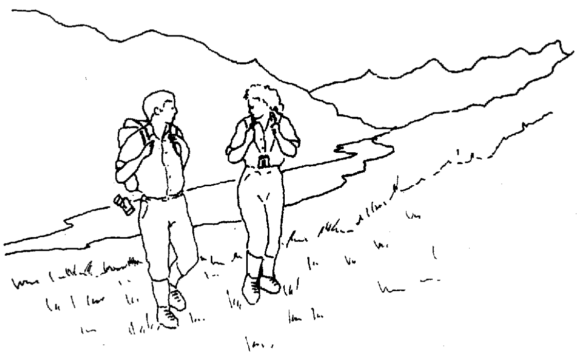
(c) $5x - y = 29$, $4x + 2y = 12$

(d) $4x + 3y = 5$, $5y + 7x = 7$

(e) $2x + 5y = 11$, $7x + 2y = 24$

(f) $3x + 2y = 2$, $15x + 10y = 10$

ACTIVITY 10.4 *Hill Walking*



Walking up a hill slows down your pace, but by how much?

We will try to provide a mathematical model to describe this situation, which can then be used to estimate the time it would take to climb a hill.

The time taken (T hours) will generally depend on 4 factors:

1. *Map distance* (horizontal distance travelled) (d miles)
2. *Height of the hill* (vertical distance climbed) (h feet)
3. *Speed of walking horizontally* (x miles per hour)
4. *Speed of climbing vertically* (y feet per hour)

The model will be of the form

$$T = \frac{d}{x} + \frac{h}{y}$$

1. The table below shows some data which was gathered experimentally.

   (a) If $X = \frac{1}{x}$ and $Y = \frac{1}{y}$, use the data opposite to form two equations involving X and Y.

   (b) Solve these equations for X and Y.

   (c) Hence find the values of x and y.

Map Distance (d miles)	*Height of Hill* (h feet)	*Time Taken* (T hours)
12	1500	$5\frac{1}{2}$
15	2000	7

2. Use the model

$$T = \frac{d}{3} + \frac{h}{1000}$$

   to determine how long it might take to climb *Mount Snowdon* (height 3560 feet) if the map distance from your starting point is 3 miles.

3. If it takes you 8 hours to climb a hill and your map distance is 4 miles, estimate the height of the hill.

Extension

1. Apply this test to a local situation, using your own experimental data, to see whether this model works in practice.

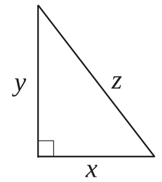
2. Produce your own model for the time it takes to come *down* a hill.

ACTIVITY 10.5 *Diophantine Equations*

Equations which have many *integer* (positive/negative whole numbers) solutions are known as *Diophantine equations*, after the Greek mathematician and philosopher, *Diophantos* of Alexandria (c. 250 A.D.). He is credited with being the founder of modern algebra. The use of symbols to represent numbers was found in hs published document, *Arithmetic*.

1. One example of a *Diophantine equation* could be Pythagoras' result that, for any right-angled triangle,

$$x^2 + y^2 = z^2.$$



Find 3 different integer solutions to $x^2 + y^2 = z^2$.

2. Another example of a *Diophantine equation* is:

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

for some number, n, and where x and y are integers.

(a) For example, when $n = 6$,

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{6}.$$

(i) One possible solution to this equation is $x = 8$, $y = 24$.
Check that these values do indeed give a solution for the equation.

(ii) A second solution is obviously $x = 24$, $y = 8$, but there are many more.
Find in total 17 solutions of this equation.
[*Hint*: Remember that x or y can be a negative integer.]

(b) How many integer solutions can you find for the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{8}$?

Extension

1. Find two solutions, using small integers, to the equation $x^3 + y^3 + z^3 = 3$.

2. An unsolved conjecture is that the *Diophantine equation*,

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{4}{n},$$

has at least *one* solution for x, y and z for any integer, $n > 1$.

Show that this conjecture is true for $n = 2$, 3 and 4.